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Applications of soft sets to q-ideals and a-ideals in BCI-algebras

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Abstract. The notion of intersectional soft q-ideals and a-ideals of a BCI-algebra are introduced, and some related properties are investigated. We show that given a subalgebra A of a BCI-algebra X, an intersectional A-soft set $\mathscr{F}_A$ over U is an intersectional A-soft a-ideal over U if and only if it is both an intersectional A-soft p-ideal over U and an intersectional A-soft q-ideal over U.

1. **Introduction**

In 1966, Imai and Iséki ([4]) and Iséki ([5]) introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. In [19], the notion of p-ideals of BCI-algebras was introduced and several properties of them were investigated. Y. S. Hwang and S. S. Ahn ([2]) defined the notion of fuzzy p-ideals of BCI-algebras with degrees in the interval $(0, 1]$.

Various problems in system identification involve characteristics which are essentially non-probabilistic in nature ([16]). In response to this situation Zadeh ([17]) introduced *fuzzy set theory* as an alternative to probability theory. Uncertainty is an attribute of information. In order to suggest a more general framework, the approach to uncertainty is outlined by Zadeh ([18]). To solve complicated problem in economics, engineering, and environment, we can't successfully use classical methods because of various uncertainties typical for those problems. There are three theories: theory of probability, theory of fuzzy sets, and the interval mathematics which we can consider as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties. Uncertainties can't be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as probability theory, theory of (intuitionistic) fuzzy sets, theory of vague sets, theory of interval mathematics, and theory of rough sets. However, all of these theories have their own difficulties which are pointed out in [14]. Maji et al. ([12]) and Molodtsov ([14]) suggested that one reason for these difficulties may be due to the inadequacy of the parametrization tool of the theory. To overcome these difficulties, Molodtsov ([14]) introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. At present, works on the soft set theory are progressing rapidly. Maji et al. ([12]) described the application of soft set

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theory to a decision making problem. Y. S. Hwang and S. S. Ahn ([3]) introduced the notion of int-soft p-ideals of a BCI-algebra, and investigated their properties.

In this paper, we introduced the notions of intersectional soft q-ideals and a-ideals of a BCI-algebra and investigate some related properties. We show that given a subalgebra A of a BCI-algebra X, an intersectional A-soft set $\mathscr{F}_A$ over U is an intersectional A-soft a-ideal over U if and only if it is both an intersectional A-soft p-ideal over U and an intersectional A-soft q-ideal over U.

2. **Preliminaries**

In this section, we recall some basic definitions and results on BCK/BCI-algebras and soft set theory.

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.

An algebra $(X; *, 0)$ of type (2, 0) is called a *BCI-algebra* if it satisfies the following conditions:

(I) $(\forall x, y, z \in X)(((x * y) * (x * z)) * (z * y) = 0)$,
(II) $(\forall x, y \in X)((x * (x * y)) * y = 0)$,
(II) $(\forall x \in X)(x * x = 0)$,
(IV) $(\forall x, y \in X)(x * y = 0, y * x = 0 \Rightarrow x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)(0 * x = 0)$,

then X is called a *BCK-algebra*.

Any BCI-algebra X has the following properties:

(a1) $(\forall x \in X)\ (x * 0 = x)$.
(a2) $(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y)$.
(a3) $(\forall x, y \in X)\ (0 * (x * y) = (0 * x) * (0 * y))$.
(a4) $(\forall x, y \in X)\ (x * (x * (x * y)) = x * y)$.
(a5) $(\forall x, y, z \in X)\ (x \leq y \ \Rightarrow\ x * z \leq y * z,\ z * y \leq z * x)$.
(a6) $(\forall x, y, z \in X)\ ((x * z) * (y * z) \leq x * y)$.

where $x \leq y$ if and only if $x * y = 0$. A non-empty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. A non-empty subset A of a BCK/BCI-algebra X is called a *BCK/BCI-ideal* of X if it satisfies:

(b1) $0 \in A$,
(b2) $(\forall x \in X)(\forall y \in A)(x * y \in A \Rightarrow x \in A)$.

A non-empty subset A of a BCI-algebra X is called a *p-ideal* ([19]) of X if it satisfies (b1) and

(b3) $(\forall x, y, z \in X)((x * z) * (y * z) \in A, y \in A \Rightarrow x \in A)$.

A non-empty subset A of a BCI-algebra X is called a *q-ideal* ([11]) of X if it satisfies (b1) and

(b4) $(\forall x, y, z \in X)(x * (y * z) \in A, y \in A \Rightarrow x * z \in A)$.

A non-empty subset A of a BCI-algebra X is called a *a-ideal* ([11]) of X if it satisfies (b1) and

(b4) $(\forall x, y, z \in X)((x * z) * (0 * y) \in A, z \in A \Rightarrow y * x \in A)$.

Note that any $p/q/a$-ideal is an ideal, but the converse is not true in general (see[11,19]).

We refer the reader to the books [1,13] for further information regarding BCK/BCI- algebras. Let U and E denote an initial universe set and a set of parameters, respectively. Molodtsov ([14]) defined the soft set in the following way:

Definition 2.1. A pair $(\mathscr{F}, E)$ is called a *soft set* over U if $\mathscr{F}$ is a mapping given by

$$\mathscr{F} : E \to \mathscr{P}(U).$$

Definition 2.2. Given a non-empty subset A of E, a soft set $(\mathscr{F}, E)$ over U satisfying the following condition:

$$\mathscr{F}(x) = \emptyset \text{ for all } x \notin A$$

is called an *A-soft set* over U and is denoted by $\mathscr{F}_A$, that is, an A-soft set $\mathscr{F}_A$ over U is a function $\mathscr{F}_A : E \to \mathscr{P}(U)$ such that $\mathscr{F}_A(x) = \emptyset$ for all $x \notin A$.

Note that an E-soft set over U is a soft set over U.

3. Intersectional soft q-ideal

Definition 3.1.([8]) Let $E = X$ be BCK/BCI-algebra. Given a subset A of E, let $\mathscr{F}_A$ be an A-soft set over U. Then $\mathscr{F}_A$ is called an *intersectional A-soft BCI-algebra* over U if it satisfies the following condition:

(3.1) $(\forall x, y \in A)(x * y \in A \Rightarrow \mathscr{F}_A(x) \cap \mathscr{F}_A(y) \subseteq \mathscr{F}_A(x * y))$.

An intersectional A-soft BCK/BCI-algebra over U with $A = E$ is called an *intersectional soft BCK/BCI-algebra* over U, and it is denoted by $\mathscr{F}_X$.

Definition 3.2.([15]) Let $E = X$ be BCK/BCI-algebra. Given a subset A of E, let $\mathscr{F}_A$ be an A-soft set over U. Then $\mathscr{F}_A$ is called an *intersectional A-soft BCI-ideal* over U if it satisfies the following condition:

(3.2) $(\forall x \in A)(\mathscr{F}_A(x) \subseteq \mathscr{F}_A(0))$.

(3.3) $(\forall x, y \in A)(x * y \in A \Rightarrow \mathscr{F}_A(x * y) \cap \mathscr{F}_A(y) \subseteq \mathscr{F}_A(x))$.

An intersectional A-soft BCK/BCI-algebra over U with $A = E$ is called an *intersectional soft BCI-ideal* over U, and it is denoted by $\mathscr{F}_X$.

Definition 3.3.([3]) Let $E = X$ be a BCK/BCI-algebra. Given a subset A of E, let $\mathscr{F}_A$ be an A-soft set over U. Then $\mathscr{F}_A$ is called an *intersectional A-soft p-ideal* over U if it satisfies (3.2) and the following condition:

(3.4) $(\forall x, y, z \in A)((x * z) * (y * z) \in A \Rightarrow \mathscr{F}_A((x * z) * (y * z)) \cap \mathscr{F}_A(y) \subseteq \mathscr{F}_A(x))$.

An intersectional A-soft BCK/BCI-algebra over U with $A = E$ is called an *intersectional soft p-ideal* over U, and it is denoted by $\mathscr{F}_X$.

Definition 3.4. Let $E = X$ be a BCK/BCI-algebra. Given a subset A of E, let $\mathscr{F}_A$ be an A-soft set over U. Then $\mathscr{F}_A$ is called an *intersectional A-soft q-ideal* over U if it satisfies (3.2) and the following condition:

(3.5) $(\forall x, y, z \in A)(x * (y * z) \in A \Rightarrow \mathscr{F}_A(x * (y * z)) \cap \mathscr{F}_A(y) \subseteq \mathscr{F}_A(x * z))$.

An intersectional A-soft BCK/BCI-algebra over U with $A = E$ is called an *intersectional soft q-ideal* over U, and it is denoted by $\mathscr{F}_X$.

Theorem 3.5. *Let $E = X$ be a BCI-algebra. Given a subalgebra A of E, every intersectional A-soft q-ideal over U is both an intersectional A-soft BCI-algebra over U and an intersectional A-soft BCI-ideal over U.*

Proof. Let $\mathscr{F}_A$ be an intersectional A-soft q-ideal over U. For any $x, z \in A$, putting $y := z$ in (3.5), we have

$$\begin{aligned}\mathscr{F}_A(x * z) \supseteq & \mathscr{F}_A(x * (z * z)) \cap \mathscr{F}_A(z) \\ = & \mathscr{F}_A(x * 0) \cap \mathscr{F}_A(z) \\ = & \mathscr{F}_A(x) \cap \mathscr{F}(z).\end{aligned}$$

Hence (3.1) holds. Thus $\mathscr{F}_A$ is an intersectional A-soft BCI-algebra over X.

For any $x, y \in A$, putting $z := 0$ in (3.5), we have

$$\begin{aligned}\mathscr{F}_A(x) \supseteq & \mathscr{F}_A(x * (y * 0)) \cap \mathscr{F}_A(y) \\ = & \mathscr{F}_A(x * y) \cap \mathscr{F}_A(y).\end{aligned}$$

Hence (3.3) holds. Thus $\mathscr{F}_A$ is an intersectional A-soft BCI-ideal over X. This completes the proof. □

The converse of Theorem 3.5 is not true in general as seen in the following example.

Example 3.6. Consider the BCI-algebra $(\mathbb{Z}; *, 0)$ as the initial universe set U, where $a*b = a-b$ for all $a, b \in \mathbb{Z}$. Let $E = X = \{0, a, b, c\}$ be a BCI-algebra ([9]) with the following Cayley table:

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

(1) Define a soft set $(\mathscr{F}_X, X)$ over U by

$$\mathscr{F}_X : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \mathbb{Z} & \text{if } x = 0, \\ 2\mathbb{Z} & \text{if } x = a \\ 3\mathbb{Z} & \text{if } x = b \\ 8\mathbb{Z} & \text{if } x = c \end{cases}$$

Then $\mathscr{F}_X$ is an intersectional soft BCI-ideal over U([15]). But it is not an intersectional soft q-ideal over U since

$$\mathscr{F}_X(a * c) = \mathscr{F}_X(b) = 3\mathbb{Z} \not\supseteq \mathscr{F}_X(a * (a * c)) \cap \mathscr{F}_X(a) = 8\mathbb{Z} \cap 2\mathbb{Z} = 8\mathbb{Z}.$$

(2) Define a soft set $(\mathscr{G}_X, X)$ over U by

$$\mathscr{G}_X : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \mathbb{Z} & \text{if } x \in \{0, a\} \\ 2\mathbb{Z} & \text{if } x \in \{b, c\} \end{cases}$$

Then $\mathscr{G}_X$ is an intersectional soft q-ideal over U.

Example 3.7. Consider the BCI-algebra $(\mathbb{Z}; *, 0)$ as the initial universe set U, where $a*b = a-b$ for all $a, b \in \mathbb{Z}$. Let $E = X = \{0, a, b, c\}$ be a BCI-algebra ([9]) with the following Cayley table:

$*$	0	a	b	c
0	0	c	b	a
a	a	0	c	b
b	b	a	0	c
c	c	b	a	0

Define a soft set $(\mathscr{F}_X, X)$ over U by

$$\mathscr{F}_X : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \mathbb{Z} & \text{if } x = 0, \\ 2\mathbb{Z} & \text{if } x \in \{a, b, c\} \end{cases}$$

Then $\mathscr{F}_X$ is both an intersectional soft BCI-algebra over U and an intersectional soft BCI-ideal over U. But it is not an intersectional soft q-ideal over U since

$$\mathscr{F}_X(c * a) = \mathscr{F}_X(b) = 2\mathbb{Z} \not\supseteq \mathscr{F}_X(c * (0 * a)) \cap \mathscr{F}_X(0) = \mathscr{F}_X(0) = \mathbb{Z}.$$

Lemma 3.8.([8]) *Let $E = X$ be a BCK/BCI-algebra. Given a subalgebra A of E, every intersectional A-soft BCK/BCI-ideal $\mathscr{F}_A$ over U satisfies the following condition:*

(1) $(\forall x, y \in A)(x \leq y \Rightarrow \mathscr{F}_A(y) \subseteq \mathscr{F}_A(x))$.
(2) $(\forall x, y, z \in X)(x * y \leq z \Rightarrow \mathscr{F}_A(y) \cap \mathscr{F}_A(z) \subseteq \mathscr{F}_A(x))$.

Theorem 3.9. *Let $E = X$ be a BCI-algebra. Given a subalgebra of X, let $\mathscr{F}_A$ be an intersectional A-soft BCI-ideal over U. Then the following are equivalent:*

(1) *$\mathscr{F}_A$ is an intersectional A-soft q-ideal over U.*
(2) $(\forall x, y \in X)(\mathscr{F}_A(x * y) \supseteq \mathscr{F}_A(x * (0 * y))$.
(3) $(\forall x, y \in X)(\mathscr{F}_A((x * y) * z) \supseteq \mathscr{F}_A(x * (y * z))$.

Proof. (1)$\Rightarrow$(2) Putting $y := 0$ and $z := y$ in (3.5), we have

$$\begin{aligned} \mathscr{F}_A(x * y) &\supseteq \mathscr{F}_A(x * (0 * y)) \cap \mathscr{F}_A(0) \\ &= \mathscr{F}_A(x * (0 * y)). \end{aligned}$$

Hence (2) holds.

(2)⇒(3) Since for any $x, y, z \in X$

$$\begin{aligned}((x*y)*(0*z))*(x*(y*z)) =&((x*y)*(x*(y*z)))*(0*z)\\ \leq&((y*z)*y)*(0*z)\\ =&(0*z)*(0*z)=0,\end{aligned}$$

we have $((x*y)*(0*z))*(x*(y*z)))=0$, it follows from Lemma 3.8(1) that $\mathscr{F}_A(x*(y*z)) \subseteq \mathscr{F}_A((x*y)*(0*z)) \subseteq \mathscr{F}_A((x*y)*z)$. Thus (3) holds.
(3)⇒(1) Using (3.3), (a2), and (3), we get

$$\begin{aligned}\mathscr{F}_A(x*z) \supseteq&\mathscr{F}_A((x*z)*y)\cap\mathscr{F}_A(y)\\ =&\mathscr{F}_A((x*y)*z)\cap\mathscr{F}_A(y)\\ \supseteq&\mathscr{F}_A(x*(y*z))\cap\mathscr{F}_A(y).\end{aligned}$$

Thus $\mathscr{F}_A$ is an intersectional A-soft q-ideal over U. □

Theorem 3.10. *Let $E = X$ be a BCI-algebra. Given a subalgebra A of X, let $\mathscr{F}_A$ be an intersectional A-soft BCI-ideal over U such that $\mathscr{F}_A(x*y) \supseteq \mathscr{F}_A(x)$ for all $x, y \in X$. Then it is an intersectional A-soft q-ideal over U.*

Proof. Using (3.3) and assumption, we have

$$\begin{aligned}\mathscr{F}_A(x*z) \supseteq&\mathscr{F}_A((x*z)*(y*z))\cap\mathscr{F}_A(y*z)\\ =&\mathscr{F}_A((x*(y*z))*z)\cap\mathscr{F}_A(y*z)\\ \supseteq&\mathscr{F}_A(x*(y*z))\cap\mathscr{F}_A(y*z)\\ \supseteq&\mathscr{F}_A(x*(y*z))\cap\mathscr{F}_A(y)\end{aligned}$$

for all $x, y, z \in X$. Hence $\mathscr{F}_A$ is an intersectional A-soft q-ideal over U. □

The converse of Theorem 3.10 is not true in general as seen in the following example.

Example 3.11. Consider the BCI-algebra $(\mathbb{Z}; *, 0)$ as the initial universe set U, where $a*b = a-b$ for all $a, b \in \mathbb{Z}$. Let $E = X = \{0, a, b\}$ be a BCI-algebra ([9]) with the following Cayley table:

$*$	0	a	b
0	0	0	b
a	a	0	b
b	b	b	0

Define a soft set $(\mathscr{F}_X, X)$ over U by

$$\mathscr{F}_X : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \mathbb{Z} & \text{if } x \in \{0, a\} \\ 5\mathbb{Z} & \text{if } x = b \end{cases}$$

Then $\mathscr{F}_X$ is an intersectional soft q-ideal over U, but it does not satisfy $\mathscr{F}_A(x*y) \supseteq \mathscr{F}_A(x)$ since $\mathscr{F}_A(0*b) = \mathscr{F}_A(b) = 5\mathbb{Z} \not\supseteq \mathscr{F}_A(0) = \mathbb{Z}$.

Definition 3.12. A BCI-algebra X is said to be *associative* if $(x * y) * z = x * (y * z)$ for any $x, y, z \in X$. A BCI-algebra X is said to be *quasi-associative* if $(x * y) * z \leq x * (y * z)$ for any $x, y, z \in X$.

Every associative BCI-algebra is quasi-associative, but the converse is not true in general.

Proposition 3.13. *Let $E = X$ be a quasi-associative BCI-algebra. Given a subalgebra A of E, every intersectional A-soft BCI-ideal over U is an intersectional A-soft q-ideal over U.*

Proof. Let $\mathscr{F}_A$ be an intersectional A-soft BCI-ideal over U. Since X is a quasi-associative BCI-algebra, we have $(x * y) * z \leq x * (y * z)$ for any $x, y, z \in X$. It follows from Lemma 3.8(1) that $\mathscr{F}_A((x * y) * z) \supseteq \mathscr{F}_A(x * (y * z))$. By Theorem 3.9, $\mathscr{F}_A$ is an intersectional A-soft q-ideal over U. □

Proposition 3.13 is not true if X is not a quasi-associative BCI-algebra as seen in the following example.

Example 3.14. Consider a BCI-algebra X and an intersectional A-soft set $\mathscr{F}_A$ as in Example 3.7. Since $(a * b) * c \not\leq a * (b * c)$, X is not a quasi-associative BCI-algebra. Then $\mathscr{F}_A$ is an intersectional A-soft BCI-ideal over U but not an intersectional A-soft q-ideal over U.

Corollary 3.15. *Let $E = X$ be an associative BCI-algebra. Given a subalgebra A of E, every intersectional A-soft BCI-ideal over X is an intersectional A-soft q-ideal over U.*

Proof. Straightforward. □

4. Intersectional soft a-ideal

Definition 4.1. Let $E = X$ be BCI-algebra. Given a subset A of E, let $\mathscr{F}_A$ be an A-soft set over U. Then $\mathscr{F}_A$ is called an *intersectional A-soft a-ideal* over U if it satisfies (3.2) and

(3.6) $(\forall x, y, z \in A)((x * z) * (0 * y) \in A \Rightarrow \mathscr{F}_A((x * z) * (0 * y)) \cap \mathscr{F}_A(z) \subseteq \mathscr{F}_A(y * x))$.

An intersectional A-soft BCK/BCI-algebra over U with $A = E$ is called an *intersectional soft a-ideal* over U, and it is denoted by $\mathscr{F}_X$.

Example 4.2. Consider the BCI-algebra $(\mathbb{Z}; *, 0)$ as the initial universe set U, where $a * b = a - b$ for all $a, b \in \mathbb{Z}$. Let $E = X = \{0, a, b, c\}$ be a BCI-algebra as in Example 3.7. Define a soft set $(\mathscr{F}_X, X)$ over U by

$$\mathscr{F}_X : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \mathbb{Z} & \text{if } x \in \{0, a\}, \\ 3\mathbb{Z} & \text{if } x \in \{b, c\} \end{cases}$$

Then $\mathscr{F}_X$ is an intersectional soft a-ideal over U.

Theorem 4.3. *Let $E = X$ be a BCI-algebra. Given a subalgebra A of E, every intersectional A-soft a-ideal over U is both an intersectional A-soft BCI-ideal over U and an intersectional A-soft BCI-algebra over U.*

Proof. Let $\mathscr{F}_A$ be an intersectional A-soft a-ideal over U. For any $x, y \in A$, putting $y = z = 0$ in (3.6), we have

$$\begin{aligned}\mathscr{F}_A(0 * x) \supseteq & \mathscr{F}_A((x * 0) * (0 * 0)) \cap \mathscr{F}_A(0) \\ = & \mathscr{F}_A(x) \cap \mathscr{F}_A(0) \\ = & \mathscr{F}_A(x).\end{aligned} \tag{4.1}$$

Taking $x = z = 0$ in (3.6), for any $y \in A$, we obtain

$$\begin{aligned}\mathscr{F}_A(y) = \mathscr{F}_A(y * 0) \supseteq & \mathscr{F}_A((0 * 0) * (0 * y)) \cap \mathscr{F}_A(0) \\ = & \mathscr{F}_A(0 * (0 * y)) \cap \mathscr{F}_A(0) \\ = & \mathscr{F}_A(0 * (0 * y)).\end{aligned} \tag{4.2}$$

Putting $y = 0$ in (3.6), for any $x, y \in A$ we get

$$\begin{aligned}\mathscr{F}_A(0 * x) \supseteq & \mathscr{F}_A((x * z) * (0 * 0)) \cap \mathscr{F}_A(z) \\ = & \mathscr{F}_A(x * z) \cap \mathscr{F}_A(z).\end{aligned} \tag{4.3}$$

Using (4.2) and (4.1), we obtain $\mathscr{F}_A(x) \supseteq \mathscr{F}_A(0 * (0 * x)) \supseteq \mathscr{F}_A(0 * x)$. By (4.3), we have $\mathscr{F}_A(x) \supseteq \mathscr{F}_A(x * z) \cap \mathscr{F}_A(z)$ and so (3.3) holds. Thus $\mathscr{F}_A$ is an intersectional A-soft BCI-ideal over U.

Using (3.3), we have

$$(\forall x, y, z \in A)(\mathscr{F}_A(x * y) \supseteq \mathscr{F}_A((x * y) * z) \cap \mathscr{F}_A(z)). \tag{4.4}$$

Putting $z := x$ in (4.4) and use (4.1), for any $x, y \in A$ we have

$$\begin{aligned}\mathscr{F}_A(x * y) \supseteq & \mathscr{F}_A((x * y) * x) \cap \mathscr{F}_A(x) \\ = & \mathscr{F}_A((x * x) * y) \cap \mathscr{F}_A(x) \\ = & \mathscr{F}_A(0 * y) \cap \mathscr{F}_A(x) \\ \supseteq & \mathscr{F}_A(y) \cap \mathscr{F}_A(x).\end{aligned}$$

Thus $\mathscr{F}_A$ is an intersectional A-soft BCI-algebra over U. $\square$

The converse of Theorem 4.3 is not true in general as seen in the following example.

Example 4.4. Consider the BCI-algebra $(\mathbb{Z}; *, 0)$ as the initial universe set U, where $a*b = a-b$ for all $a, b \in \mathbb{Z}$. Let $E = X = \{0, a, b\}$ be a BCI-algebra as in Example 3.11. Define a soft set $(\mathscr{F}_X, X)$ over U by

$$\mathscr{F}_X : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \mathbb{Z} & \text{if } x = 0 \\ 7\mathbb{Z} & \text{if } x \in \{a, b\} \end{cases}$$

Then $\mathscr{F}_X$ is both an intersectional A-soft BCI-algebra and an intersectional A-soft BCI-ideal over U. But it is not an intersectional A-soft a-ideal over U since $\mathscr{F}_A(a * 0) \not\supseteq \mathscr{F}_A((0 * 0) * (0 * a)) \cap \mathscr{F}_A(0)$.

Lemma 4.5.([15]) *Let $E = X$ be a BCK/BCI-algebra. Given a subalgebra A of E, every intersectional A-soft BCK/BCI-algebra $\mathscr{F}_A$ over U satisfies:*

$$(\forall x \in X)(\mathscr{F}_A(x) \subseteq \mathscr{F}_A(0)).$$

Next we give the characterizations of intersectional A-soft a-ideal over U.

Theorem 4.6. *Let $E = X$ be a BCI-algebra. Given a subalgebra A of E, let $\mathscr{F}_A$ be an intersectional A-soft BCI-ideal over U. Then the following are equivalent:*

(1) *$\mathscr{F}_A$ is an intersectional A-soft a-ideal over U.*
(2) $(\forall x, y, z \in A)(\mathscr{F}_A(y * (x * z)) \supseteq \mathscr{F}_A((x * z) * (0 * y)))$.
(3) $(\forall x, y \in A)(\mathscr{F}_A(y * x) \supseteq \mathscr{F}_A(x * (0 * y))$.

Proof. (1)⇒(2) Let $s := (x * z) * (0 * y)$ for any $x, y, z \in X$. Then $((x * z) * s) * (0 * y) = ((x * z) * (0 * y)) * s = 0$. Using (3.6), for any $x, y, z \in A$ we have

$$\begin{aligned}\mathscr{F}_A(y * (x * z)) &\supseteq \mathscr{F}_A(((x * z) * s) * (0 * y)) \cap \mathscr{F}_A(s)\\ &= \mathscr{F}_A(0) \cap \mathscr{F}_A(s)\\ &= \mathscr{F}_A(s)\\ &= \mathscr{F}_A((x * z) * (0 * y)).\end{aligned}$$

Hence (2) holds.
(2)⇒(3) Let $z := 0$ in (2). We obtain (3).
(3)⇒(1) Let $x, y, z \in X$. Using (a6) and (II), we have $(x*(0*y))*((x*z)*(0*y)) \leq x*(x*z) \leq z$ and so $(x * (0 * y)) * ((x * z) * (0 * y)) \leq z$. It follows from Lemma 3.8(2) that $\mathscr{F}_A((x * (0 * y)) \supseteq \mathscr{F}_A((x * z) * (0 * y)) \cap \mathscr{F}_A(z)$. Using (3), we obtain

$$\mathscr{F}_A(y * x) \supseteq \mathscr{F}_A(x * (0 * y)) \supseteq \mathscr{F}_A((x * z) * (0 * y)) \cap \mathscr{F}_A(z).$$

Hence (3.6) holds. Thus $\mathscr{F}_A$ is an intersectional A-soft a-ideal over U. □

Now, we discuss the relations among an intersectional A-soft a-ideals, an intersectional A-soft p-ideals and an intersectional A-soft q-ideals over U and give another characterization of an intersectional A-soft a-ideals over U.

Theorem 4.7.([3]) *Let $E = X$ be a BCI-algebra. Given a subalgebra A of X, an intersectional A-soft BCI-ideal over U is an intersectional A-soft p-ideal over U if and only if it satisfies the following*

$$(\forall(x \in A)(\mathscr{F}_A(x) \supseteq \mathscr{F}_A(0 * (0 * x)).$$

Theorem 4.8. *Let $E = X$ be a BCI-algebra. Given a subalgebra A of E, every intersectional A-soft a-ideal over U is an intersectional A-soft p-ideal over U.*

Proof. Let $\mathscr{F}_A$ be an intersectional A-soft a-ideal over U. Then it is an intersectional A-soft BCI-ideal over U by Theorem 4.3. Setting $x = z = 0$ in Theorem 4.6(2), we have

$$\mathscr{F}_A(y * (0 * 0)) \supseteq \mathscr{F}_A((0 * 0) * (0 * y)),$$

i.e., $\mathscr{F}_A(y) \supseteq \mathscr{F}_A(0 * (0 * y))$. By Theorem 4.7, $\mathscr{F}_A$ is an intersectional A-soft p-ideal over U. □

The converse of Theorem 4.8 is not true in general as the following example.

Example 4.9. Consider the BCI-algebra $(\mathbb{Z}; *, 0)$ as the initial universe set U, where $a*b = a-b$ for all $a, b \in \mathbb{Z}$. Let $E = X = \{0, a, b\}$ be a BCI-algebra ([9]) with the following Cayley table:

$*$	0	a	b
0	0	b	a
a	a	0	b
b	b	a	0

Define a soft set $(\mathscr{F}_X, X)$ over U by

$$\mathscr{F}_X : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \mathbb{Z} & \text{if } x = 0 \\ 4\mathbb{Z} & \text{if } x \in \{a, b\} \end{cases}$$

Then $\mathscr{F}_X$ is an intersectional soft p-ideal over U, but it is not an intersectional A-soft a-ideal over U since $\mathscr{F}_A(b * a) \not\supseteq \mathscr{F}_A((a * 0) * (0 * b)) \cap \mathscr{F}_A(0)$.

Theorem 4.10. *Let $E = X$ be a BCI-algebra. Given a subalgebra A of E, every intersectional A-soft a-ideal over U is an intersectional A-soft q-ideal over U.*

Proof. Let $\mathscr{F}_A$ be an intersectional A-soft a-ideal over U. Then it is an intersectional A-soft BCI-ideal over U by Theorem 4.3. In order to prove that $\mathscr{F}_A$ is an intersectional A-soft q-ideal from Theorem 3.9(2), it suffice to show that $\mathscr{F}_A(x * y) \supseteq \mathscr{F}_A(x * (0 * y))$ for all $x, y \in X$. Since for any $x, y \in X$

$$\begin{aligned}(0 * (0 * (y * (0 * x))))&*(x * (0 * y)) \\ &=[(0 * (0 * y)) * (0 * (0 * (0 * x)))] * (x * (0 * y)) \\ &=((0 * (0 * y)) * (0 * x)) * (x * (0 * y)) \\ &\leq (x * (0 * y)) * (x * (0 * y)) = 0,\end{aligned}$$

we have $(0 * (0 * (y * (0 * x)))) * (x * (0 * y)) = 0$ and so $0 * (0 * (y * (0 * x))) \leq x * (0 * y)$. It follows from Theorem 4.8, Theorem 4.7 and Lemma 3.8(1) that

$$\mathscr{F}_A(y * (0 * x)) \supseteq \mathscr{F}_A(0 * (0 * (y * (0 * x)))) \supseteq \mathscr{F}_A(x * (0 * y)).$$

Using Theorem 4.6(3), we have $\mathscr{F}_A(x * y) \supseteq \mathscr{F}_A(y * (0 * x))$. Hence $\mathscr{F}_A(x * y) \supseteq \mathscr{F}_A(x * (0 * y))$. By Theorem 3.9(2), $\mathscr{F}_A$ is an intersectional A-soft q-ideal over U. □

The converse of Theorem 4.9 is not true in general as seen in the following example.

Example 4.11. Consider the BCI-algebra $(\mathbb{Z}; *, 0)$ as the initial universe set U, where $a*b = a-b$ for all $a, b \in \mathbb{Z}$. Let $E = X = \{0, a, b\}$ be a BCI-algebra as in Example 3.11. Define a soft set $(\mathscr{F}_X, X)$ over U by

$$\mathscr{F}_X : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \mathbb{Z} & \text{if } x = 0 \\ 3\mathbb{Z} & \text{if } x \in \{a, b\} \end{cases}$$

Then $\mathscr{F}_X$ is an intersectional A-soft q-ideal over U. But it is not an intersectional A-soft a-ideal over U since $\mathscr{F}_A(a * 0) \not\supseteq \mathscr{F}_A((0 * 0) * (0 * a)) \cap \mathscr{F}_A(0)$.

Lemma 4.12. *Let $E = X$ be a BCI-algebra. Given a subalgebra A of E, let $\mathscr{F}_A$ be an intersectional A-soft BCI-algebra and an intersectional A-soft BCI-ideal over U. Then $\mathscr{F}_A(0 * x) \supseteq \mathscr{F}_A(x)$ for all $x \in X$.*

Proof. Put $x := 0$ in (3.1). Then for all $y \in X$, we have $\mathscr{F}_A(0 * y) \supseteq \mathscr{F}_A(0) \cap \mathscr{F}_A(y) = \mathscr{F}_A(y)$. This completes the proof. □

Theorem 4.13. *Let $E = X$ be a BCI-algebra. Given a subalgebra A of E, let $\mathscr{F}_A$ be an intersectional A-soft set over U. Then $\mathscr{F}_A$ is an intersectional A-soft a-ideal over U if and only if it is both an intersectional A-soft p-ideal and an intersectional A-soft q-ideal over U.*

Proof. Assume that $\mathscr{F}_A$ is both an intersectional A-soft p-ideal and an intersectional A-soft q-ideal over U. Then $\mathscr{F}_A$ is both an intersectional A-soft BCI-algebra and an intersectional A-soft BCI-ideal over U by Theorem 3.5. In order to prove that $\mathscr{F}_A$ is an intersectional A-soft a-ideal over U from Theorem 4.6(3), it suffices to show that $\mathscr{F}_A(y * x) \supseteq \mathscr{F}_A(x * (0 * y))$ for all $x, y \in A$. Since for all $x, y, z \in X$

$$\begin{aligned}(0 * (y * x)) * (x * y) &= ((0 * y) * (0 * x)) * (x * y) \\ &= ((0 * (x * y)) * y) * (0 * x) \\ &= (((0 * x) * (0 * y)) * y) * (0 * x) \\ &= (0 * (0 * y)) * y \\ &= (0 * y) * (0 * y) = 0,\end{aligned}$$

we obtain $0*(y*x) \le x*y$. It follows from Lemma 3.8(1) that $\mathscr{F}_A(x*y) \subseteq \mathscr{F}_A(0*(y*x))$. Lemma 4.12 and Theorem 4.7, we have $\mathscr{F}_A(x*y) \subseteq \mathscr{F}_A(0*(y*x)) \subseteq \mathscr{F}_A(0*(0*(y*x))) \subseteq \mathscr{F}_A(y*x)$. By Theorem 3.9(2), $\mathscr{F}_A(x*(0*y)) \subseteq \mathscr{F}_A(x*y) \subseteq \mathscr{F}_A(y*x)$. By Theorem 4.6, $\mathscr{F}_A$ is an intersectional A-soft a-ideal over U.

Conversely, if $\mathscr{F}_A$ is an intersectional A-soft a-ideal over U, then it is both an intersectional A-soft p-ideal and on intersectional A-soft q-ideal over U by Theorem 4.8 and Theorem 4.10. □

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Sharkovskii's theorem of continuous multi-valued maps on the compact intervals *

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Abstract Let I be a compact interval and $(\mathbb{I}, D)$ be the set of all nonempty connected closed subsets of I with the Hausdoff metric D. Suppose that f is a continuous map from I to $\mathbb{I}$. In this paper, we show that if f has n-orbits and $n \triangleright m$ (in the Sharkovskii ordering), then it also has m-orbits.

Keywords: Multi-valued map, Sharkovskii's theorem, orbit.

MR(2000)MSC: Primary 37E25, 37B40, 54H20.

1. Introduction

In 1964, A. N. Sharkovskii in [1] (also see [2]) introduced a new ordering of all positive integers,

$$3 \triangleright 5 \triangleright 7 \triangleright \cdots \triangleright 2 \cdot 3 \triangleright 2 \cdot 5 \triangleright 2 \cdot 7 \triangleright \cdots \triangleright 2^k \cdot 3 \triangleright 2^k \cdot 5 \triangleright 2^k \cdot 7 \triangleright \cdots \triangleright 2^3 \triangleright 2^2 \triangleright 2 \triangleright 1$$

and obtained the following celebrated theorem.

Theorem A Let f be a continuous self-map on the compact interval I. If f has n-periodic points and $n \triangleright m$, then it also has m-periodic points.

By a period, we mean the least period, i.e. a point $x \in I$ is said to be a n-periodic point of f if $f^n(x) = x$ and $f^j(x) \neq x$ for $0 < j < n$.

T. Y. Li and J. A. Yorke in [3] proved that if f has 3-periodic points, then f has n-periodic points for every positive integer n, and f is chaotic. It is well known (see [4] or [5]) that f is also chaotic if it has a periodic point of a period which is not a power of 2.

Our purpose in this paper will be to discuss the dynamics of continuous multi-valued maps on the compact intervals. Throughout we assume that $\mathbf{N}$ denotes the set of natural numbers. For any $n \in \mathbf{N}$, set $\mathbf{N}_n = \{1, 2, \cdots, n\}$. Let I be a compact interval of the set of real numbers

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and $\mathbb{I}$ denote the set of all nonempty connected closed subsets of I. The Hausdoff metric D on $\mathbb{I}$ is defined by

$$D(J, L) = \max\{\sup_{v\in J} d(v, L), \sup_{w\in L} d(w, J)\}, \quad \text{for any } J, L \in \mathbb{I},$$

where $d(x, Y) = \inf\{d(x, y) : y \in Y\}$ for any $x \in I$ and $Y \in \mathbb{I}$ (see [6]). Let $C^0(I, \mathbb{I})$ denote the set of all continuous maps from I to $\mathbb{I}$ (i.e. If $f \in C^0(I, \mathbb{I})$, then for any $x \in I$, $f(x)$ is either a single point or a nondegenerale closed interval of I, and for any $\varepsilon > 0$, there exists $\delta = \delta(x, \varepsilon) > 0$ such that $D(f(x), f(y)) < \varepsilon$ when $d(x, y) < \delta$). For any $f \in C^0(I, \mathbb{I})$ and any $J \subset I$, we write $f(J) = \bigcup_{x\in J} f(x)$.

Definition 1 Let $f \in C^0(I, \mathbb{I})$. A sequence $\mathcal{O} = (x_1, x_2, \cdots, x_n)$ of points in I is said to be a n-orbit of f if $x_{j+1} \in f(x_j)$ for every $j \in \mathbf{N}_n$, where $x_{n+1} = x_1$ and $\mathcal{O}$ is not a shorter m-orbit traversed p-times with $mp = n$. If the points x_j (for any $j \in \mathbf{N}_n$) in $\mathcal{O}$ are mutually different (i.e. $x_i \neq x_j$ for any $i, j \in \mathbf{N}_n$ with $i \neq i$), then $\mathcal{O}$ is said to be a primary orbit.

Recently, there has been a lot of works on the dynamics of multi-valued maps (see [7-10]). In this paper, we study the orbits of continuous multi-valued maps on the compact intervals, our main result is the following theorem.

Theorem 1 Let $f \in C^0(I, \mathbb{I})$. If f has n-orbits and $n \rhd m$, then f also has m-orbits.

2. Some basic properties

In this section we discuss some basic properties of continuous multi-valued maps on the compact intervals.

Proposition 1 Let $f \in C^0(I, \mathbb{I})$. Suppose that points $x_1, x_2, \cdots \in I$ and point $A_n \in f(x_n)$ for any $n \in \mathbf{N}$. If $\lim_{n\longrightarrow\infty} x_n = x$ and $\lim_{n\longrightarrow\infty} A_n = A$, then $A \in f(x)$.

Proof For every A_n, there exists $B_n \in f(x)$ such that $d(A_n, B_n) = d(A_n, f(x))$ since $f(x)$ is closed. By taking subsequence we let $B_n \longrightarrow B \in f(x)$. Note that

$$d(A_n, B_n) \leq D(f(x_n), f(x)).$$

We have

$$d(A, B) = \lim_{n\longrightarrow\infty} d(A_n, B_n) \leq \lim_{n\longrightarrow\infty} d(f(x_n), f(x)) = 0,$$

which implies $A = B \in f(x)$. The proof is completed.

Proposition 2 Let $f \in C^0(I, \mathbb{I})$ and J be a closed interval of I. Then there exist $a, b \in J$ such that $\inf f(J) \in f(a)$ and $\sup f(J) \in f(b)$.

Proof Let $A = \inf f(J)$ and $B = \sup f(J)$. Then there exist $a_n, b_n \in J$ and $A_n \in f(a_n)$ and $B_n \in f(b_n)$ such that $A_n \longrightarrow A$ and $B_n \longrightarrow B$. By taking subsequence we let $a_n \longrightarrow a$ and $b_n \longrightarrow b$. Then $a, b \in J$. By Proposition 1 we have $A \in f(a)$ and $B \in f(b)$. The proof is completed.

Proposition 3 Let $f \in C^0(I, \mathbb{I})$ and $a, b \in I$ with $a \neq b$. Assume that $A \in f(a)$ and $B \in f(b)$ with $A < B$. Then

(1) For any $\varepsilon > 0$, there exists $\delta > 0$, such that $d(a, x) < \delta$ implies that there exists $X \in f(x)$ satisfying $d(A, X) < \varepsilon$.

(2) For any $C \in (A, B)$, there exists $c \in (a, b)$ if $a < b$ (or $c \in (b, a)$ if $b < a$) such that $C \in f(c)$.

Proof (1) Since f is continuous, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $D(f(a), f(x)) < \varepsilon$ if $d(a, x) < \delta$. Let $X \in f(x)$ satisfying $d(A, X) = d(A, f(x))$ since $f(x)$ is closed. Then $d(A, X) = d(A, f(x)) \leq D(f(a), f(x)) < \varepsilon$.

(2) We may consider only the case $a < b$. Let $\varepsilon = \min\{d(A, C)/2, d(C, B)/2\}$. By Proposition 3 (1) there exist $u, v \in (a, b)$ with $u < v$, and $U \in f(u)$ and $V \in f(v)$ such that $U < A + \varepsilon < C < B - \varepsilon < V$.

We claim that $C \in f(c)$ for some $c \in [u, v]$. Indeed, if $C \notin f(x)$ for any $x \in [u, v]$, then let $a_1 = u < v = b_1$. If $\max f((a_1 + b_1)/2) < C$, then write $a_1 < a_2 = (a_1 + b_1)/2 < b_2 = b_1$. If $\min f((a_1 + b_1)/2) > C$, then write $a_1 = a_2 < (a_1 + b_1)/2 = b_2 < b_1$. Continuing in this fashion, we can obtain two sequences of points $a_n, b_n (n \in \mathbf{N})$ such that

(i) $a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots \leq b_n \leq \cdots \leq b_2 \leq b_1$ with $b_n - a_n = (b - a)/2^{n-1} \longrightarrow 0$.

(ii) $\max f(a_n) < C < \min f(b_n)$ for every $n \in \mathbf{N}$.

Let $a_n \longrightarrow c$. Then $c \in [u, v]$ and $b_n \longrightarrow c$. By taking subsequence we let $\max f(a_n) \longrightarrow P$ and $\min f(b_n) \longrightarrow Q$. Then $P \leq C \leq Q$. On the other hand, by Proposition 1 we have $P, Q \in f(c)$, which implies $C \in f(c)$. A contradiction. The proof is completed.

From Proposition 2 and 3 we obtain

Corollary 1 Let $f \in C^0(I, \mathbb{I})$ and J be a closed interval of I. Then $f(J)$ is also a closed interval of I.

Proposition 4 Let $f \in C^0(I, \mathbb{I})$. Suppose that $a, b \in I$ with $a < b$, and $A \in f(a)$ and $B \in f(b)$. If $(A - a)(B - b) < 0$, then there exists $p \in (a, b)$ such that $p \in f(p)$.

Proof We may consider only the case $A < a < b < B$. Let $\varepsilon = \min\{d(A, a)/2, d(B, b)/2\}$. By Proposition 3 (1) there exist $u, v \in (a, b)$ with $u < v$, and $U \in f(u)$ and $V \in f(v)$ such that $U < A + \varepsilon < a < u < v < b < B - \varepsilon < V$.

We claim that $p \in f(p)$ for some $p \in [u, v]$. Indeed, if $x \notin f(x)$ for any $x \in [u, v]$, then let $a_1 = u < v = b_1$. If $\max f((a_1 + b_1)/2) < (a_1 + b_1)/2$, then write $a_1 < a_2 = (a_1 + b_1)/2 < b_2 = b_1$. If $\min f((a_1 + b_1)/2) > (a_1 + b_1)/2$, then write $a_1 = a_2 < (a_1 + b_1)/2 = b_2 < b_1$. Continuing in this fashion, we can obtain two sequences of points $a_n, b_n (n \in \mathbf{N})$ such that

(i) $a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots \leq b_n \leq \cdots \leq b_2 \leq b_1$ with $b_n - a_n = (b - a)/2^{n-1} \longrightarrow 0$.

(ii) $\max f(a_n) < a_n$ and $b_n < \min f(b_n)$ for every $n \in \mathbf{N}$.

Let $a_n \longrightarrow p$. Then $p \in [u, v]$ and $b_n \longrightarrow p$. By taking subsequence we let $\max f(a_n) \longrightarrow P$ and $\min f(b_n) \longrightarrow Q$. Then $P \leq p \leq Q$. On the other hand, by Proposition 1 we have $P, Q \in f(p)$, which implies $p \in f(p)$. A contradiction. The proof is completed.

Let $f \in C^0(I, \mathbb{I})$. For any $n \geq 2$ and $x \in I$, set $f^n(x) = \bigcup_{y \in f^{n-1}(x)} f(y)$, which is said to be the n-fold composition of f with itself.

Proposition 5 Let $f \in C^0(I, \mathbb{I})$. Then $f^n \in C^0(I, \mathbb{I})$ for any $n \geq 2$.

Proof Since I is closed, it follows that f is uniformly continuous. Then for any $\varepsilon > 0$, there

exists $\delta = \delta(\varepsilon) > 0$ such that $D(f(x), f(y)) \leq \varepsilon$ if $d(x, y) \leq \delta$.

We claim that if J and L are two closed intervals of I satisfying $D(J, L) \leq \delta$, then $D(f(J), f(L)) \leq \varepsilon$. Indeed, for any $A \in f(J)$, let $x_A \in J$ such that $A \in f(x_A)$. Let $y_A \in L$ such that $d(x_A, y_A) = d(x_A, L) \leq \delta$, which implies $D(f(x_A), f(y_A)) \leq \varepsilon$. Thus $d(A, f(y_A)) \leq \varepsilon$ and $d(A, f(L)) \leq \varepsilon$. In a similar fashion, we can show that $d(B, f(J)) \leq \varepsilon$ for any $B \in f(L)$. From which it follows $D(f(J), f(L)) \leq \varepsilon$.

Let $0 < \delta_1 \leq \delta_2 \leq \cdots \leq \delta_n \leq \delta_{n+1} = \varepsilon$ such that $D(f(x), f(y)) \leq \delta_{i+1}$ if $d(x, y) \leq \delta_i$ for any $i \in \mathbf{N}_n$. By the above claim we have $D(f^i(x), f^i(y)) \leq \delta_{i+1}$ if $d(x, y) \leq \delta_1$, which implies $f^n \in C^0(I, \mathbb{I})$. The proof is completed.

3. Proof of main theorem

Let $f \in C^0(I, \mathbb{I})$ and L, J be two closed intervals of I, we write $J \longrightarrow L$ if $f(J) \supset L$. In this section, we shall show the main theorem. To do this we need the following lemmas.

Lemma 1 Let $f \in C^0(I, \mathbb{I})$ and L, J be two closed intervals of I. If $J \longrightarrow L$, then one of the following holds:

(1) There exists some $u \in J$ such that $f(u) \supset L$.

(2) There exists a closed subinterval $J_1 \subset J$ such that $f(J_1) = L$.

Proof If (1) does not hold, then $L \not\subset f(x)$ for any $x \in J$. Let $L = [A, B]$. Then there exist $a, b \in J$ with $a \neq b$ such that $A \in f(a)$ and $B \in f(b)$.

If $a < b$, then let $u = \sup\{x \in [a, b] : A \in f(x)\}$ and $v = \inf\{x \in [u, b] : B \in f(x)\}$. By Proposition 1 we have $A \in f(u)$ and $B \in f(v)$ with $u < v$. Thus it follows that $f([u, v]) = L$ (indeed, if there exists $C < A$ (or $C > B$) such that $C \in f([u, v])$, then by Proposition 3 we can take $w \in (u, v)$ such that $A \in f(w)$ (or $B \in f(w)$), this is a contradiction with the definition of u (or v)).

If $b < a$, then let $u = \inf\{x \in [b, a] : A \in f(x)\}$ and $v = \sup\{x \in [b, u] : B \in f(x)\}$. By Proposition 1 we have $A \in f(u)$ and $B \in f(v)$ with $v < u$. Thus it follows from Proposition 3 that $f([v, u]) = L$. The proof is completed.

Lemma 2 Let $f \in C^0(I, \mathbb{I})$. Assume that I_i ($i \in \mathbf{N}_n$) are closed intervals such that $I_1 \longrightarrow I_2 \longrightarrow \cdots \longrightarrow I_n \longrightarrow I_{n+1} = I_1$. Then there exists $x_i \in I_i$ for any $i \in \mathbf{N}_n$ such that $x_{i+1} \in f(x_i)$, where $x_{n+1} = x_1$.

Proof Set $E_{n+1} = I_{n+1}$. If $A_n = \{u \in I_n : f(u) \supset E_{n+1}\} \neq \emptyset$, then choose a $u_n \in A_n$ and set $E_n = \{u_n\}$. Otherwise, by Lemma 1 there exists a closed subinterval $J_n \subset I_n$ such that $f(J_n) = E_{n+1}$, and set $E_n = J_n$. Note $I_{n-1} \longrightarrow E_n$. If $A_{n-1} = \{u \in I_{n-1} : f(u) \supset E_n\} \neq \emptyset$, then choose a $u_{n-1} \in A_{n-1}$ and set $E_{n-1} = \{u_{n-1}\}$. Otherwise, by Lemma 1 there exists a closed subinterval $J_{n-1} \subset I_{n-1}$ such that $f(J_{n-1}) = E_n$, and set $E_{n-1} = J_{n-1}$. Continuing in this fashion, for every $i \in \mathbf{N}_n$, we obtain $E_i \subset I_i$ such that if $A_i = \{u \in I_i : f(u) \supset E_{i+1}\} \neq \emptyset$, then choose a $u_i \in A_i$ and set $E_i = \{u_i\}$; if $A_i = \emptyset$, by Lemma 1 there exists a closed subinterval $J_i \subset I_i$ such that $f(J_i) = E_{i+1}$, and set $E_i = J_i$.

If $E_i = \{u_i\}$ for some $i \in \mathbf{N}_n$, then let $r = \max\{i \in \mathbf{N}_n : E_i = \{u_i\}\}$. Thus $E_i = \{u_i\}$ for every $i \in \mathbf{N}_r$. Set $x_{n+1} = u_1$. For every $r+1 \leq i \leq n$, by Proposition 3 we can take $x_i \in E_i = J_i$

such that $x_{i+1} \in f(x_i)$. Set $x_i = u_i$ for every $i \in \mathbf{N}_r$.

If $E_i \neq \{u_i\}$ for every $i \in \mathbf{N}_n$, then $f(E_i) = E_{i+1}$ $(i \in \mathbf{N}_n)$. Thus $f^n(E_1) = I_1$ and $E_1 \subset I_1$. By Proposition 4 and 5 we see that there exists $x_1 \in E_1$ such that $x_1 \in f^n(x_1)$. Thus for every $i \in \mathbf{N}_n$ there exists $x_i \in f^{i-1}(x_1) \subset E_i \subset I_i$ such that $x_{i+1} \in f(x_i)$, where $x_{n+1} = x_1$. The proof is completed.

Definition 2 $f \in C^0(I, \mathbb{I})$ is said to be turbulent if there exist $a, b, c \in I$ with $a < b < c$ such that $f([a, b]) \cap f([b, c]) \supset [a, c]$.

Lemma 3 If $f \in C^0(I, \mathbb{I})$ is turbulent, then f has m-orbits for any $m \in \mathbf{N}$.

Proof Let f be turbulent. Then there exist $a, b, c \in I$ with $a < b < c$ such that $f([a, b]) \cap f([b, c]) \supset [a, c]$. It is obvious that f has 1-orbits since $f(I) \subset I$. Now we suppose that $m \geq 2$.

Since $b \in (a, c)$, it follows from Proposition 3 that we can take point $u \in (a, c)$ and $v \in (b, c)$ such that $b \in f(u) \cap f(v)$.

If there exists $x \in [a, b)$ such that $c \in f(x)$, then write $I_0 = I_m =< u, x >$ and $I_i = [b, c]$ for any $i \in \mathbf{N}_{m-1}$, where $< \alpha, \beta >$ denotes the closed interval with endpoints α and β. We have

$$I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_m = I_0.$$

By Lemma 2, we obtain a m-orbit $(x_1, x_2, \cdots, x_m)$ with $x_i \in I_{i-1}$ for any $i \in \mathbf{N}_m$. In a similar fashion, we can obtain a m-orbit if there exists $y \in (b, c]$ such that $a \in f(y)$.

If $a, c \in f(b)$, then $u, b \in f(b)$ since $f(b)$ is a closed interval. It is obvious that $(u, \overbrace{b, \cdots, b}^{m-1})$ is a m-orbit. The proof is completed.

Definition 3 Let $f \in C^0(I, \mathbb{I})$. A sequence $\mathcal{P}_n = (x_0, x_1, x_2, \cdots, x_n)$ of points in I with $x_i \in f(x_{i-1})$ for any $i \in \mathbf{N}_n$ is said to be a return trajectory if $x_1 < x_0 \leq x_n$ or $x_1 > x_0 \geq x_n$.

Lemma 4 Let $f \in C^0(I, \mathbb{I})$ and $\mathcal{P}_n = (x_0, x_1, x_2, \cdots, x_n)$ is a return trajectory. Then

(1) If there exists $p \in I$ such that $p \in f(p) \cap < x_0, x_n >$, then f has m-orbits for any $m \in \mathbf{N}$.

(2) f has m-orbits for some $2 \leq m \leq n$. Furthermore, if $n \geq 3$ is odd, then m is also odd.

Proof (1) We may assume without loss of generality that $x_n \leq x_0 < x_1$ and $p = \max\{x \in [x_n, x_0] : x \in f(x)\}$. Set $k = \min\{i \in \mathbf{N}_n : x_i \leq p\}$. Then $k \geq 2$ and $x_{k-1} > p$. Write $r = \min\{i \in \mathbf{N}_{k-1} : x_i \geq x_{k-1}\}$. Then $r \geq 1$ and $p \leq x_{r-1} < x_{k-1}$ with $f([p, x_{r-1}]) \cap f([x_{r-1}, x_{k-1}]) \supset [p, x_{k-1}]$. If $p < x_{r-1}$, then it follows Lemma 3 that f has m-orbits for any $m \in \mathbf{N}$. If $p = x_{r-1}$, then there exists $q \in (p, x_{k-1})$ such that $q \in f(q)$. Since $x_k \in f(x_{k-1})$, we see that there exists $w \in [q, x_{k-1}]$ such that $p \in f(w)$. Thus $(\overbrace{p, \cdots, p}^{m-1}, w)$ is a m-orbit if $m > 1$ since $w \in [p, x_{k-1}] \subset f(p)$.

(2) We may assume without loss of generality that $x_n \leq x_0 < x_1$. By Lemma 4 (1) we may suppose that $x \notin f(x)$ for any $x \in [x_n, x_0]$. Write $c = \min I$. Since $x_n (\leq x_0) \in f^n(x_0)$ and $f^n(c) \subset I$, it follows from Proposition 4 that there exists $q \in [c, x_0]$ such that $q \in f^n(q)$. We may assume without loss of generality that $q = \max\{x \in [c, x_0] : x \in f^n(x)\}$.

We claim that $q \notin f(q)$. Indeed, if $q \in f(q)$, then $q < x_n$. By Proposition 3, we can take points $q < z_0 < z_1 < z_2 < \cdots < z_{n-1} = x_0 < z_n = x_1$ such that $z_i \in f(z_{i-1})$ for every $i \in \mathbf{N}_n$. Thus by Proposition 4 it follows that there exists $s \in [z_0, x_0]$ such that $s \in f^n(s)$ since $z_0 < x_1 \in f^n(z_0)$ and $x_0 \geq x_n \in f^n(x_0)$, which contradicts the definition of q.

Write $y_1 = y_{n+1} = q$. Then we can obtain a sequence $\mathcal{O} = (y_1, y_2, \cdots, y_n)$ of points in I such that $y_{i+1} \in f(y_i)$ for every $i \in \mathbf{N}_n$ and $y_2 \neq q$. Write $m = \min\{r \in \mathbf{N}_n : \mathcal{O}$ consists entirely of $(y_1, y_2, \cdots, y_r)$ traversed s-times with $rs = n\}$. Then $2 \leq m \leq n$ and $(y_1, y_2, \cdots, y_m)$ is a m-orbit of f. Furthermore, if n is odd, then m is odd. The proof is completed.

Corollary 2 Let $f \in C^0(I, \mathbb{I})$. If f has a n-orbit $\mathcal{O}$ for some $n \geq 2$, then f has 2-orbits.

Proof We may assume that $\mathcal{O} = (x_1, x_2, \cdots, x_n)$ is primary. Indeed, if $\mathcal{O}$ is not primary, then there exist $i, j \in \mathbf{N}_n$ with $i < j$ such that $x_i = x_j$. We consider $(x_i, \cdots, x_{j-1})$ and $(x_j, \cdots, x_n, x_1, \cdots, x_{i-1})$. Continuing in this fashion, we can obtain a m-orbit which is primary for some $1 < m \leq n$.

Let $x_r = \max\{x_i \in \mathcal{O} : x_i < x_{i+1}\}$ and $x_l = \min\{x_i \in \mathcal{O} : x_i > x_r\}$. It follows from Proposition 4 that there exists $p \in (x_r, x_l)$ such that $p \in f(p)$. Since $x_l \in f([x_r, p])$, there exists $y_0 \in [x_r, p]$ such that $x_l \in f(y_0)$. Thus (y_0, x_l, x_{l+1}) is a return trajectory. It follows from Lemma 4 that f has 2-orbits. The proof is completed.

Lemma 5 Let $f \in C^0(I, \mathbb{I})$ and $n > 1$ is odd. Assume that f has a n-orbit $\mathcal{O}$ and has no l-orbit for every odd integer l with $1 < l < n$.

(1) If $n = 3$, then $\mathcal{O}$ may be written as (x_1, x_1, x_2) with $x_2 < x_1$ or $x_2 > x_1$, or $\mathcal{O}$ may be written as (x_1, x_2, x_3) with $x_2 < x_1 < x_3$ or $x_2 > x_1 > x_3$.

(2) If $n \geq 5$, then $\mathcal{O}$ may be written as $(x_1, x_2, \cdots, x_n)$ with $x_{n-1} < x_{n-3} < \cdots < x_2 < x_1 < x_3 < \cdots < x_{n-2} < x_n$ or $x_{n-1} > x_{n-3} > \cdots > x_2 > x_1 > x_3 > \cdots > x_{n-2} > x_n$.

Proof We may assume $n \geq 5$ since the case $n = 3$ is trivial. Let $\mathcal{O} = (y_1, y_2, \cdots, y_n)$ and write $y_{n+i} = y_i$ for any $i \in \mathbf{N}_{2n}$. We claim that $\mathcal{O}$ is primary. Indeed. If $y_i = y_{i+1} \neq y_{i+2}$ for some $i \in \mathbf{N}_n$, then $(y_{i+1}, y_{i+2}, \cdots, y_{n+i+1})$ is a return trajectory with $y_{n+i} = y_{n+i+1} \in f(y_{n+i})$. By Lemma 4 it follows that f has 3-orbits. This is a contradiction with the assumption of the maximality of n. If $y_i \neq y_{i+1}$ for every $i \in \mathbf{N}_n$, and $y_j = y_k$ for some $j, k \in \mathbf{N}_n$ with $k > j + 1$, then we consider $(x_j, \cdots, x_{k-1})$ and $(x_k, \cdots, x_n, x_1, \cdots, x_{j-1})$. Continuing in this fashion, we can obtain a shorter odd orbit. This is also a contradiction with the assumption of the maximality of n.

Let $y_s = \max\{y_i \in \mathcal{O} : y_{i+1} > y_i\}$ and $y_t = \min\{y_i \in \mathcal{O} : y_i > y_s\}$. Then there exists $p \in (y_s, y_t)$ such that $p \in f(p)$. We may assume without loss of generality that $|\{y_i \in \mathcal{O} : y_i \leq y_s\}| > |\{y_i \in \mathcal{O} : y_i \geq y_t\}|$, where $|S|$ denote the number of elements of S. Write $x_i = y_{s-1+i}$ if $1 \leq i \leq n - s + 1$ and $x_i = y_{s-1+i-n}$ if $n - s + 2 \leq i \leq n$.

We claim that $x_i > x_1$ for every even integer $i \in \mathbf{N}_n$. Indeed, if $x_k < x_1$ for some even integer $k \in \mathbf{N}_n$, then $k \geq 4$ and $(x_1, x_2, \cdots, x_k)$ is a return trajectory. By Lemma 4 we see that f has m-orbits for some odd integer $3 \leq m \leq k - 1$. This is a contradiction with the assumption of the maximality of n. Thus $x_i > x_1$ if $i \in \mathbf{N}_n$ is even and $x_i \leq x_1$ if $i \in \mathbf{N}_n$ is odd.

We claim also that $x_1 < x_i < x_{i+2}$ for every even integer $i \in \mathbf{N}_{n-2}$. Indeed, if $x_1 < x_{r+2} < x_r$ for some even integer $r \in \mathbf{N}_{n-2}$, then there exist points $z_i \in (x_i, p)$ if $1 \leq i \leq r$ is odd and $z_i \in (p, x_i)$ if $1 \leq i \leq r - 1$ is even such that $z_{i+1} \in f(z_i)$ for any $i \in \mathbf{N}_{r-2}$ and $x_{r+2} \in f(z_{r-1})$. Thus $(z_1, z_2, \cdots, z_{r-1}, x_{r+2}, \cdots, x_n, x_1)$ is a return trajectory. By Lemma 4 we see that f has m-orbits for some odd integer $3 \leq m \leq n - 2$, which is a contradiction with the assumption of the maximality of n. In a similar fashion, we can show that $x_i > x_{i+2}$ for every odd integer

$i \in \mathbf{N}_{n-2}$. Thus $x_n < x_{n-2} < \cdots < x_3 < x_1 < x_2 < \cdots < x_{n-3} < x_{n-1}$. The proof is completed.

Lemma 6 Let $f \in C^0(I, \mathbb{I})$ and $n > 1$ is odd. If f has n-orbits and $n \rhd m$, then f has m-orbits.

Proof We may suppose that n is minimal and $\mathcal{O}$ is a n-orbit. By Proposition 4 and Corollary 2 we see that f has 1-orbits and 2-orbits since $n > 1$ and $f(I) \subset I$. In the following we assume that $m \geq 3$.

If $\mathcal{O}$ is not primary, then $n = 3$ and we may assume that $\mathcal{O} = (x_1, x_1, x_2)$ with $x_1 \neq x_2$. Thus $(\overbrace{x_1, \cdots, x_1}^{m-1}, x_2)$ is a m-orbit.

If $\mathcal{O}$ is primary, then we may assume without loss of generality that $\mathcal{O} = (x_1, x_2, \cdots, x_n)$ with $x_{n-1} < x_{n-3} < \cdots < x_2 < x_1 < x_3 < \cdots < x_{n-2} < x_n$. It follows from Proposition 3 that there exist $x_0 \in (x_2, x_1)$ and $x_{-1} \in (x_{n-2}, x_n)$ such that $x_1 \in f(x_0)$ and $x_0 \in f(x_{-1})$. Furthermore, there exists $x_0' \in (x_{n-2}, x_n)$ such that $x_{n-3} \in f(x_0')$ when $n \geq 5$.

For every integer $m \geq n + 1$, we consider

$$[x_{-1}, x_n] \longrightarrow [x_0, x_1] \longrightarrow \overbrace{[x_2, x_1] \longrightarrow \cdots \longrightarrow [x_2, x_1]}^{m-n+1} \longrightarrow [x_1, x_3] \longrightarrow$$

$$\cdots \longrightarrow [x_{n-4}, x_{n-2}] \longrightarrow [x_{n-1}, x_{n-3}] \longrightarrow [x_{-1}, x_n].$$

By Lemma 2 we see that f has m-orbits.

For every even integer m with $4 \leq m \leq n - 1$ when $n \geq 5$, we consider

$$[x_0', x_n] \longrightarrow [x_{n-m+1}, x_{n-m-1}] \longrightarrow [x_{n-m}, x_{n-m+2}] \longrightarrow$$

$$\cdots \longrightarrow [x_{n-4}, x_{n-2}] \longrightarrow [x_{n-1}, x_{n-3}] \longrightarrow [x_0', x_n].$$

By Lemma 2 we see that f has m-orbits. The proof is completed.

Lemma 7 Let $f \in C^0(I, \mathbb{I})$, and $n, r, s \in \mathbf{N}$ and $q > 1$ is odd.

(1) If f^{2^n} has $2^s q$-orbits, then f has $2^{n+s} q$-orbits.

(2) If f^{2^n} has $q(> 1)$-orbits and f has no l-orbit for any $l \rhd 2^n \cdot 3$, then f has $2^n q$-orbits.

(3) If f^{2^n} has 2^r-orbits, then f has 2^{r+n}-orbits.

Proof (1) By the hypothesis, let $\mathcal{T} = (x_1, x_{2^n+1}, \cdots, x_{(2^s q-1)2^n+1})$ be a $2^s q$-orbit of f^{2^n} and let $\mathcal{S} = (x_1, x_2, \cdots, x_{2^n}, x_{2^n+1}, \cdots, x_{(2^s q-1)2^n}, x_{(2^s q-1)2^n+1}, \cdots, x_{2^{n+s} q})$ such that $x_{i+1} \in f(x_i)$ for every $i \in \mathbf{N}_{2^{n+s} q}$, where $x_{2^{n+s} q+1} = x_1$. Write

$$m = \min\{h \in \mathbf{N}_{2^{n+s} q} : \mathcal{S} \text{ consists entirely of } (x_1, x_2, \cdots, x_h)$$

$$\text{traversed } l\text{-times with } hl = 2^{n+s} q\}.$$

Then $m = 2^\mu p$, where $0 \leq \mu \leq n + s$ and q is divided by p.

We claim that $\mu = n+s$ and $p = q$. Indeed. If $\mu < n+s$, then $x_{(i-1)2^n+1} = x_{(i-1+2^{s-1} q)2^n+1}$ for any $i \in \mathbf{N}_{2^{s-1} q}$, which means that $\mathcal{T}$ consists entirely of $(x_1, x_{2^n+1}, \cdots, x_{(2^{s-1} q-1)2^n+1})$ traversed 2-times, a contradiction. If $p < q$, then $x_{(i-1)2^n+1} = x_{(i-1+2^s p)2^n+1}$ for any $i \in \mathbf{N}_{2^s q-2^s p}$ and $\mathcal{T}$ consists entirely of $(x_1, x_{2^n+1}, \cdots, x_{(2^s p-1)2^n+1})$ traversed q/p-times, a contradiction. Hence $\mathcal{S}$ is a $2^{n+s} q$-orbit of f.

(2) By the hypothesis, let $\mathcal{T} = (x_1, x_{2^n+1}, \cdots, x_{(q-1)2^n+1})$ be a q-orbit of f^{2^n} and let $\mathcal{S} = (x_1, x_2, \cdots, x_{2^n}, x_{2^n+1}, \cdots, x_{(q-1)2^n}, x_{(q-1)2^n+1}, \cdots, x_{2^nq})$ such that $x_{i+1} \in f(x_i)$ for every $i \in \mathbf{N}_{2^nq}$, where $x_{2^nq+1} = x_1$. Write

$$m = \min\{h \in \mathbf{N}_{2^nq} : \mathcal{S} \text{ consists entirely of } (x_1, x_2, \cdots, x_h)$$

$$\text{traversed } l\text{-times with } hl = 2^nq\}.$$

Then it follows from the hypothesis that $m = 2^np$, where q is divided by p. If $q \neq p$, then $x_{(i-1)2^n+1} = x_{(i-1+p)2^n+1}$ for any $i \in \mathbf{N}_{q-p}$ and $\mathcal{T}$ consists entirely of $(x_1, x_{2^n+1}, \cdots, x_{(p-1)2^n+1})$ traversed q/p-times, a contradiction. Hence $m = 2^nq$ and $\mathcal{S}$ is a 2^nq-orbit of f.

(3) By the hypothesis, let $\mathcal{T} = (x_1, x_{2^n+1}, \cdots, x_{(2^r-1)2^n+1})$ be a 2^r-orbit of f^{2^n} and let $\mathcal{S} = (x_1, x_2, \cdots, x_{2^n}, x_{2^n+1}, \cdots, x_{(2^r-1)2^n}, x_{(2^r-1)2^n+1}, \cdots, x_{2^{n+r}})$ such that $x_{i+1} \in f(x_i)$ for every $i \in \mathbf{N}_{2^{n+r}}$, where $x_{2^{n+r}+1} = x_1$. Write

$$m = \min\{h \in \mathbf{N}_{2^{n+r}} : \mathcal{S} \text{ consists entirely of } (x_1, x_2, \cdots, x_h)$$

$$\text{traversed } l\text{-times with } hl = 2^{n+r}\}.$$

Then $m = 2^\mu$ with $0 \le \mu \le n + r$. If $\mu < n + r$, then $x_{(i-1)2^n+1} = x_{(i-1+2^{r-1})2^n+1}$ for any $i \in \mathbf{N}_{2^{r-1}}$, which means that $\mathcal{T}$ is a shorter orbit traversed several times, a contradiction. Hence $\mathcal{S}$ is a 2^{n+r}-orbit of f. The proof is completed.

Lemma 8 Let $f \in C^0(I, \mathbb{I})$ and $q > 1$ is odd and $n \in \mathbf{N}$. If f has 2^nq-orbits, then f^{2^n} has q-orbits.

Proof Let$\mathcal{S} = (x_1, x_2, \cdots, x_{2^n}, x_{2^n+1}, \cdots, x_{(q-1)2^n}, x_{(q-1)2^n+1}, \cdots, x_{2^nq})$ be a 2^nq-orbit of f. For any $k \in \mathbf{N}_{2^n}$ we write

$$\mathcal{S}_k = (x_k, x_{2^n+k}, \cdots, x_{(q-1)2^n+k}).$$

If $\mathcal{S}_k$ is 1-orbit (x_k) traversed q-times for every $k \in \mathbf{N}_{2^n}$, then $\mathcal{S}$ consists entirely of $(x_1, x_2, \cdots, x_{2^n})$ traversed q-times, a contradiction. Thus there exists some $s \in \mathbf{N}_{2^n}$ such that $x_{i2^n+s} \neq x_{(i+1)2^n+s}$ for some $0 \le i \le q-2$. Note that $\mathcal{P}_q = (x_{i2^n+s}, x_{(i+1)2^n+s}, \cdots, x_{(q-1)2^n+s}, x_s, \cdots, x_{i2^n+s})$ is a return trajectory of f^{2^n}. By Lemma 4 we see that f^{2^n} has p-orbits for some odd integer $3 \le p \le q$. Thus it follows from Lemma 6 that f^{2^n} has q-orbits. The proof is completed.

Proof of Theorem 1 Let $n = 2^kq$, where $k \ge 0$ and $q \in \mathbf{N}$ is odd. It follows from Proposition 4 that f has 1-orbits. In the following we assume that $m \ge 2$.

Case 1 $q \ge 3$: When $k = 0$, it follows from Lemma 6 that Theorem 1 is true. Now assume that $k \in \mathbf{N}$. Furthermore, we can assume that f has no l-orbit for any $l \rhd n$. It follows from Lemma 8 that f^{2^k} has q-orbits, and by Lemma 6 f^{2^k} has l-orbits for any $q \rhd l$. Subsequently, using Lemma 7, f has l-orbits for any $n \rhd l$ with $l \notin \{2^r : r \in \mathbf{N}_k\}$. In the following we show that f has l-orbits for any $l \in \{2^r : r \in \mathbf{N}_k\}$.

Write $g = f^{2^{r-1}}$ for any $r \in \mathbf{N}_k$. Let

$$\mathcal{S} = (x_1, x_2, \cdots, x_{2^{r-1}}, x_{2^{r-1}+1}, \cdots, x_{(2^{k-r+2}-1)2^{r-1}}, x_{(2^{k-r+2}-1)2^{r-1}+1}, \cdots, x_{2^{k+1}})$$

be a 2^{k+1}-orbit of f. For any $s \in \mathbf{N}_{2^{r-1}}$ we write

$$\mathcal{S}_s = (x_s, x_{2^{r-1}+s}, \cdots, x_{(2^{k-r+2}-1)2^{r-1}+s}).$$

If $\mathcal{S}_s$ is 1-orbit (x_s) traversed 2^{k-r+2}-times for every $s \in \mathbf{N}_{2^{r-1}}$, then $\mathcal{S}$ consists entirely of $(x_1, x_2, \cdots, x_{2^{r-1}})$ traversed 2^{k-r+2}-times, a contradiction. Thus there exists some $j \in \mathbf{N}_{2^{r-1}}$ such that $x_{i2^{r-1}+j} \neq x_{(i+1)2^{r-1}+j}$ for some $0 \leq i \leq 2^{k-r+2} - 2$. From $\mathcal{S}_j$ we can obtain l-orbits of g for some l with $1 < l \leq 2^{k-r+2}$. Thus it follows from Corollary 2 that $g = f^{2^{r-1}}$ has 2-orbits. Applying again Lemma 7, f has 2^r-orbits.

Case 2 $q = 1$: With respect to Case 1, we can assume that f has no l-orbit for any $l \rhd 2^k$ and $k > 1$. Write $g = f^{2^{r-1}}$ for any $r \in \mathbf{N}_{k-1}$. Analogously as in the second part of Case 1, we can show that g has 2-orbits, which and Lemma 7 implies that f has 2^r-orbits. The proof is completed.

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Asymptotic equivalence of systems of difference equations with unbounded solutions

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Abstract

We study an asymptotic behavior of solutions of a nonlinear system by comparing with a linear system, whose asymptotic behavior of solutions is similar to the behavior of solutions of the original system.
Keywords: System of difference equations, Asymptotic equivalence, Exponentially dichotomous system.

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1 Introduction

The method of asymptotic equivalence of two systems is well known. According to it, the asymptotic behavior of solutions is investigated by comparing the original system with some simpler system. If we know the asymptotic behavior of the solutions of one of the systems, then we can obtain information about the asymptotic behavior of the solutions of the other system. First results in this direction were obtained by Wintner [18], Yakybovich [19] and Levinson [12].

Further, the problem of asymptotic equivalence for different classes of differential equations, including linear, nonlinear, functional and stochastic equations was studied by many authors. For ordinary differential equations, we mention the works [2,5,6,10,11]. For impulse systems, we refer to the monograph [16] . Stochastic systems were investigated in [17].

This method can be applied to the study of difference equations [1,8,13]. Similar topics for difference equations can be found in [3,4,7,10,14,15].

This work continues the authors' research in this area started in [3,4,9]. But now we consider a nonlinear case. A nonlinear difference system is compared with a linear system, whose asymptotic behavior of solutions is similar to the behavior of solutions of the original system. We study the conditions of asymptotic equivalence of these systems.
Another difference is that now the linear system can have unbounded solutions.

2 Statement of the problem and Auxiliary results

Consider the linear system of difference equations

$$x_{n+1} = x_n + A_n x_n \tag{1}$$

and the corresponding nonlinear system

$$y_{n+1} = y_n + A_n y_n + f_n(y_n) \tag{2}$$

where A_n is a $d\times d$ matrix, $x_n, y_n \in R^d$, $f_n(y_n)$ is a $d-$dimensional vector, $n \in N_0 = \{0, 1, 2, ...\}$. Let $|.|$ denote the norm on R^d , and let $\|.\|$ denote the induced norm on the space of $d \times d$ matrices.

A solution of the system (2) for which $y_n(n_0, y) = y$ is denoted by $y_n(n_0, y)$.

Definition 2.1 Systems (1) and (2) are asymptotically equivalent for $n \to \infty$ if there exists a one-to-one correspondence between their solutions x_n and y_n such that

$$\lim_{n\to\infty} |x_n - y_n| = 0 \tag{3}$$

Let $X_{n,k}$ be the fundamental matrix of the system (1) . $X_{k,k} = I$, where I is identity matrix. Let $X_{k,0} = X_k$

Assume that the following conditions hold

(a) the matrix A_n is defined and bounded, so that $a = \sup_{n\in N} \|A_n\| < \infty$;

(b) the function $f_n(y)$ is defined for $n \in N_0$, $y \in R^d$ and it satisfies the condition

$$|f_n(y_1) - f_n(y_2)| \le \eta_n |y_1 - y_2| ,$$

where $n \in N_0$, y_1 , $y_2 \in R^d$, η_n is a nonnegative sequence of real numbers;

(c) $a_1 = \sum_{n=0}^{\infty} \eta_n < \infty$;

(d) $f_n(0) = 0$;

(e) det $(I + A_n) \ne 0$.

Definition 2.2. System (1) is called exponentially dichotomous on N if there exist two complementing projectors P_1 and P_2 and also positive constants υ_1, υ_2 , N_1 , N_2 such that the following inequalities hold:

$$\left\|X_n P_1 X_{n_0}^{-1}\right\| \le N_1(1 + a)^{-\upsilon_1(n-n_0)}, \quad n \ge n_0 \tag{4}$$

and

$$\left\|X_n P_2 X_{n_0}^{-1}\right\| \le N_2(1 + a)^{-\upsilon_2(n_0-n)}, \quad n_0 \ge n \tag{5}$$

We will need the following lemmas.

Lemma 2.1. Under the condition (a), the fundamental matrix of the system (1) satisfies the inequality

$$\|X_{n,n_0}\| \le (1 + a)^{(n-n_0)}, \quad n \ge n_0 \tag{6}$$

The proof of this statement follows from the representation of the fundamental matrix in the form

$$X_{n,n_0} = (I + A_{n-1})(I + A_{n_0}) \ . \ . \ . \ (I + A_{n-1})$$

Lemma 2.2. Any solution y_n of the system (2) can be represented as

$$y_n = X_{n,n_0} y_{n_0} + \sum_{k=n_0}^{n} X_{n,k} f_{k-1}(y_{k-1}) \tag{7}$$

Proof. Substituting (7) into (2), we get

$$X_{n+1,n_0}y_{n_0} + \sum_{k=n_0}^{n+1} X_{n+1,k}f_{k-1}(y_{k-1})$$

$$= (I + A_n)X_{n,n_0}y_{n_0} + (I + A_n)\sum_{k=n_0}^{n} X_{n,k}f_{k-1}(y_{k-1}) + f_n(y_n)$$

or

$$X_{n+1,n_0}y_{n_0} + X_{n+1,n+1}f_n(y_n) + \sum_{k=n_0}^{n} X_{n+1,k}f_{k-1}(y_{k-1})$$

$$= (I + A_n)X_{n,n_0}y_{n_0} + (I + A_n)\sum_{k=n_0}^{n} X_{n,k}f_{k-1}(y_{k-1}) + f_n(y_n)$$

Hence, we have

$$(I + A_n)X_{n,n_0}y_{n_0} + f_n(y_n) + \sum_{k=n_0}^{n} (I + A_n)X_{n,k}f_{k-1}(y_{k-1})$$

$$= (I + A_n)X_{n,n_0}y_{n_0} + (I + A_n)\sum_{k=n_0}^{n} X_{n,k}f_{k-1}(y_{k-1}) + f_n(y_n).$$

This completes the proof.

Lemma 2.3. Under the conditions (a) - (e), there exists a positive constant $a_2 > 0$, such that any solution of system (2) satisfies the inequality

$$|y_n| \le a_2 |y_{n_0}| (1 + a)^{n-n_0} \quad , \quad n \ge n_0. \tag{8}$$

Proof. From the conditions of Lemma 2 it follows that any solution of (2) can be represented as

$$y_n = X_{n,n_0}y_{n_0} + \sum_{k=n_0}^{n} X_{n,k}f_{k-1}(y_{k-1}).$$

Therefore

$$|y_n| \le \|X_{n,n_0}\| |y_{n_0}| + \sum_{k=n_0}^{n} \|X_{n,k}\| |f_{k-1}(y_{k-1})| .$$

Using (6), together with conditions (b) and (d) , we have

$$|y_n| \le (1 + a)^{n-n_0} |y_{n_0}| + \sum_{k=n_0}^{n} (1 + a)^{n-k}\eta_{k-1} |y_{k-1}| .$$

Multiplying the last inequality by $(1 + a)^{-(n-n_0)}$, we get

$$|y_n| (1 + a)^{-(n-n_0)} \le |y_{n_0}| + \sum_{k=n_0}^{n} (1 + a)^{-(k-n_0)}\eta_{k-1} |y_{k-1}| . \tag{9}$$

Using the discrete version of the Gronwall-Bellman inequality [1,13], we obtain the estimate

$$|y_n|\,(1+a)^{-(n-n_0)} \leq |y_{n_0}|\, e^{\sum\limits_{k=n_0}^{n} \eta_{k-1}} \leq |y_{n_0}|\, e^{a_1}$$

or

$$|y_n| \leq |y_{n_0}|\, a_2(1+a)^{n-n_0} \quad ,$$

where $a_2 = e^{a_1}$.
The proof is completed .

Denote

$$\begin{aligned} X^1_{n,n_0} &= X_n P_1 X^{-1}_{n_0}, \\ X^2_{n,n_0} &= X_n P_2 X^{-1}_{n_0}. \end{aligned}$$

Lemma 2.4. The matrices X_{n,n_0} , X^1_{n,n_0} , X^2_{n,n_0} satisfy the following relations:
1. $X^1_{n,n_0} + X^2_{n,n_0} = X_{n,n_0}$.

2. $X^i_{n,k} = X^i_{n,n_0} X^i_{n_0,k}$, for all n , n_0, $k \in N$, $i = 1,2$.
3. $X^i_{n,k} = X_{n,n_0} X^i_{n_0,k}$, $i = 1,2$.
Proof. The first statement follows from the definition of the matrices X_{n,n_0} , X^1_{n,n_0} , X^2_{n,n_0} , the equality $X_{n,n_0} = X_n\, X^{-1}_{n_0}$ and mutually complementing property of the projectors $P_1 + P_2 = I$.

$$X^1_{n,n_0} + X^2_{n,n_0} = X_n P_1 X^{-1}_{n_0} + X_n P_2 X^{-1}_{n_0} = X_n X^{-1}_{n_0} = X_{n,n_0}.$$

The second statement follows from

$$X^i_{n,n_0} X^i_{n_0,k} = X_n P_i X^{-1}_{n_0} X_{n_0} P_i X^{-2}_k = X_n P^2_i X^{-1}_k = X^i_{n,k} \quad , \qquad i = 1,2.$$

The third statement follows analogously

$$X_{n,n_0} X^i_{n_0,k} = X_n X^{-1}_{n_0} X_{n_0} P_i X^{-1}_k = X_n P_i X^{-1}_k = X^i_{n,k} \quad , \qquad i = 1,2 \ .$$

The lemma is proved .

3 Main result

Now, we prove the main result of this work. We prove the conditions of asymptotic equivalence of systems (1) and (2).

Theorem 3.1. Suppose that conditions (a) ,(b) ,(c) ,(d) and (e) are satisfied . Further, suppose that system (1) is exponentially dichotomous on N. Additionally, if

$$\sum_{k=0}^{\infty} e^{ak} \eta_k < \infty, \tag{10}$$

then systems (1) and (2) are asymptotically equivalent for $n \to \infty$.
Proof. Let y_n be an arbitrary solution of system (2). Using Lemma 2 and Lemma 4, we have, for $n_0 \geq 0$

$$
\begin{aligned}
y_n &= X_{n,n_0} y_{n_0} + \sum_{k=n_0}^{n} X_{n,k} f_{k-1}(y_{k-1}) \\
&= X_{n,n_0} y_{n_0} + \sum_{k=n_0}^{n} X^1_{n,k} f_{k-1}(y_{k-1}) + \sum_{k=n_0}^{n} X^2_{n,k} f_{k-1}(y_{k-1}) \\
&= X_{n,n_0} y_{n_0} + \sum_{k=n_0}^{n} X^1_{n,k} f_{k-1}(y_{k-1}) + \sum_{k=n_0}^{\infty} X^2_{n,k} f_{k-1}(y_{k-1}) - \sum_{k=n+1}^{\infty} X^2_{n,k} f_{k-1}(y_{k-1}) \qquad (11) \\
&= X_{n,n_0} \left[y_{n_0} + \sum_{k=n_0}^{\infty} X^2_{n,k} f_{k-1}(y_{k-1}) \right] + \sum_{k=n_0}^{n} X^1_{n,k} f_{k-1}(y_{k-1}) - \sum_{k=n+1}^{\infty} X^2_{n,k} f_{k-1}(y_{k-1}).
\end{aligned}
$$

Note, that convergence of $\sum_{k=0}^{\infty} e^{ak}\eta_k$ implies convergence of $\sum_{k=0}^{\infty} (1+a)^k \eta_k$.

Then from (5), lemma 3 and conditions (b) and (d), we estimate

$$
\begin{aligned}
\sum_{k=n_0}^{\infty} \left\| X^2_{n_0,k} \right\| |f_{k-1}(y_{k-1})| &\leq \sum_{k=n_0}^{\infty} N_2 (1+a)^{-\upsilon_2(k-n_0)} \eta_{k-1} |y_{k-1}| \\
&\leq N_2 a_2 \sum_{k=n_0}^{\infty} (1+a)^{-\upsilon_2(k-n_0)} \eta_{k-1} |y_{n_0}| \ (1+a)^{k-n_0} \\
&\leq N_2 a_2 |y_{n_0}| \sum_{k=n_0}^{\infty} \eta_k \ (1+a)^k < \infty \ .
\end{aligned}
$$

Thus, the series in (11) are absolutely convergent.

Note also, that condition (10) implies (c).

The solutions x_n and y_n of systems (1) and (2) are uniquely defined by their initial conditions. Thus, for each solution y_n of system (2) with initial condition $y_{n_0} = y_0$, we put into correspondence the solution x_n of system (1) with condition $x_{n_0} = x_0$ given by

$$
x_{n_0} = y_{n_0} + \sum_{k=n_0}^{\infty} X^2_{n_0,k} f_{k-1}(y_{k-1}). \qquad (12)
$$

Next, we prove that the correspondence given by (12) is one - to - one under the proper choice of n_0.

Denote

$$
\Phi_{n_0}(y_0) = \sum_{k=n_0}^{\infty} X^2_{n_0,k} f_{k-1}(y_{k-1}). \qquad (13)
$$

Then, (12) can rewritten in the form

$$
x_0 = y_0 + \Phi_{n_0}(y_0). \qquad (14)
$$

It can be considered as an equation with respect to $y_0 \in R^d$ for some n_0.Now, we prove, that equation (14) has a unique solution for every x_0.

Rewrite (14) in the form $y_0 = x_0 - \Phi_{n_0}(y_0)$.

Now, we show that

$$x_0 - \Phi_{n_0}(y_0) \tag{15}$$

is contraction for every $x_0 \in R^d$ and some $n_0 \in N_0$.
Indeed, for all $y_0, y_1 \in R^d$, we have

$$\begin{aligned} |x_0 - \Phi_{n_0}(y_0) - x_0 + \Phi_{n_0}(y_1)| &= |\Phi_{n_0}(y_1) - \Phi_{n_0}(y_0)| \\ &\leq \sum_{k=n_0}^{\infty} \left\| X_{n_0,k}^2 \right\| \eta_{k-1} \left| y_{k-1}(n_0, y_1) - y_{k-1}(n_0, y_0) \right| . \end{aligned} \tag{16}$$

Solutions of systems (1) and (2) can be represented in the form

$$y_{k-1}(n_0, y_0) = y_0 + \sum_{j=n_0}^{k-2} \left[A_j y_j(n_0, y_0) + f_j(y_j(n_0, y_0)) \right] \quad ,$$

and

$$y_{k-1}(n_0, y_1) = y_1 + \sum_{j=n_0}^{k-2} \left[A_j y_j(n_0, y_1) + f_j(y_j(n_0, y_1)) \right] \quad .$$

Subtracting the second equation from the first equation, we get

$$|y_{k-1}(n_0, y_1) - y_{k-1}(n_0, y_0)| \leq |y_1 - y_0| +$$

$$+ \sum_{j=n_0}^{k-2} \left[\|A_j\| \left| y_j(n_0, y_1) - y_j(n_0, y_0) \right| + \eta_j \left| y_j(n_0, y_1) - y_j(n_0, y_0) \right| \right]$$

$$\leq |y_1 - y_0| + \sum_{j=n_0}^{k-2} (a + \eta_j) \left| y_j(n_0, y_1) - y_j(n_0, y_0) \right| .$$

Using the discrete version of the Gronwall-Bellman inequality, we obtain the estimate

$$|y_{k-1}(n_0, y_1) - y_{k-1}(n_0, y_0)| \leq |y_1 - y_0| \, e^{\sum_{j=n_0}^{k-2} (a+\eta_j)} \leq |y_1 - y_0| \, e^{a(k-n_0)+a_1}. \tag{17}$$

Substituting (17) into (16), we get

$$\begin{aligned} |\Phi_{n_0}(y_1) - \Phi_{n_0}(y_0)| &\leq \sum_{k=n_0}^{\infty} \left\| X_{n_0,k}^2 \right\| \eta_{k-1} \left| y_1 - y_0 \right| e^{a(k-n_0)+a_1} \\ &\leq \sum_{k=n_0}^{\infty} N_2 (1+a)^{-\upsilon_2 (k-n_0)} \eta_{k-1} \left| y_1 - y_0 \right| e^{a(k-n_0)+a_1} \\ &\leq N_2 e^{a_1} \sum_{k=n_0}^{\infty} e^{k-n_0} \left| y_1 - y_0 \right| \leq N_2 e^{a_1} e^{a(1-n_0)} \sum_{k=n_0}^{\infty} \eta_{k-1} e^{a(k-1)} \left| y_1 - y_0 \right| . \end{aligned} \tag{18}$$

By the condition (10) , we can choose n_o such that

$$N_2 e^{a_1} e^{a(1-n_0)} \sum_{k=n_0}^{\infty} \eta_{k-1} e^{a(k-1)} < 1. \tag{19}$$

Then from (18) and (19) , it follows that mapping $x_0 - \Phi_{n_0}(y_0)$ is a contraction in R^d . So, the equation (14) has a unique solution in R^d for certain $n_0 \in N_0$ and any $x_0 \in R^d$.

Therefore, the correspondence between the solutions of system of systems (1) and (2) given by (12) is one - to - one .

Now, it remains to prove (3).

Let x_n and y_n be solutions of systems (1) and (2), respectively.

Since any solution of (1) has the form $x_n = X_{n,n_0} x_{n_0}$ and x_{n_0} is defined by (12), then from (11), we have

$$|x_n - y_n| \leq \left| \sum_{k=n_0}^{n} X^1_{n,k} f_{k-1}(y_{k-1}) \right| + \left| \sum_{k=n+1}^{\infty} X^2_{n,k} f_{k-1}(y_{k-1}) \right| . \tag{20}$$

We estimate the first series

$$\begin{aligned}
\left| \sum_{k=n_0}^{n} X^1_{n,k} f_{k-1}(y_{k-1}) \right| &\leq \sum_{k=n_0}^{n} \left\| X^1_{n,k} \right\| \eta_{k-1} \left| y_{k-1} \right| \\
&\leq \sum_{k=n_0}^{n} (N_1 (1+a)^{-\upsilon_1 (n-k)} \eta_{k-1} a_2 (1+a)^{k-1-n_0} \left| y_{n_0} \right|) \\
&= \sum_{k=n_0}^{\left[\frac{n-n_0}{2}\right]} (N_1 (1+a)^{-\upsilon_1 (n-k)} \eta_{k-1} a_2 (1+a)^{k-1-n_0} \left| y_{n_0} \right|) \\
&\quad + \sum_{k=\left[\frac{n-n_0}{2}\right]+1}^{n} (N_1 (1+a)^{-\upsilon_1 (n-k)} \eta_{k-1} a_2 (1+a)^{k-1-n_0} \left| y_{n_0} \right|)
\end{aligned}$$

$$\begin{aligned}
&\leq N_1 a_2 \left| y_{n_0} \right| \ (1+a)^{-\upsilon \frac{n-n_0}{2}} \sum_{k=n_0}^{\left[\frac{n-n_0}{2}\right]} \eta_{k-1} (1+a)^{k-1} + N_1 a_2 \left| y_{n_0} \right| \sum_{k=\left[\frac{n-n_0}{2}\right]+1}^{n} \eta_{k-1} (1+a)^{k-1} \\
&\leq N_1 a_2 \left| y_{n_0} \right| \ (1+a)^{-\upsilon \frac{n-n_0}{2}} \sum_{k=n_0}^{\infty} \eta_{k-1} (1+a)^{k-1} + N_2 a_2 \left| y_{n_0} \right| \sum_{k=\left[\frac{n-n_0}{2}\right]+1}^{n} \eta_{k-1} (1+a)^{k-1} \quad \to 0,
\end{aligned}$$

$n \ \to \ \infty$

Here $\left[\frac{n-n_0}{2}\right]$ is an integer part of $\frac{n-n_0}{2}$.

Then, show that the second series approaches to zero, too.

$$\begin{aligned}
\left|\sum_{k=n+1}^{\infty} X_{n,k}^{2} f_{k-1}(y_{k-1})\right| &\leq \sum_{k=n+1}^{n} \left\|X_{n,k}^{2}\right\| \eta_{k-1} \left|y_{k-1}\right| \\
&\leq \sum_{k=n+1}^{\infty} N_2(1+a)^{-\upsilon_2(k-n)} \eta_{k-1} a_2 (1+a)^{(k-1-n_0)} \left|y_{n_0}\right| \\
&\leq N_2 a_2 \sum_{k=n+1}^{\infty} \eta_{k-1}(1+a)^{k-1} \left|y_{n_0}\right| \ \to 0, \\
n &\to \infty.
\end{aligned}$$

Thus $\underset{n\to\infty}{Lim} |x_n - y_n| = 0$. This completes the proof of the theorem.

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ε-uniform convergence of the midpoint upwind scheme on the Bakhvalov-Shishkin mesh for singularly perturbed problems *

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Abstract: In this paper, we investigate the midpoint upwind scheme on the Bakhvalov-Shishkin mesh for solving singularly perturbed convection-diffusion boundary value problems. Its elaborate ε-uniform pointwise convergence is proved by using the comparison principle and constructing barrier functions. The numerical experiments indicate that the estimate is sharp and high accuracy is achieved for resolving the boundary layer.

Keywords: Singularly perturbed problem; Convection-diffusion; Midpoint upwind scheme; Bakhvalov-Shishkin mesh; Uniform convergence

1 Introduction

Singularly perturbed second-order boundary value problems arise in many branches of science and engineering. Let's consider it in the following form:

$$\begin{cases} Lu(x) := -\varepsilon u''(x) + b(x)u'(x) + c(x)u(x) = f(x), x \in (0,1), \\ u(0) = A, u(1) = B, \end{cases} \tag{1}$$

where $0 < \varepsilon \ll 1$ is a small perturbation parameter, A and B are given constants, and functions $b(x), c(x)$ and $f(x)$ are sufficiently smooth with $b(x) > \beta > 0$ and $c(x) \geq 0$. Under these conditions, singularly perturbed problem (1) has a unique solution with boundary layer on the right side. A wide variety of numerical methods were established to solve the problem in the past few decades (see [1-3]).

The layer-adapted graded mesh was first introduced for solving the singularly perturbed reaction-diffusion problem by Bakhvalov [4], where a nonlinear equation should be solved for the transition point. The special piecewise uniform mesh by Shishkin [5] livened up further discussion because of its simple structure. By the technique of the comparison principle and barrier functions used by Kellogg and Tsan [6] on equidistant meshes, the Bakhvalov-Shishkin

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mesh as a combination of the above two meshes was proved ε-uniform convergence of order $O(N^{-1})$ for the simple upwind scheme and the Galerkin FEM by Linß [7, 8]. Roos and Linß [9] summarized the simple upwind difference scheme and the Galerkin finite element method on Shishkin-type meshes, generalized ε-uniform convergence of order $O(N^{-1}\ln N)$ on the S mesh and $O(N^{-1})$ on the B-S mesh. Stynes and Roos [10] investigated the midpoint upwind scheme on the S mesh and obtained the ε-uniform convergence of order $O(N^{-1}(\varepsilon+N^{-1}))$ on the nodes in coarse part and $O(N^{-1}\ln N)$ on the nodes in fine part. Furthermore, considering the midpoint upwind scheme on the B-S mesh for solving $-\varepsilon u''(x)+b(x)u'(x)=f(x)$, Liang, Li and Jiang [11] only proved the ε-uniform convergence of order $O(N^{-1})$ on the whole nodes.

In this paper, we study the midpoint upwind scheme on Bakhvalov-Shishkin mesh for solving the general problem (1). In section 2, the properties of the exact solution and the B-S mesh are introduced. In section 3, we prove the ε-uniform pointwise convergence of order $O(N^{-2})$ on the nodes in coarse part and $O(N^{-1})$ on the nodes in fine part. In section 4, several numerical examples support the elaborate error estimate and demonstrate the efficiency of the method.

2 Properties of the solution and the mesh

In the following, Lemma 1 is classical and Lemma 2 is easy to be proved by calculation, see, e.g., [2, Chapter 1] and [7], respectively.

Lemma 1 For any positive integer $q>0$, if $u(x)$ is the solution of problem (1) with sufficiently smooth data, then $u(x)$ can be decomposed as $u=S+E$, where the smooth part S satisfies

$$LS(x)=f(x) \text{ and } \mid S^{(i)}(x)\mid\leq C, 0\leq i\leq q,$$

while the layer part E satisfies

$$LE(x)=0 \text{ and } \mid E^{(i)}(x)\mid\leq C\varepsilon^{-i}\exp(-\tfrac{\beta(1-x)}{\varepsilon}), 0\leq i\leq q.$$

Let $\tau=\min\{\frac{1}{2},\frac{2\varepsilon\ln N}{\beta}\}$ and $1-\tau$ be the transition point, where $\varepsilon\leq N^{-1}$ as generally in practice. Partition $[0,1-\tau]$ uniformly into $\frac{N}{2}$ subintervals, and $[1-\tau,1]$ into $\frac{N}{2}$ subintervals by inverting the exponential function such that $e^{-\beta(1-x_i)/(2\varepsilon)}=A\frac{i}{N}+B, i=\frac{N}{2},...,N$, with $x_{N/2}=1-\tau$ and $x_N=1$. Thus, the Bakhvalov-Shishkin mesh: $x_i=x(t_i)$, $t_i=\frac{i}{N}, i=0,1,...,N$, follows by the mesh generating function

$$x(t)=\begin{cases} 2(1-\dfrac{2\varepsilon\ln N}{\beta})t, 0\leq t\leq\dfrac{1}{2}, \\ 1+\frac{2\varepsilon\ln[1-2(1-N^{-1})(1-t)]}{\beta}, \frac{1}{2}\leq t\leq 1, \end{cases} \tag{2}$$

which maps the equidistant mesh $\{t_i\}$ onto the layer-adapted mesh $\{x_i\}$. Denote $h_i=x_i-x_{i-1}$ and $x_{i-1/2}=(x_{i-1}+x_i)/2$.

Lemma 2. $N^{-1}\leq h_i<2N^{-1}$, $\frac{(2i-1)\varepsilon}{i^2\beta}<h_{N/2+i}<\frac{4\varepsilon}{i\beta}\leq CN^{-1}, i=1,2,...,N/2.$

Throughout the paper, C is a generic positive constant that is independent of ε and h_i, and note that C can take different values at each occurrence, even in the same argument.

3 The scheme and its uniform convergence

Let's investigate the midpoint upwind scheme as follows:

$$L^N u_i^N := -\varepsilon D^+D^-u_i^N + b_{i-1/2}D^-u_i^N + c_{i-1/2}(u_i^N + u_{i-1}^N)/2 = f_{i-1/2}, \tag{3}$$

where $D^+u_i^N = \frac{u_{i+1}^N - u_i^N}{h_{i+1}}, D^-u_i^N = \frac{u_i^N - u_{i-1}^N}{h_i}$ and $D^+D^-u_i^N = \frac{2(D^+u_i^N - D^-u_i^N)}{h_{i+1}+h_i}$.

Since the matrix associated with L^N is M-matrix, the discrete comparison principle holds for the midpoint upwind scheme.

By direct computation and Taylor formula as usual, we have the following two lemmas.

Lemma 3. If $Z_0 = 1, Z_i = \prod\limits_{j=1}^{i}(1+\frac{\beta h_j}{2\varepsilon}), i = 1, 2, ..., N$, then $L^N Z_i \geq \frac{CZ_i}{\max\{\varepsilon, h_i\}}$.

Lemma 4. $\mid L^N(u_i - u_i^N) \mid \leq C[\varepsilon\int_{x_{i-1}}^{x_{i+1}} \mid u'''(t) \mid \mathrm{d}t + h_i\int_{x_{i-1}}^{x_i} \mid u''(t) \mid \mathrm{d}t]$.

As in the continuous case, decompose the numerical solution into the smooth part and the layer part by $u_i^N = S_i^N + E_i^N$, where S_i^N and E_i^N satisfy $L^N S_i^N = f_i, S_0^N = S_0, S_N^N = S_N; L^N E_i^N = 0, E_0^N = E_0, E_N^N = E_N$. Therefore, $|u_i - u_i^N| \leq |S_i - S_i^N| + |E_i - E_i^N|$.

For the smooth part, by Lemma 1 and 4, we have

$$|L^N(S_i - S_i^N)| \leq C[\varepsilon\int_{x_{i-1}}^{x_{i+1}} |S'''(t)|\mathrm{d}t + h_i\int_{x_{i-1}}^{x_i} |S''(t)|\mathrm{d}t] \leq CN^{-1}(\varepsilon + N^{-1}),$$

for $i = 1, \cdots, N-1$. Setting $w_i = C_0N^{-1}(\varepsilon + N^{-1})x_i$ for all i, we have

$$L^N w_i = b_iC_0N^{-1}(\varepsilon + N^{-1}) + c_iw_i \geq CN^{-1}(\varepsilon + N^{-1}) \geq |L^N(S_i - S_i^N)|,$$

where C_0 is a sufficiently large constant. Clearly $w_0 = 0 = |S_0 - S_0^N|$ and $w_N = C_0N^{-1}(\varepsilon + N^{-1}) \geq 0 = |S_N - S_N^N|$. By the discrete comparison principle, we get

$$|S_i - S_i^N| \leq w_i \leq C_0N^{-1}(\varepsilon + N^{-1}) \leq CN^{-2}, i = 0, 1, \cdots, N. \tag{4}$$

For the layer part, by Lemma 1, we have

$$|E_i| \leq Ce^{-\beta[1-(1-\tau)]/\varepsilon} = CN^{-2}, i = 0, 1, \cdots, N/2. \tag{5}$$

Lemma 5. There exists a constant C such that

$$|E_i^N| \leq CN^{-2}, \quad i = 0, 1, \cdots, N/2.$$

Proof. Let $z_i = \prod\limits_{j=1}^{i}(1+\frac{\beta h_j}{\varepsilon})$. Since $e^t > 1+t$, $t > 0$, we have

$$\frac{z_i}{z_N} = \prod_{j=i+1}^{N}(1+\frac{\beta h_j}{\varepsilon})^{-1} \geq \prod_{j=i+1}^{N} e^{-\beta h_j/\varepsilon} = e^{-\beta(1-x_i)/\varepsilon}.$$

Set $Y_i = C_0 z_i/z_N$ for $i = 0, 1, \cdots, N$. Then $L^N Y_i = (C_0/z_N)L^N z_i \geq 0 = |L^N E_i^N|$ for $i = 1, \cdots, N-1$. By Lemma 1, $Y_N = C_0 \geq |E(1)| = |E_N^N|$ and $Y_0 = \frac{C_0 z_0}{z_N} \geq C_0 e^{-\beta/\varepsilon} \geq |E(0)| = |E_0^N|$, provided that the constant C_0 is chosen sufficiently large. So, Y_i is a discrete barrier function for E_i^N and $|E_i^N| \leq Y_i = \frac{C_0 z_i}{z_N}$ for all i. The proof is completed by combining it with

$$\frac{z_i}{z_N} \leq \frac{z_{N/2}}{z_N} = \prod_{j=N/2+1}^{N}(1+\frac{\beta h_j}{\varepsilon})^{-1} \leq Ce^{-\frac{\beta(1-x_{N/2})}{\varepsilon}} = CN^{-2}, i = 0, 1, \cdots, N/2.$$

The above inequality is valid, because we have from Lemma 2 that

$$\ln \prod_{j=\frac{N}{2}+1}^{N}(1+\frac{\beta h_j}{\varepsilon}) \geq \sum_{j=\frac{N}{2}+1}^{N}(\frac{\beta h_j}{\varepsilon} - \frac{1}{2}(\frac{\beta h_j}{\varepsilon})^2) \geq \frac{\beta(1-x_{\frac{N}{2}})}{\varepsilon} - \sum_{i=1}^{\frac{N}{2}}\frac{8}{i^2} = \frac{\beta(1-x_{\frac{N}{2}})}{\varepsilon} - C. \quad \square$$

Lemma 6. There exists a constant C such that

$$| E_i - E_i^N | \leq CN^{-2}, \quad i = 0, 1, \cdots, N/2.$$

Proof. From (5) and Lemma 5, the proof is completed. $\square$

Lemma 7. There exists a constant C such that

$$| E_i - E_i^N | \leq CN^{-1}, \quad i = N/2+1, \cdots, N.$$

Proof. By using Lemma 4, Lemma 1, (2), Lemma 2 and Lemma 3 successively, and noting that

$$\frac{Z_i}{Z_N} = \prod_{j=i+1}^{N}(1+\frac{\beta h_j}{2\varepsilon})^{-1} \geq \prod_{j=i+1}^{N} e^{-\beta h_j/(2\varepsilon)} = e^{-\beta(1-x_i)/(2\varepsilon)},$$

we have

$$\begin{aligned}
| L^N(E_i - E_i^N) | &\leq C[\varepsilon \textstyle\int_{x_{i-1}}^{x_{i+1}} | E'''(x) | \,\mathrm{d}x + h_i \int_{x_{i-1}}^{x_i} | E''(x) | \,\mathrm{d}x] \\
&\leq C \textstyle\int_{x_{i-1}}^{x_{i+1}} \varepsilon^{-2} \exp(-\frac{\beta(1-x)}{\varepsilon})\mathrm{d}x \\
&= C\varepsilon^{-2} \textstyle\int_{t_{i-1}}^{t_{i+1}} [1-2(1-N^{-1})(1-t)]^2 \frac{2\varepsilon}{\beta} \frac{2(1-N^{-1})}{1-2(1-N^{-1})(1-t)}\mathrm{d}t \\
&\leq C\varepsilon^{-1} \textstyle\int_{t_{i-1}}^{t_{i+1}} \exp(-\frac{\beta(1-x(t))}{2\varepsilon})\mathrm{d}t \\
&\leq C\varepsilon^{-1}N^{-1} \exp(-\frac{\beta(1-x_{i+1})}{2\varepsilon}) \\
&\leq C\varepsilon^{-1}N^{-1} \frac{Z_i}{Z_N}(1+\frac{\beta h_{i+1}}{2\varepsilon}) \\
&\leq C\varepsilon^{-1}N^{-1} \frac{Z_i}{Z_N} \\
&\leq CN^{-1}L^N(1+\frac{Z_i}{Z_N}), \quad i = N/2+1, \cdots, N-1.
\end{aligned}$$

Setting $\phi_i = CN^{-1}(1+\frac{Z_i}{Z_N})$, provided that the constant C is chosen sufficiently large, we have $\phi_{\frac{N}{2}} = CN^{-1}(1+\frac{Z_{\frac{N}{2}}}{Z_N}) \geq CN^{-1} \geq |E_{\frac{N}{2}} - E_{\frac{N}{2}}^N|$, $\phi_N = 2CN^{-1} \geq 0 = |E_N - E_N^N|$,

$L^N\phi_i = CN^{-1}L^N(1+\frac{Z_i}{Z_N}) \geq |L^N(E_i - E_i^N)|$. Therefore, $| E_i - E_i^N | \leq \phi_i \leq CN^{-1}$ by the discrete comparison principle. $\square$

Theorem 1. The midpoint upwind scheme on the Bakhvalov-Shishkin mesh satisfies

$$|u_i - u_i^N| \leq \begin{cases} CN^{-2}, 0 \leq i \leq \frac{N}{2}, \\ CN^{-1}, \frac{N}{2} < i \leq N. \end{cases} \tag{6}$$

Proof. Combining (4), Lemma 6 and Lemma 7, the proof is completed. $\square$

4 Numerical examples

The numerical results are illustrated in Tables 1-4 and Figs. 1-4. The numerical convergence orders are computed by $\log_2 \frac{\max|u_i - u_i^N|}{\max|u_i - u_i^{2N}|}$ on the coarse part and the fine part, respectively. The numerical convergence constants on the B-S mesh are computed by $\max\limits_{i\leq N/2}|u_i - u_i^N|/N^{-2}$ and $\max\limits_{i>N/2}|u_i - u_i^N|/N^{-1}$ on the fine part and coarse part, respectively. Those on the S mesh are computed by $\max\limits_{i\leq N/2}|u_i - u_i^N|/N^{-2}$ and $\frac{\max\limits_{i>N/2}|u_i - u_i^N|}{N^{-1}\ln N}$, respectively.

Problem 1 (see [12]).

$$\begin{cases} -\varepsilon y'' + (1 + x(1-x))y' = f(x), 0 < x < 1, \\ y(0) = y(1) = 0, \end{cases}$$

where $f(x)$ is chosen such that $y(x) = \frac{1-e^{\frac{-(1-x)}{\varepsilon}}}{1-e^{\frac{-1}{\varepsilon}}} - \cos\frac{\pi x}{2}$ is the exact solution.

Numerical results on the B-S mesh and the S mesh are shown in Table 1 and Table 2. The behaviors of the solutions are depicted in Figs. 1 and 2.

Table 1. The comparison on the B-S mesh and the S mesh for Problem 1 with $\varepsilon = 10^{-8}$

	B-S mesh						S mesh					
N	$i \leq N/2$	order	const	$i > N/2$	order	const	$i \leq N/2$	order	const	$i > N/2$	order	const
64	9.705e-4	2.39	3.98	0.0296	0.95	1.89	9.548e-5	3.10	.391	0.0560	0.63	.861
128	1.627e-4	2.58	2.67	0.0150	0.98	1.92	2.448e-5	1.96	.401	0.0343	0.71	.906
256	2.498e-5	2.70	1.64	0.0076	0.99	1.94	6.198e-6	1.98	.406	0.0203	0.77	.936
512	2.492e-6	2.84	.915	0.0038	1.00	1.95	1.559e-6	1.99	.409	0.0116	0.81	.955
1024	4.160e-7	3.07	.436	0.0019	1.00	1.95	3.910e-7	2.00	.410	0.0065	0.84	.967
2048	9.788e-8	2.09	.411	9.509e-4	1.00	1.95	9.789e-8	2.00	.411	0.0036	0.85	.973
4096	2.449e-8	2.00	.411	4.757e-4	1.00	1.95	2.449e-8	2.00	.411	0.0020	0.85	.977

Table 2. The maximal error and its convergence order on the B-S mesh for Problem 1

	$\varepsilon = 10^{-6}$						$\varepsilon = 10^{-10}$					
N	$i \leq N/2$	order	const	$i > N/2$	order	const	$i \leq N/2$	order	const	$i > N/2$	order	const
64	9.704e-4	2.39	3.97	0.0296	0.95	1.89	9.705e-4	2.39	3.98	0.0296	0.95	1.89
128	1.627e-4	2.58	2.67	0.0150	0.98	1.92	1.627e-4	2.58	2.67	0.0150	0.98	1.92
256	2.497e-5	2.70	1.64	0.0076	0.98	1.94	2.498e-5	2.70	1.64	0.0076	0.99	1.94
512	3.491e-6	2.84	.915	0.0038	1.00	1.94	3.492e-6	2.84	.915	0.0038	1.00	1.94
1024	4.164e-7	3.07	.437	0.0019	1.00	1.95	4.160e-7	3.07	.436	0.0019	1.00	1.95
2048	9.673e-8	2.11	.406	9.509e-4	1.00	1.95	9.790e-8	2.09	.411	9.509e-4	1.00	1.95
4096	2.396e-8	2.01	.402	4.757e-4	1.00	1.95	2.449e-8	2.00	.411	4.757e-4	1.00	1.95

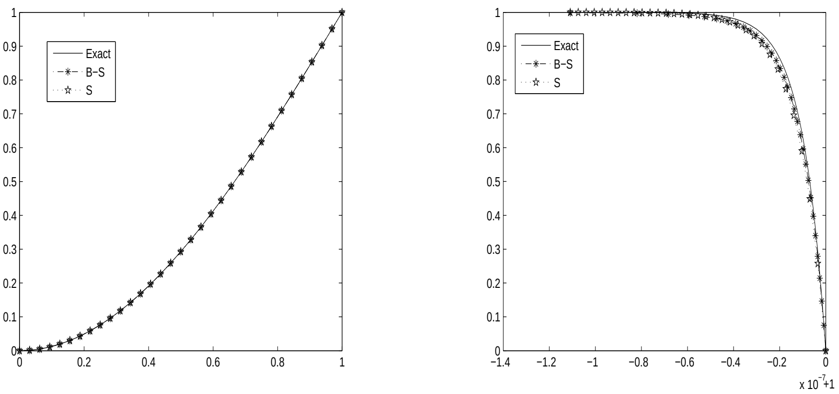

Fig. 1 The solutions of Problem 1 with $\varepsilon = 10^{-8}$ on $[0, 1-\tau]$ and $[1-\tau, 1]$, respectively.

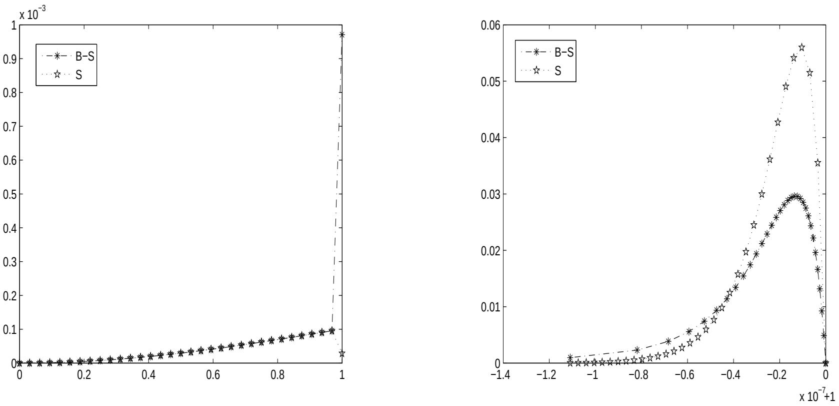

Fig. 2 The errors for Problem 1 with $\varepsilon = 10^{-8}$ on $[0, 1-\tau]$ and $[1-\tau, 1]$, respectively.

Problem 2 (see [10]).

$$\begin{cases} -\varepsilon y'' + \dfrac{1}{x+1} y' + \dfrac{1}{x+2} y = f(x), 0 < x < 1, \\ y(0) = 1 + 2^{\frac{-1}{\varepsilon}}, y(1) = e + 2, \end{cases}$$

where $f(x) = (-\varepsilon + \frac{1}{x+1} + \frac{1}{x+2})e^x + \frac{1}{x+2} 2^{\frac{-1}{\varepsilon}} (x+1)^{1+\frac{1}{\varepsilon}}$ and $y(x) = e^x + 2^{\frac{-1}{\varepsilon}} (x+1)^{1+\frac{1}{\varepsilon}}$. Numerical results are illustrated in Table 3 and Fig. 3.

Table 3. The maximal error and its convergence order on the B-S mesh for Problem 2

	$\varepsilon = 10^{-6}$						$\varepsilon = 10^{-10}$					
N	$i \le N/2$	order	const	$i > N/2$	order	const	$i \le N/2$	order	const	$i > N/2$	order	const
64	0.0020	2.38	8.11	0.0592	0.95	3.79	0.0020	2.38	8.11	0.0592	0.95	3.79
128	3.361e-4	2.57	5.51	0.0300	0.98	3.84	3.363e-4	2.57	5.51	0.0300	0.98	3.84
256	5.269e-5	2.67	3.45	0.0151	0.99	3.87	5.275e-5	2.67	3.46	0.0151	0.99	3.87
512	7.674e-6	2.78	2.01	0.0076	0.99	3.88	7.691e-6	2.78	2.02	0.0076	0.99	3.88
1024	1.005e-6	2.93	1.05	0.0038	1.00	3.89	1.010e-6	2.93	1.06	0.0038	1.00	3.89
2048	1.501e-7	2.74	.630	0.0019	1.00	3.89	1.514e-7	2.74	.635	0.0019	1.00	3.89
4096	3.732e-8	2.01	.626	9.514e-4	1.00	3.90	3.788e-8	2.00	.636	9.517e-4	1.00	3.90

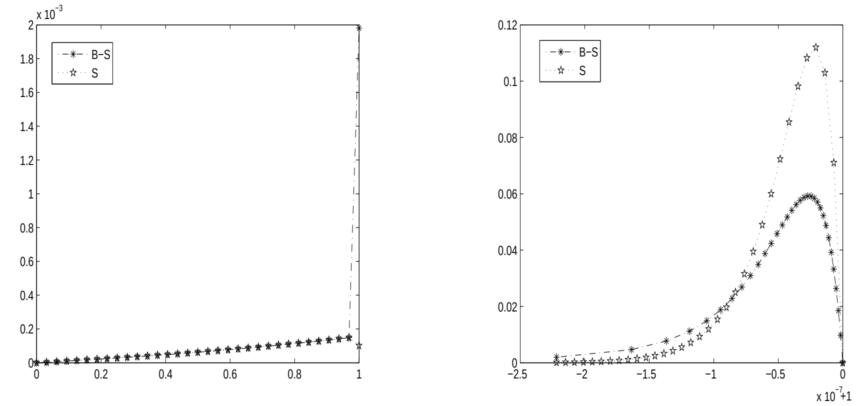

Fig. 3 The errors for Problem 2 with $\varepsilon = 10^{-8}$ on $[0, 1-\tau]$ and $[1-\tau, 1]$, respectively.

Problem 3 (see [11]).

$$\begin{cases} -\varepsilon y'' + y' + (1+\varepsilon)y = 0, 0 < x < 1, \\ y(0) = 1 + e^{-\frac{1+\varepsilon}{\varepsilon}}, y(1) = 1 + \frac{1}{e}, \end{cases}$$

Its exact solution is given by $y(x) = e^{-x} + e^{\frac{(1+\varepsilon)(x-1)}{\varepsilon}}$. Numerical results are illustrated in Table 4 and Fig. 4.

Table 4. The maximal error and its convergence order on the B-S mesh of Problem 3

	$\varepsilon = 10^{-6}$						$\varepsilon = 10^{-10}$					
N	$i \le N/2$	order	const	$i > N/2$	order	const	$i \le N/2$	order	const	$i > N/2$	order	const
64	2.993e-5	2.00	.123	0.0384	0.89	2.46	2.993e-5	2.00	.123	0.0384	0.89	2.46
128	7.486e-6	2.00	.123	0.0200	0.94	2.56	7.484e-6	2.00	.123	0.0200	0.94	2.56
256	1.872e-6	2.00	.123	0.0102	0.97	2.61	1.871e-6	2.00	.123	0.0102	0.97	2.61
512	4.685e-7	2.00	.123	0.0052	0.97	2.64	4.678e-7	2.00	.123	0.0052	0.97	2.64
1024	1.173e-7	2.00	.123	0.0026	1.00	2.65	1.170e-7	2.00	.123	0.0026	1.00	2.65
2048	2.941e-8	2.00	.123	0.0013	1.00	2.66	2.924e-8	2.00	.123	0.0013	1.00	2.66
4096	7.398e-9	1.99	.124	6.5082e-4	1.00	2.67	7.309e-9	2.00	.123	6.5082e-4	1.00	2.67

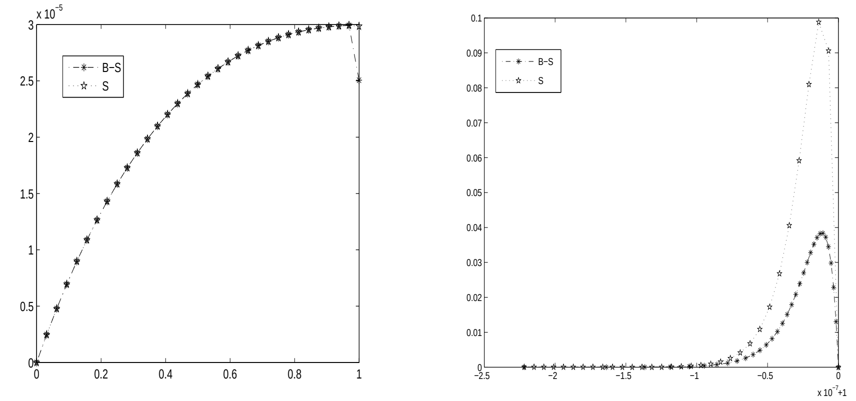

Fig. 4 The errors for Problem 3 with $\varepsilon = 10^{-8}$ on $[0, 1-\tau]$ and $[1-\tau, 1]$, respectively.

The examples for the midpoint upwind scheme on the B-S mesh show that the numerical convergence is of second-order on the nodes in coarse part and first-order on the nodes in fine part with the bounded coefficient for the error estimate formulas.

5 Conclusions

The ε-uniform pointwise convergence of the midpoint upwind scheme on the B-S mesh for solving singularly perturbed convection-diffusion problems is proved by using the comparison principle and constructing the barrier functions. The numerical experiments indicate that the estimate is sharp and high accuracy is achieved for resolving the boundary layer.

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Monotone Iterative Technique of Periodic Solutions for Impulsive Evolution Equations in Banach Space[†]

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Abstract: In this paper, we discuss the problem of impulsive evolution equation in an ordered Banach space. Under impulsive function satisfied ordered conditions, and under compact semigroup, or noncompactness measure, or normal cone, we obtained the existence of ω-periodic mild solutions for impulsive evolution equation by using the monotone iterative technique. The results improve and extend the evolution equations without impulse and some relevant results in ordinary differential equations.

Keywords: Semilinear impulsive evolution equation; Coupled upper and lower solutions; Monotone iterative technique; Periodic boundary value problems

1 Introduction

The study of impulsive differential equations is a new and important branch of differential equation theory,for studying evolution processes of real life phenomena not only in natural science but also in social science such as climate, food supplement, insecticide population, sustainable development that are subjected to sudden changes at certain instants,(See[1, 2, 3, 4]). Recently, many authors considered the applications of the theory of impulsive differential equations to different areas and had obtained some basic results on impulsive differential equations,(See [2, 3, 4, 5, 6, 7, 8, 9, 10])and references therein. The monotone iterative technique was applied in impulsive differential equations by many authors in (See [1, 2, 3, 4, 10]), but the research on the problem of periodic solutions for impulsive evolution equations is seldom,(See [7, 9]). In this paper, by using a mixed monotone iterative technique in the presence of coupled lower and upper L-quasisolutions, we are concerned with periodic solutions for impulsive evolution equations

$$\begin{cases} u'(t) + Au(t) = f(t, u(t)), \quad t \geq 0, \qquad t \neq t_k, \\ \Delta u \mid_{t=t_k} = I_k(u(t_k)), \qquad k \in \mathbb{N}. \end{cases} \tag{1.1}$$

in an ordered Banach space X, where $A : D(A) \subset X \to X$ is a closed linear operator and $-A$ generates a positive C_0-semigroup $T(t)(t \geq 0)$ in X; and $f : [0, +\infty) \times X \to X$ is a continuous function, and is $\omega-$periodic about t. $J = [0, \omega]$, ω is a constant; $0 < t_1 < t_2 < \cdots < t_p < \omega$. $I_k : X \to X$ $(k = 1, 2, \cdots, p)$ is impulsive function. $\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-)$ denotes the jump of $u(t)$ at $t = t_k$, where $u(t_k^+)$, $u(t_k^-)$ represent the right and left limits of $u(t)$ at $t = t_k$ $(k \in \mathbb{N})$, respectively. Let $J' = J \setminus \{t_1,\ t_2,\ \cdots,\ t_p\}$, $J'' = J \setminus \{0,\ t_1,\ t_2,\ \cdots,\ t_p\}$, and $J_0 = [0, t_1]$, $J_1 = (t_1, t_2]$, $\cdots$, $J_p = (t_p, \omega]$. Evidently, $PC(J, X) = \big\{u : J \to X \mid u(t)$ is continuous in J', and left continuous at t_k ,and $u(t_k^+)$ exists, $k = 1, 2, \cdots, p\big\}$. $PC(J, X)$ is a Banach space with the norm $\|\cdot\|_{PC} = \sup\{\|u\| \mid t \in [0, \omega]\}$. Denote by X_1 the Banach space generated by $D(A)$ with the norm $\|\cdot\|_1 = \|\cdot\| + \|A\cdot\|$.

Obviously, the periodic problem of impulsive evolution equations (1.1) is equal to the periodic bound-

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ary value problem (PBVP) of impulsive evolution equations in J

$$\begin{cases} u'(t)+Au(t)=f(t,u(t)), \quad t\in J, \qquad t\neq t_k, \\ \Delta u\mid_{t=t_k}=I_k(u(t_k)), \quad k=1,2,\ \cdots,p, \\ u(0)=u(\omega). \end{cases} \tag{1.2}$$

Without impulse, the PBVP (1.2) has been studied by many authors, see([10, 12, 13]) and the references therein. In particular, Li([12]) considered the existence of coupled mild ω-periodic quasisolution pair for the following periodic boundary value problem (PBVP) in X:

$$\begin{cases} u'(t)+Au(t)=f(t,u(t)), \quad t\in J, \\ u(0)=u(\omega). \end{cases} \tag{1.3}$$

where $f:\mathbf{R}\times X\to X$ is continuous. Under one of the following situations:

(i) $T(t)(t\geq 0)$ is a compact semigroup,

(ii) K is regular in X and $T(t)$ is continuous in operator norm for $t>0$, they built a mixed monotone iterative method for the PBVP (1.3), and they proved that, if the PBVP(1.3) has coupled lower and upper quasisolutions v_0 and w_0, nonlinear term f satisfies the following condition:

(F) There exists $C>0$, for $\forall u,v\in[v_0,w_0]$, when $u\leq v$, has

$$f(t,v)-f(t,u)\geq -C(v-u).$$

Then the PBVP (1.3) has minimal and maximal coupled mild ω -periodic quasisolutions between v_0 and w_0, which can be obtained by monotone iterative sequences from v_0 and w_0.

In this paper, we will consider the existence of mild ω-periodic solutions for the impulsive evolution Equation (1.2) by means of a mixed monotone iterative technique under a new concept of upper and lower solutions, in an ordered Banach space X. In our results, we obtain the mild ω-periodic solutions of the problem with impulse when the operator semigroup $T(t)(t\geq 0)$ satisfied conditions (i) or (ii). In addition, we delete the conditions (i) and (ii), and demand that the nonlinear term f and impulsive function I_k satisfy the noncompactness measure condition

$$\alpha\big(f(t,G(t))+CG(t)\big)\leq L\alpha\big(G(t)\big).$$

and

$$\alpha\big(I_k(G(t))\big)\leq M_k\alpha\big(G(t)\big).$$

where $G=\{u_n\}\subset[v_0,w_0]$ is equicontinuous sequence in $t\in J$, L, M_k are positive constants and satisfy

$$\left[\frac{2L}{C-\nu_0}+\frac{\sum\limits_{k=1}^{p}M_k}{1-e^{-(C-\nu_0)\omega}}\right]M<1.$$

Our main results are as follows:

Theorem 1.1 *Let X be an ordered Banach space, whose positive cone K is normal, $A:D(A)\subset X\to X$ be a closed linear operator and $-A$ generate a compact and positive C_0-semigroup $T(t)(t\geq 0)$ in X. $f\in C(J\times X,X)$ and f is $\omega-$periodic about t, $I_k\in C(X,X), k=1,2,\ \cdots,p$. Assume that the PBVP (1.2) has coupled lower and upper L-quasisolutions v_0 and w_0 with $v_0(t)\leq w_0(t)\,(t\in J)$. Suppose that the following conditions are satisfied:*

(H_1) There exists a constant $C\geq 0$ such that

$$f(t,x_2)-f(t,x_1)\geq -C(x_2-x_1), \qquad t\in J,$$

for $\forall t\in J, v_0(t)\leq x_1\leq x_2\leq w_0(t)$.

(H_2) For $\forall t\in J, v_0(t)\leq x_1\leq x_2\leq w_0(t)$, impulsive function I_k satisfies

$$I_k(x_1)\leq I_k(x_2), \qquad k=1,2,\cdots,p.$$

Then the PBVP (1.2) has minimal and maximal coupled mild ω -periodic L-quasisolutions $\overline{u}$ and $\underline{u}$ between v_0 and w_0, which can be obtained by monotone iterative sequences starting from v_0 and w_0.

Remark 1.1 *If $I_k \equiv 0$, then Theorem 1.1 in this paper is Theorem 3.1 in([12]).*

Theorem 1.2 *Let X be an ordered Banach space,whose positive cone K is normal, $A : D(A) \subset X \to X$ be a closed linear operator and $-A$ generate a positive C_0-semigroup $T(t)(t \geq 0)$ be continuous in operator norm for $t > 0$ in X. $f \in C(J \times X, X)$ and f is $\omega-$periodic about t, and $I_k \in C(X, X), k = 1, 2, \cdots, p$. Assume that the PBVP (1.2) has coupled lower and upper L-quasisolutions v_0 and w_0 with $v_0(t) \leq w_0(t)(t \in J)$, the nonlinear term f and impulsive functions I_k satisfy the following assumptions:*

(H_3) $\exists 0 < L$, for $t \in J$ and any equicontinuous sequence $G = \{u_n\} \subset [v_0, w_0]$, such that

$$\alpha\big(f(t, G(t)) + CG(t)\big) \leq L\alpha\big(G(t)\big).$$

(H_4) $\exists 0 < M_k$, for $t \in J$ and any equicontinuous sequence $G = \{u_n\} \subset [v_0, w_0]$, such that

$$\alpha\big(I_k(G(t))\big) \leq M_k\alpha\big(G(t)\big).$$

(H_5) $\left[\frac{2L}{C-\nu_0} + \frac{\sum\limits_{k=1}^{p} M_k}{1-e^{-(C-\nu_0)\omega}}\right] M < 1.$

(H_1) and (H_2), where $C > \nu_0$.

Then the PBVP (1.2) has minimal and maximal coupled mild ω -periodic L-quasisolutions $\overline{u}$ and $\underline{u}$ between v_0 and w_0, and at least has one mild ω -periodic solution between $\overline{u}$ and $\underline{u}$.

The proof of Theorem 1.1 will be shown in the next section. In Section 2, we also discuss the existence of mild ω-periodic solutions for the PBVP (1.2) between coupled lower and upper L-quasisolutions (see Theorem 2,Theorem 3).

2 Preliminaries

Let X be an ordered Banach space with the norm $\|\cdot\|$ and partial order $\leq$, whose positive cone $K = x \in X | x \geq \theta$ is normal with normal constant N . We use X_1 to denote the Banach space $D(A)$ with the graph norm $\|\cdot\|_1 = \|\cdot\| + \|A\cdot\|$. Let $C(J, X)$ denote the Banach space of all continuous X -value functions on interval J with the norm $\|u\|_C = \max\limits_{t\in J} \|u(t)\|$. Then $C(J, X)$ is an ordered Banach space reduced by the convex cone $K_C = \{u \in C(J, X) \mid u(t) \geq 0,\ t \in J\}$, and K_C is also a normal cone. Let $\alpha(\cdot)$ denote the Kuratowski measure of noncompactness of the bounded set. For the details of the definition and properties of the measure of noncompactness, see [16]. For any $B \in C(J, X)$ and $t \in J$, we set $B(t) = u(t)|u \in B \subset X$. If B is bounded in $C(J, X)$, then $B(t)$ is bounded in X, and $\alpha(B(t)) \leq \alpha(B)$.

The following lemmas are needed in our argument.

Lemma 2.1 *([17]) Let $B \subset C(J, X)$ be bounded and equicontinuous.Then $\alpha(B(t))$ is continuous on J , and $\alpha(B) = \max\limits_{t\in J} \alpha\big(B(t)\big) = \alpha(B(J))$.*

Lemma 2.2 *([19]) Let $B = u_n \subset PC(J, X)$ be a bounded and countable set. Then $\alpha(B(t))$ is Lebesgue integral on J, and*

$$\alpha\bigg(\bigg\{\int_J u_n(t)dt | n \in \mathbb{N}\bigg\}\bigg) \leq 2\int_J \alpha\big(B(t)\big)dt.$$

Lemma 2.3 *([20]) Let $T : X \to X$ is a linear and bounded operator, and $D \subset X$ is a bounded set. Then*

$$\alpha\big(T(D)\big) \leq \|T\| \cdot \alpha(D).$$

Let X be a Banach space, $A : D(A) \subset X \to X$ be a closed linear operator and $-A$ generate a C_0-semigroup $T(t)(t \geq 0)$ in X. Then there exist constants $M > 0$ and $\nu \in \mathbf{R}$, such that

$$\|T(t)\| \leq Me^{\nu t}, t \geq 0, \tag{2.3}$$

$$\nu_0 = \inf\big\{\nu \in \mathbb{R} \mid \exists M > 0, \|T(t)\| \leq Me^{\nu t}, \forall t \geq 0\big\}$$

is called growth index of C_0-semigroup $T(t)(t \geq 0)$.

Definition 2.1 *A C_0-semigroup $T(t)(t \geq 0)$ is said to be exponentially stable in X if there exist constants $M \geq 1$ and $\nu > 0$ such that*

$$\|T(t)\| \leq Me^{-\nu t}, t \geq 0. \tag{2.4}$$

It is easy to see that for any $C > \nu_0$ and $-(A + CI)$, they also generates a exponentially stable C_0-semigroup $S(t) = e^{-Ct}T(t)(t \geq 0)$ in X, growth index of $S(t)$ is $-C + \nu_0$. It is well-known that for $\forall \nu \in (0, C - \nu_0)$, there exists $M > 0$ and $S(t)$ satisfies

$$\|S(t)\| \leq Me^{-\nu t}, \qquad t \geq 0. \tag{2.4}$$

We define an equivalent norm in X by

$$|x| = \sup_{t \geq 0} \|e^{\nu t} S(t)x\|, \tag{2.5}$$

then $\|x\| \leq |x| \leq M\|x\|$.

Lemma 2.4 *([18]) Let $C > \nu_0$, exponentially stable C_0-semigroup $S(t) = e^{-Ct}T(t)$ be generated by $-(A + CI)$ satisfies inequality:*

$$|(I - S(t))^{-1}| \leq \frac{1}{1 - e^{-(C-\nu_0)\omega}}.$$

Definition 2.2 *Let $C \geq 0$ be a constant. If functions $v_0, w_0 \in PC(J, X) \cap C^1(J', X) \cap C(J', X_1)$ satisfy*

$$\begin{cases} v_0'(t) + Av_0(t) \leq f(t, v_0(t)) + Cv_0(t), \qquad t \in J, \qquad t \neq t_k, \\ \Delta v_0 \mid_{t=t_k} \leq I_k(v_0(t_k)), \quad k = 1, 2, \cdots, p, \\ v_0(0) \leq v_0(\omega), \end{cases} \tag{2.6}$$

and

$$\begin{cases} w_0'(t) + Aw_0(t) \geq f(t, w_0(t)) + Cw_0(t), \qquad t \in J, \qquad t \neq t_k, \\ \Delta w_0 \mid_{t=t_k} \geq I_k(w_0(t_k)), \quad k = 1, 2, \cdots, p, \\ w_0(0) \geq w_0(\omega), \end{cases} \tag{2.7}$$

we call v_0 and w_0 is the coupled lower and upper L-quasisolutions of the PBVP(1.1). Only choosing " = " in (2.6) and (2.7), we call v_0 and w_0 is the coupled ω-periodic and L-quasisolution pair of the PBVP(1.1). Furthermore, if $v_0 = w_0 = u_0$, we call u_0 an ω -periodic solution of the PBVP(1.1).

Let $I_0 = [t_0, T]$. Denote by $C(I_0, X)$ the Banach space of all continuous X-value functions on interval I_0 with the norm $\|u\|_C = \max_{t \in I_0} \|u(t)\|$. It is well-known ([18], Chapter 4, Theorem 2.9)that for any $x_0 \in D(A)$ and $h \in C^1(I_0, X)$, the initial value problem(IVP) of linear evolution equation

$$\begin{cases} u'(t) + Au(t) = h(t), \qquad t \in I_0, \\ u(t_0) = x_0, \end{cases} \tag{2.8}$$

has a unique classical solution $u \in C^1(I_0, X) \cap C(I_0, X_1)$ expressed by

$$u(t) = T(t - t_0)x_0 + \int_{t_0}^{t} T(t - s)h(s)\, \mathrm{d}s, t \in I_0 \tag{2.9}$$

If $x_0 \in X$ and $h \in C(I_0, X)$, the function u given by (2.9) belongs to $C(I_0, X)$. We call it a mild solution of the IVP(2.8). To prove Theorem 1, for any $h \in PC(J, X)$, we consider the periodic boundary value problem (PBVP) of linear impulsive evolution equation in X

$$\begin{cases} u'(t) + Au(t) = h(t), \quad t \in J, \quad t \neq t_k, \\ \Delta u \mid_{t=t_k} = a_k, \quad k = 1, 2, \cdots, p, \\ u(0) = u(\omega). \end{cases} \tag{2.10}$$

where $a_k \in X, k = 1, 2, \cdots, p$.

Lemma 2.5 *Let $T(t)(t \geq 0)$ be an exponentially stable C_0 -semigroup in X . Then for any $h \in PC(J,X)$ and $a_k \in X, k = 1,2,\cdots,p.$, the linear PBVP(2.10) has a unique mild solution $u \in PC(J,X)$ given by*

$$u(t) = T(t)B_1(h) + \int_0^t T(t-s)h(s)\,ds + \sum_{0<t_k<t} T(t-t_k)a_k, t \in J \tag{2.11}$$

where

$$B_1(h) = \big(I - T(\omega)\big)^{-1}\Big[\int_0^\omega T(\omega-s)h(s)\,ds + \sum_{0<t_k<\omega} T(\omega-t_k)a_k\Big].$$

Proof 2.1 *For any $h \in PC(J,X)$, we first show that the initial value problem (IVP) of linear impulsive evolution equation*

$$\begin{cases} u'(t) + Au(t) = h(t), & t \in J, \quad t \neq t_k, \\ \Delta u\mid_{t=t_k} = a_k, & k = 1,2,\cdots,p, \\ u(0) = x_0. \end{cases} \tag{2.12}$$

has a unique mild solution $u \in PC(J,X)$ given by

$$u(t) = T(t)x_0 + \int_0^t T(t-s)h(s)\,ds + \sum_{0<t_k<t} T(t-t_k)a_k, t \in J \tag{2.13}$$

where $x_0 \in X$ and $a_k \in X, k = 1,2,\cdots,p.$

Let $J_k = [t_k, t_{k+1}), k = 1,2,\cdots,p$. Let $y_0 = 0$. If $u \in PC(J,X)$ is a mild solution of the linear IVP(2.12), then the restriction of u on J_k satisfies the initial value problem (IVP) of linear evolution equation without impulse

$$\begin{cases} u'(t) + Au(t) = h(t), & t \in J_k, \\ u(t_k^{\ +}) = u(t_k) + a_k. \end{cases} \tag{2.14}$$

Hence, on $(t_k, t_{k+1}], u(t)$ can be expressed by

$$u(t) = T(t-t_k)(u(t_k) + a_k) + \int_{t_k}^t T(t-s)h(s)\,ds, \tag{2.15}$$

Iterating successively in the above equality with $u(t_j)$ for $j = k, k-1, ..., 1, 0$, we see that u satisfies (2.13).

Inversely, we can verify directly that the function $u \in PC(J,X)$ defined by (2.13) is a solution of the linear IVP(2.12). Hence the linear IVP(2.12) has a unique mild solution $u \in PC(J,X)$ given by (2.13).

Next,we show that the linear PBVP(2.10) has a unique mild solution $u \in PC(J,X)$ given by (2.11). If a function $u \in PC(J,X)$ defined by (2.13) is a solution of the linear PBVP(2.10), then $x_0 = u(\omega)$, namely,

$$(I - T(\omega))x_0 = \int_0^\omega T(\omega-s)h(s)\,ds + \sum_{0<t_k<\omega} T(\omega-t_k)a_k. \tag{2.15}$$

Since $T(t)(t \geq 0)$ is exponentially stable, we define an equivalent norm in X by

$$|x| = \sup_{t\geq 0} \|e^{\nu t}T(t)x\|. \tag{2.16}$$

Then $\|x\| \leq |x| \leq M\|x\|$ and $|T(t)| < e^{-\nu t}(t \geq 0)$, and especially, $|T(\omega)| < e^{-\nu\omega} < 1$. It follows that $I - T(\omega)$ has a bounded inverse operator $(I - T(\omega))^{-1}$, which is a positive operator when $T(t)(t \geq 0)$ is a positive semigroup. Hence we choose $x_0 = (I - T(\omega))^{-1}[\int_0^\omega T(\omega-s)h(s)\,ds + \sum\limits_{0<t_k<\omega} T(\omega-t_k)a_k] \triangleq B_1(h)$.

Then x_0 is the unique initial value of the IVP(2.12) in X , which satisfies $u(0) = x_0 = u(\omega)$. Combining this fact with (2.13), it follows that (2.11) is satisfied. Inversely, we can verify directly that the function $u \in PC(J,X)$ defined by (2.11) is a solution of the linear PBVP(2.10). Therefore, the conclusion of Lemma 3 holds.

3 Main results

Proof of Theorem 1: Let $C > \nu_0$, and $-(A+CI)$ generates an exponentially stable, compact and positive C_0-semigroup $S(t) = e^{-Ct}T(t)(t \geq 0)$ in X . Denote $D = [v_0, w_0]$. For $\forall h \in D$, we consider the periodic boundary value problem (PBVP) of linear impulsive evolution equation in X

$$\begin{cases} u'(t) + Au(t) + Cu(t) = f(t,\, h(t)) + Ch(t), \quad t \in J, \quad t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(h(t_k)), \quad k = 1, 2, \cdots, p, \\ u(0) = u(\omega). \end{cases} \tag{3.16}$$

where $\forall t \in [0, \omega]$, denote $f_1(t, x) = f(t, x) + Cx$. From Lemma 5, the PBVP(3.16)has a unique mild solution $u \in PC(J, X)$ given by:

$$u(t) = S(t)B_2(h) + \int_0^t S(t-s)f_1(s,\, h(s))\,\mathrm{d}s + \sum_{0<t_k<t} S(t-t_k)I_k(h(t_k)) \triangleq Q(h), \tag{3.17}$$

$$B_2(h) = (I - S(\omega))^{-1}[\int_0^\omega S(\omega-s)f_1(s,\, h(s))\,\mathrm{d}s + \sum_{k=1}^p S(\omega-t_k)I_k(h(t_k))].$$

Since f and I_k are continuous, so $Q : D \to PC(J, X)$ is continuous.

Clearly, the coupled mild ω-periodic L-quasi-solutions of the PBVP (3.2) is equivalent to the coupled fixed points of operator Q .

(i) We show $Q : D \to PC(J, X)$ is a monotone operator, $v_0 \leq Qv_0$, $\quad Qw_0 \leq w_0$.

In fact, for $\forall h_1,\, h_2 \in D$ and $h_1 \leq h_2$, from the assumptions (H_1) and (H_2), we have

$$f_1(t,\, h_1(t)) = f(t,\, h_1(t)) + Ch_1(t) \leq f(t,\, h_2(t)) + Ch_2(t) = f_1(t,\, h_2(t)), \quad t \in J.$$

$$I_k(h_1(t_k)) \leq I_k(h_2(t_k)), \quad k = 1, 2, \cdots, p.$$

Since $S(t)$ is an exponentially stable and positive C_0-semigroup, by Lemma 4,it follows that $(I-S(\omega))^{-1} = \sum\limits_{n=0}^{\infty} S(n\omega)$ is a positive operator. So, we have

$$\begin{aligned} &\big(I - S(\omega)\big)^{-1}\Big[\int_0^\omega S(\omega-s)f_1(s,\, h_1(s))\,\mathrm{d}s + \sum_{k=1}^p S(\omega-t_k)I_k(h_1(t_k))\Big] \\ \leq &\big(I - S(\omega)\big)^{-1}\Big[\int_0^\omega S(\omega-s)f_1(s,\, h_2(s))\,\mathrm{d}s + \sum_{k=1}^p S(\omega-t_k)I_k(h_2(t_k))\Big], \end{aligned}$$

i.e. $B_2(h_1) \leq B_2(h_2)$,then $S(t)B_2(h_1) \leq S(t)B_2(h_2)$. Hence from (3.17), we see that $Qh_1 \leq Qh_2$, which implies that Q is a monotone operator. Since

$$\begin{cases} \overline{h}(t) \triangleq v_0'(t) + Av_0(t) + Cv_0(t) \leq f(t, v_0(t)) + Cv_0(t), \ t \in J, \quad t \neq t_k, \\ g_k \triangleq \Delta v_0 \mid_{t=t_k} \leq I_k(v_0(t_k)), \quad k = 1, 2, \ \cdots, p. \end{cases} \tag{3.18}$$

from Lemma 5 and (2.6), we have

$$\begin{aligned} v_0(t) = &S(t)B_3(\overline{h}) + \int_0^t S(t-s)\overline{h}(s)\,\mathrm{d}s + \sum_{0<t_k<t} S(t-t_k)g_k \\ \leq &S(t)B_3(\overline{h}) + \int_0^t S(t-s)f_1(s,\, v_0(s))\,\mathrm{d}s + \sum_{0<t_k<t} S(t-t_k)I_k(v_0(t_k)), \end{aligned} \tag{3.19}$$

$B_3(\overline{h}) = \big(I - S(\omega)\big)^{-1}\big[\int_0^\omega S(\omega-s)\overline{h}(s)\,\mathrm{d}s + \sum\limits_{k=1}^p S(\omega-t_k)g_k\big]$. for $t \in J$. Especially, we have

$$v_0(\omega) \leq S(\omega)B_3(\overline{h}) + \int_0^\omega S(\omega-s)f_1(s,\, v_0(s))\,\mathrm{d}s + \sum_{k=1}^p S(\omega-t_k)I_k(v_0(t_k)). \tag{3.20}$$

From (3.19) , we have $v_0(0) = B_3(\overline{h})$. Since $v_0(0) \le v_0(\omega)$, from (3.20) , so

$$B_3(\overline{h}) \le \big(I - S(\omega)\big)^{-1}\Big[\int_0^{\omega} S(\omega - s) f_1(s,\, v_0(s))\,\mathrm{d}s + \sum_{k=1}^{p} S(\omega - t_k) I_k(v_0(t_k))\Big] = B_2(v_0).$$

On the other hand, from (3.17), we have

$$Qv_0 = S(t)B_2(v_0) + \int_0^t S(t-s) f_1(s,\, v_0(s))\,\mathrm{d}s + \sum_{0<t_k<t} S(t-t_k) I_k(v_0(t_k)). \tag{3.21}$$

Therefore, $Qv_0(t) - v_0(t) \ge S(t)(B_2(v_0) - B_3(\overline{h})) \ge \theta$. It implies that $v_0(t) \le Qv_0(t)$. Similarly, we can prove that $Qw_0(t) \le w_0(t)$.

(ii) Next, we will prove that the operator Q has coupled fixed points on $[v_0, w_0]$.

Now, we define sequences $\big\{v_n(t)\big\}$ and $\big\{w_n(t)\big\}$ by the iterative scheme

$$v_n = Qv_{n-1},\ \ w_n = Qw_{n-1}, \qquad n = 1, 2, \cdots \tag{3.22}$$

Then from the monotonicity of operator Q, we have

$$v_0 \le v_1 \le v_2 \le \cdots \le v_n \le \cdots \le w_n \le \cdots \le w_2 \le w_1 \le w_0. \tag{3.23}$$

Denote $G = \big\{v_n \mid n \in \mathbb{N}\big\},\ G_0 = \big\{v_{n-1} \mid n \in \mathbb{N}\big\}$, then $G_0 = \big\{v_0\big\} \cup G,\ G = Q(G_0)$.

For $\forall v_{n-1} \in G_0$, let

$$W(v_{n-1})(t) = \int_0^t S(t-s) f_1(s,\, v_{n-1}(s))\,\mathrm{d}s + \sum_{0<t_k<t} S(t-t_k) I_k(v_{n-1}(t_k)). \tag{3.24}$$

then $Q(v_{n-1})(t) = S(t)B_2(v_{n-1}) + W(v_{n-1})(t)$. First, we will prove that for any $0 < t < \omega$, $Y(t) \triangleq \big\{W(v_{n-1})(t) \mid v_{n-1} \in G_0\big\}$ is relatively compact in X . Let $0 < \epsilon < t$ and

$$\begin{aligned} W_\epsilon(v_{n-1})(t) =& \int_0^{t-\epsilon} S(t-s) f_1(s,\, v_{n-1}(s))\,\mathrm{d}s + \sum_{0<t_k<t-\epsilon} S(t-t_k) I_k(v_{n-1}(t_k)) \\ =& S(\epsilon)\Big[\int_0^{t-\epsilon} S(t-\epsilon-s) f_1(s,\, v_{n-1}(s))\,\mathrm{d}s + \sum_{0<t_k<t-\epsilon} S(t-\epsilon-t_k) I_k(v_{n-1}(t_k))\Big]. \end{aligned} \tag{3.25}$$

from the conditions(H_1), we know that

$$f(t, v_0(t)) + Cv_0(t) \le f(t, v_{n-1}(t)) + Cv_{n-1}(t) \le f(t, w_0(t)) + Cw_0(t).$$

$$I_k(v_0(t_k)) \le I_k(v_{n-1}(t_k)) \le I_k(w_0(t_k)), \quad k = 1, 2, \cdots, p.$$

Since $f(t, v_0(t))$ and $f(t, w_0(t))$ are continuous in compact set $[0, \omega]$, so their image sets are compact sets in X, namely image sets are bounded. Combining this fact with the normality of cone K in X, we have $\exists M_1 > 0$, $\forall v_{n-1} \in G_0$ there exists $M_2 > 0$, such that

$$\|f_1(t, v_{n-1}(t))\| \le \|f_1(t, v_0(t))\| + N_0\|f_1(t, w_0(t)) - f_1(t, v_0(t))\| \le M_1.$$

and

$$\|I_k(v_{n-1}(t_k))\| \le \|I_k(v_0(t_k))\| + N_0\|I_k(w_0(t_k)) - I_k(v_0(t_k))\| \le M_2.$$

From (3.15), noticing that $S(\epsilon)$ is compact set, then $Y_\epsilon(t) = \big\{W_\epsilon(v_{n-1})(t) \mid v_{n-1} \in G_0\big\}$ is relatively compact in X. For sufficient small ϵ and $t, t-\epsilon \in J_k (k = 0, 1, 2, \cdots, p)$, then we have

$$\begin{aligned} \|W(v_{n-1})(t)) - W_\epsilon(v_{n-1})(t)\| =& \|\int_0^t S(t-s) f_1(s,\, v_{n-1}(s))\,\mathrm{d}s - \int_0^{t-\epsilon} S(t-s) f_1(s,\, v_{n-1}(s))\,\mathrm{d}s\| \\ \le& \int_{t-\epsilon}^t \|S(t-s)\|\|f_1(s,\, v_{n-1}(s))\|\,\mathrm{d}s \le MM_1\epsilon, \end{aligned}$$

Hence $Y(t)$ is totally bounded in X, thus it is relatively compact. Especially, $Y(\omega)$ is compact in X, and then $\{S(t)B_2(v_{n-1}) \mid v_{n-1} \in G_0\}$ is relatively compact.

Noticing that

$$\{Q(t) \mid v_{n-1} \in G_0\} = \{S(t)B_2(v_{n-1}) + W(v_{n-1})(t) \mid v_{n-1} \in G_0\},$$

and $Q(v_{n-1})(0) = B_2(v_{n-1})$, we consider

$$\begin{aligned}
&Q(v_{n-1})(\omega)\\
=&\big[S(\omega)(I - S(\omega))^{-1} + I\big]\Big[\int_0^\omega S(\omega - s)f_1(s,\, v_{n-1}(s))\,\mathrm{d}s + \sum_{k=1}^{p} S(\omega - t_k)I_k(v_{n-1}(t_k))\Big]\\
=&B_2(v_{n-1}),
\end{aligned}$$

then $Q(v_{n-1})(0) = Q(v_{n-1})(\omega) = B_2(v_{n-1})$, namely $\{Q(v_{n-1})(0) \mid v_{n-1} \in G_0\} = B_4(G_0)$ is relatively compact.

Therefor $\{v_n(t)\} = \{Q(v_{n-1})(t) \mid v_{n-1} \in G_0,\, t \in J\}$ is relatively compact in X, combining this fact with the monotonicity of $\{v_n\}$, we easily prove that $\{v_n(t)\}$ is convergent. Set$\{v_n(t)\} \to \underline{u}(t)$ in $t \in J$. The same idea can be used to prove that $\{w_n(t)\} \to \overline{u}(t)$ in $t \in J$.

Evidently $\{v_n(t)\}, \{w_n(t)\} \in PC(J, X)$, so $\underline{u}(t)$ and $\overline{u}(t)$ is bounded integrable in $J_k (k = 1, 2, \cdots, p)$. Since for any $t \in J_k$, $v_n(t) = Q(v_{n-1})(t)$, $w_n(t) = Q(w_{n-1})(t)$, letting $n \to \infty$, by the Lebesgue dominated convergence theorem, we have $\underline{u}(t) = Q(\underline{u})(t)$, and $\overline{u}(t) = Q(\overline{u})(t)$, and $\underline{u}(t), \overline{u}(t) \in PC(J, X)$. Combining this with monotonicity (3.23), we have $v_0(t) \le \underline{u}(t) \le \overline{u}(t) \le w_0(t)$. By the monotonicity of Q, it is easy to see that $\underline{u}(t)$ and $\overline{u}(t)$ are the minimal and maximal coupled fixed points of Q in $[v_0, w_0]$, and therefore, they are the minimal and maximal coupled mild ω -periodic L-quasisolutions of the PBVP (1.2) in in $[v_0, w_0]$, respectively.

Now, we discuss the existence of mild ω -periodic L-quasisolutions of the PBVP (1.2) in in $[v_0, w_0]$ .We assume that the noncompactness measure conditions are satisfied. Then we have the following existence result in general ordered Banach space.

Proof of Theorem 2 Denote $D = [v_0, w_0]$. From Theorem 1, we know that $Q : D \to D$ is a continuous monotone operator. Now, we define two sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ in $[v_0, w_0]$ by the iterative scheme

$$v_n = Qv_{n-1},\ w_n = Qw_{n-1}, \qquad n = 1, 2, \cdots$$

Then from the mixed monotonicity of Q, it follows that

$$v_0 \le v_1 \le v_2 \le \cdots \le v_n \le \cdots \le w_n \le \cdots \le w_2 \le w_1 \le w_0.$$

We prove that $\{v_n(t)\}$ and $\{w_n(t)\}$ are convergent in J. For convenience, let $G = \{v_n \mid n \in \mathbb{N}\}$, $G_0 = \{v_{n-1} \mid n \in \mathbb{N}\}$, then $G_0 = \{v_0\} \cup G$, $G = Q(G_0)$. For $\forall t', t'' \in J_k (k = 0, 1, 2, \cdots, p)$, and $t' < t''$, we have

$$\begin{aligned}
&\|Q(v_{n-1})(t'') - Q(v_{n-1})(t')\|\\
\le&\|S(t'')B_2(v_{n-1}) + \int_0^{t''} S(t'' - s)f_1(s,\, v_{n-1}(s))\,\mathrm{d}s + \sum_{0<t_k<t''} S(t'' - s)I_k(v_{n-1}(t_k))\\
&- S(t')B_2(v_{n-1}) - \int_0^{t'} S(t' - s)f_1(s,\, v_{n-1}(s))\,\mathrm{d}s - \sum_{0<t_k<t'} S(t' - s)I_k(v_{n-1}(t_k))\|\\
\le&\|S(t'') - S(t')\|\|B_2(v_{n-1})\| + \int_0^{t'} \|S(t'' - s) - S(t')\|\|f_1(s,\, v_{n-1}(s))\|\,\mathrm{d}s + MM_1(t'' - t')\\
&+ \sum_{0<t_k<t'} \|S(t' - s) - S(t')\|\|I_k(v_{n-1}(t_k))\|\\
\le&\|S(t'') - S(t')\|\|B_2(v_{n-1})\| + M_1 \int_0^{t'} \|S(t'' - s) - S(t')\|\,\mathrm{d}s + MM_1(t'' - t')\\
&+ M_2 \sum_{0<t_k<t'} \|S(t' - s) - S(t')\| \to 0 (t'' - t' \to 0),
\end{aligned}$$

then $Q(G_0)$ is equicontinuous in any $J_k(k=0,1,2,\cdots,p)$, and then G is an equicontinous and bounded set in J_k . Combining this by the property of noncompactness, $\alpha(G)=\sup\limits_{t\in J}\alpha\big(G(t)\big)$. From Lemma 4, we obtain $\|x\|<|x|<M\|x\|$. For convenience, Let $\alpha_1(\cdot)$ denote the Kuratowski measure of noncompactness of the space $(X,|\cdot|)$ then $\alpha(G)\le\alpha_1(G)\le M\alpha(G)$.

Combining Lemma 2 and Lemma 3 by the condition (H$_3$) and condition (H$_4$), we have

$$
\begin{aligned}
&\alpha\big(G(t)\big)\le\alpha_1\big(G(t)\big)=\alpha_1\big(Q(G_0(t))\big)\\
=&\alpha_1\Big\{S(t)B_2(v_{n-1})+\int_0^t S(t-s)f_1(s,\,v_{n-1}(s))\,\mathrm{d}s+\sum_{0<t_k<t}S(t-t_k)I_k(v_{n-1}(t_k))\mid n\in\mathbb{N}\Big\}\\
\le&\alpha_1\big(S(t)B_2(G_0)\big)+\alpha_1\bigg(\int_0^t S(t-s)f_1(s,\,G_0(s))\,\mathrm{d}s\bigg)+\alpha_1\bigg(\sum_{0<t_k<t}S(t-t_k)I_k(G_0(t_k))\bigg)\\
\le&|S(t)||(I-S(\omega))^{-1})|\bigg[2\int_0^\omega|S(\omega-s)|\alpha_1\big(f_1(s,\,G_0(s))\big)\,\mathrm{d}s+\sum_{k=1}^{p}|S(\omega-t_k)|\alpha_1\big(I_k(G_0(t_k))\big)\bigg]\\
&+2\int_0^t|S(t-s)|\alpha_1\big(f_1(s,\,G(s))\big)\,\mathrm{d}s+\sum_{0<t_k<t}|S(t-t_k)|\alpha_1\big(I_k(G_0(t_k))\big)
\end{aligned}
$$

and

$$
\begin{aligned}
\le&\frac{e^{-(C-\nu_0)t}}{1-e^{-(C-\nu_0)\omega}}\bigg[2\int_0^\omega e^{-(C-\nu_0)(\omega-s)}ML\alpha\big(G_0(s)\big)\,\mathrm{d}s+\sum_{k=1}^{p}e^{-(C-\nu_0)(\omega-t_k)}MM_k\alpha\big(G_0(t_k)\big)\bigg]\\
&+2\int_0^t e^{-(C-\nu_0)(t-s)}ML\alpha\big(G_0(s)\big)\,\mathrm{d}s+\sum_{0<t_k<t}e^{-(C-\nu_0)(t-t_k)}MM_k\alpha\big(G_0(t_k)\big)\\
\le&\frac{e^{-(C-\nu_0)t}}{1-e^{-(C-\nu_0)\omega}}\bigg[\frac{2ML}{C-\nu_0}(1-e^{-(C-\nu_0)\omega})+\sum_{k=1}^{p}e^{-(C-\nu_0)(\omega-t_k)}MM_k\bigg]\cdot\sup_{t\in J}\alpha\big(G(t)\big)\\
&+\bigg[\frac{2ML}{C-\nu_0}(1-e^{-(C-\nu_0)t})+\sum_{k=1}^{p}e^{-(C-\nu_0)(t-t_k)}MM_k\bigg]\cdot\sup_{t\in J}\alpha\big(G(t)\big)\\
=&\Bigg[\frac{2ML}{C-\nu_0}+\frac{\sum\limits_{k=1}^{p}MM_ke^{-(C-\nu_0)(t-t_k)}}{1-e^{-(C-\nu_0)\omega}}\Bigg]\cdot\sup_{t\in J}\alpha\big(G(t)\big)\\
<&\Bigg(\frac{2L}{C-\nu_0}+\frac{\sum\limits_{k=1}^{p}M_k}{1-e^{-(C-\nu_0)\omega}}\Bigg)M\cdot\sup_{t\in J}\alpha\big(G(t)\big).
\end{aligned}
$$

Using assumption (H$_5$), then $\alpha\big(G(t)\big)=0$ in $t\in J$. Hence $\big\{v_n\big\}$ is relatively compact in $PC(J,X)$. Combining this fact with the monotonicity of $\big\{v_n\big\}$, we easily prove that $\big\{v_n(t)\big\}$ is convergent. Set$\big\{v_n(t)\big\}\to\underline{u}(t)$ in $t\in J$. The same idea can be used to prove that $\big\{w_n(t)\big\}\to\overline{u}(t)$ in $t\in J$. Similarly, in general, $v_n\le u\le w_n$, letting $n\to\infty$, we get $\underline{u}(t)\le u(t)\le\overline{u}(t)$. Therefore, the PBVP (1.2) at least has one mild ω -periodic solution between $\overline{u}$ and $\underline{u}$.

Remark 3.1 *If the semigroup $T(t)$ is continuous in operator norm for $t>0$, then $T(t)$ is called equicontinuous semigroup. Analytic semigroup and differentiable semigroup are equicontinuous semigroup ([18]). In the application of partial differential equations, such as parabolic and strongly damped wave equations, the corresponding solution semigroup are analytic semigroup. So, Theorem 2 in this paper has extensive applicability.*

Theorem 3.1 *Let X be an ordered Banach space,whose positive cone K is regular , $A:D(A)\subset X\to X$ be a closed linear operator and $-A$ generate a positive C_0-semigroup $T(t)(t\ge0)$. $f\in C(J\times X,X)$ and f is $\omega-$ periodic about t, $I_k\in C(X,X),k=1,2,\cdots,p$. Assume that the PBVP (1.2) has coupled*

lower and upper L-quasisolutions v_0 and w_0 with $v_0(t) \leq w_0(t)\,(t \in J)$, nonlinear term f and impulsive functions I_k satisfy the assumptions (H_1) and (H_2) . then the PBVP (1.2) at least has one mild ω-periodic solution between v_0 and w_0 .

Proof 3.1 *Denote $D = [v_0, w_0]$. From Theorem 1, we know that $Q : D \to D$ is a continuous monotone operator. Now, we define two sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ in $[v_0, w_0]$ by the iterative scheme*

$$v_n = Qv_{n-1},\ w_n = Qw_{n-1}, \qquad n = 1, 2, \cdots$$

Then from the mixed monotonicity of Q, it follows that

$$v_0 \leq v_1 \leq v_2 \leq \cdots \leq v_n \leq \cdots \leq w_n \leq \cdots \leq w_2 \leq w_1 \leq w_0.$$

Using the regularity of the cone K ,then $\exists v^*(t),\, w^*(t),\, v_n(t) \to v^*(t),\, w_n(t) \to w^*(t)$ in $t \in J'$,where $v^*(t),\, w^*(t)$ are bounded and strongly measurable, combining the definition of Q with (3.12), we get

$$v_n(t) = Q(v_{n-1}(t)) = S(t)B_2(v_{n-1}) + \int_0^t S(t-s)f_1(s,\, v_{n-1}(s))\, ds + \sum_{0<t_k<t} S(t-t_k)I_k(v_{n-1}(t_k)).$$

Noticing $S(t) = e^{-Ct}T(t)(t \geq 0)$ is an exponentially stable and positive C_0-semigroup in X, letting $n \to \infty$, by the Lebesgue dominated convergence theorem, we have

$$v^*(t) = Q(v^*(t)) = S(t)B_2(v^*) + \int_0^t S(t-s)f_1(s,\, v^*(s))\, ds + \sum_{0<t_k<t} S(t-t_k)I_k(v^*(t_k)),$$

and $v^*(t) = Qv^*(t) \in PC(J, X)$.

Similarly, we prove that $w^*(t) \in PC(J, X)$ and $w^*(t) = Qw^*(t)$.

By (3.23), we know $v_0(t) \leq v^*(t) \leq w^*(t) \leq w_0(t)$.

If $u(t)$ is also the fixed point of Q , evidently

$$v_0(t) \leq u(t) \leq w_0(t) \tag{3.26}$$

is satisfied.From (3.22) ,by repeating the function of the operator Q on the above inequality (3.26) , we can obtain that $v_n(t) \leq u(t) \leq w_n(t)$. Letting $n \to \infty$, we get $v^*(t) \leq u(t) \leq w^*(t)$. Therefore, the PBVP (1.2) at least has one mild ω-periodic solution between v^* and w^*.

Remark 3.2 *If $I_k \equiv 0$, then Theorem 3 in this paper is Theorem 3.2 in ([12]).*

Remark 3.3 *In Theorem 3, we do not assume that the semigroup $T(t)$ is continuous in operator norm for $t > 0$,we only demand that the cone K is regular. So, if $I_k \equiv 0$, then Theorem 3 in this paper extensively generalizes the main results in ([12]).*

In Theorem 3, if X is weakly sequentially complete, it is well know that the cone K is regular. From Theorem 3, we obtain the following corollary.

Corollary 3.1 *Let X be an ordered and weakly sequentially complete Banach space, whose positive cone K is normal , $A : D(A) \subset X \to X$ be a closed linear operator and $-A$ generate a positive C_0- semigroup $T(t)(t \geq 0)$. $f \in C(J \times X, X)$ and f is $\omega-$ periodic about t , $I_k \in C(X, X), k = 1, 2, \ \cdots, p$ .If the PBVP (1.2) has coupled lower and upper L-quasisolutions v_0 and w_0 with $v_0(t) \leq w_0(t)\,(t \in J)$,nonlinear term f and impulsive functions I_k satisfy the assumptions (H_1) and (H_2) . then the PBVP (1.2) at least has one mild ω -periodic solution between v_0 and w_0.*

Remark 3.4 *If $f(t, u, u) = f(t, u)$, then Corollary 1 in this paper is the main result in ([21]).*

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ISOMETRIES ON PRODUCTS OF COMPOSITION AND INTEGRAL OPERATORS FROM H_α^∞ TO BLOCH TYPE SPACE

GENG-LEI LI

Abstract. In this paper, we characterize the isometries on the products of composition operators and integral operators from the space of all weighted bounded analytic functions to Bloch type space in the disk.

1. Introduction

Let $\mathbb{D}$ be the unit disk of the complex plane, and $S(\mathbb{D})$ be the set of analytic self-maps of $\mathbb{D}$. The algebra of all holomorphic functions with domain $\mathbb{D}$ will be denoted by $H(\mathbb{D})$.

For $0<\alpha<\infty$, by $H_\alpha^\infty(\mathbb{D})$ denote the space of all weighted bounded analytic functions on the unit disk with the norm $\|f\|_\alpha=\sup\limits_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f(z)|$.

We recall that the Bloch type space $\mathcal{B}^\beta$ $(\beta>0)$ consists of all $f\in H(\mathbb{D})$ such that

$$\mathcal{B}_\beta(f)=\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta\,|f'(z)|<\infty,$$

then $\mathcal{B}_\beta(f)$ defines a complete semi-norm on $\mathcal{B}^\beta$, which is Möbius invariant.

It is well known that $\mathcal{B}^\beta$ is a Banach space under the norm

$$\|f\|_{\mathcal{B}^\beta}=|f(0)|+\mathcal{B}_\beta(f).$$

Let φ be an analytic self-map of $\mathbb{D}$, the composition operator C_φ induced by φ is defined by

$$(C_\varphi f)(z)=f(\varphi(z))$$

for $z\in\mathbb{D}$ and $f\in H(\mathbb{D})$.

Let $g\in H(D)$, the integral type operator J_g is defined by

$$J_gf(z)=\int_0^z f(\xi)g'(\xi)d\xi$$

for $z\in\mathbb{D}$ and $f\in H(\mathbb{D})$.

The products of composition and integral type operators are defined by

$$(C_\varphi J_gf)(z)=\int_0^{\varphi(z)} f(\xi)g'(\xi)d\xi$$

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and

$$(J_g C_\varphi f)(z) = \int_0^z (f \circ \varphi)(\xi) g'(\xi) d\xi.$$

The boundedness and compactness of the products were discussed by Li and Stević [13, 14, 15].

Let X and Y be two Banach spaces, recall that a linear isometry is a linear operator T from X to Y such that $\|Tf\|_Y = \|f\|_X$ for all $f \in X$.

In [3], Banach raised the question about concerning the form of an isometry on a specific Banach space. In most cases the isometries of a space of analytic functions on the disk or the ball have the canonical form of weighted composition operators, which is also true for most symmetric function spaces. For example, the surjective isometries of Hardy and Bergman spaces are certain weighted composition operators. See [9, 10, 11].

The description of all isometric composition operators is known for the Hardy space H^2 (see [8]). An analogous statement for the Bergman space A^2_α with standard radial weights has recently been obtained in [7], and there is a unified proof for all Hardy spaces and also for arbitrary Bergman spaces with reasonable radial weights [18]. For the Dirichlet space and Bloch space, the reader is referred to [19], [17], and for the BMOA, to see [12].

Continued the work, in 2008, Bonet, Lindström and Wolf [4] discussed isometric weighted composition operators on weighted Banach spaces of type H^∞. In 2008, Cohen and Colonna [6] discussed the spectrum of an isometric composition operators on the Bloch space of the polydisk. In 2009, Allen and Colonna [1] investigated the isometric composition operators on the Bloch space in $\mathcal{C}^n$. They [2] also discussed the isometries and spectra of multiplication operators on the Bloch space in the disk. Isometries of weighted spaces of holomorphic functions on unbounded domains were discussed by Boyd and Rueda in [5]. In 2013, Li and Zhou discussed the Isometries of composition and Differentiation operators from Bloch type space to H^∞_α in [16].

Building on those foundation, the paper continues this line of research, and discusses the isometries on the products of composition operators and integral operators from the space of all weighted bounded analytic functions to Bloch type space in the disk.

2. Notation and Lemmas

To begin the discussion, let us introduce some notation and state a couple of lemmas.

For $a \in \mathbb{D}$, the involution φ_a which interchanges the origin and point a, is defined by

$$\varphi_a(z) = \frac{a - z}{1 - \overline{a}z}.$$

For z, w in $\mathbb{D}$, the pseudo-hyperbolic distance between z and w is given by

$$\rho(z, w) = |\varphi_z(w)| = \left| \frac{z - w}{1 - \overline{z}w} \right|,$$

and the hyperbolic metric is given by

$$\beta(z, w) = \inf_\gamma \int_\gamma \frac{|d\xi|}{1 - |\xi|^2} = \frac{1}{2} \log \frac{1 + \rho(z, w)}{1 - \rho(z, w)},$$

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where γ is any piecewise smooth curve in $\mathbb{D}$ from z to w.

The following lemma is well known [21].

Lemma 1. *For all $z, w \in \mathbb{D}$, we have*

$$1-\rho^2(z,w)=\frac{(1-|z|^2)(1-|w|^2)}{|1-\overline{z}w|^2}.$$

For $\varphi \in S(D)$, the Schwarz-Pick lemma shows that $\rho(\varphi(z),\varphi(w)) \le \rho(z,w)$, and if equality holds for some $z \neq w$, then φ is an automorphism of the disk. It is also well known that for $\varphi \in S(\mathbb{D})$, C_φ is always bounded on $\mathcal{B}$.

Lemma 2. *For $0<\alpha<\infty$, there exists a constant $C>0$ such that*

$$\left|(1-|z|^2)^\alpha f(z)-(1-|w|^2)^\alpha f(w)\right| \le C\,\|f\|_\alpha \cdot \rho(z,w)$$

for all $z, w \in \mathbb{D}$ and $f \in H^\infty_\alpha$.

Proof. An exercise in [21] shows that:

$$\left|(1-|z|^2)^\alpha f(z)-(1-|w|^2)^\alpha f(w)\right| \le C||f||_\alpha \cdot \beta(z,w).$$

If $\rho(z,w)<\frac{1}{2}$, note that for $0\le x<\frac{1}{2}$, $\ln\frac{1+x}{1-x}\le 3x$, the lemma is true. If $\rho(z,w)\ge\frac{1}{2}$, then we have

$$\left|(1-|z|^2)^\alpha f(z)-(1-|w|^2)^\alpha f(w)\right| \le 2||f||_\alpha \le 4||f||_\alpha \cdot \rho(z,w).$$

The lemma follows by combining the two cases above. $\square$

Throughout the remainder of this paper, C will denote a positive constant, the exact value of which will vary from one appearance to the next.

3. Main theorems

Theorem 1. *Let $g \in H(\mathbb{D})$ and φ be analytic self maps of the unit disk such that φ fixes the origin. Then the operator $C_\varphi J_g : H^\infty_\alpha \to \mathcal{B}^\beta$ is an isometry if and only if the following conditions hold:*

(I) $\sup\limits_{z\in D}\frac{(1-|z|^2)^\beta|\varphi'(z)|}{(1-|\varphi(z)|^2)^\alpha}|g'(\varphi(z))|\le 1$;

(II) *For every $a\in\mathbb{D}$, there at least exists a sequence $\{z_n\}$ in $\mathbb{D}$, such that* $\lim\limits_{n\to\infty}\rho(\varphi(z_n),a)=0$ *and* $\lim\limits_{n\to\infty}\frac{(1-|z_n|^2)^\beta|\varphi'(z_n)|}{(1-|\varphi(z_n)|^2)^\alpha}|g'(\varphi(z_n))|=1.$

Proof. We prove the sufficiency first.

By condition (I), for every $f\in H^\infty_\alpha(\mathbb{D})$, we have

$$\begin{aligned}||C_\varphi J_g f||_{\mathcal{B}^\beta} &= \sup_{z\in D}(1-|z|^2)^\beta\,|g'(\varphi(z))|\,|\varphi'(z)|\,|f(\varphi(z))|\\ &= \sup_{z\in D}\frac{(1-|z|^2)^\beta\,|\varphi'(z)|}{(1-|\varphi(z)|^2)^\alpha}\,|g'(\varphi(z))|\,(1-|\varphi(z)|^2)^\alpha\,|f(\varphi(z))|\\ &\le \|f\|_\alpha.\end{aligned}$$

Next we show that the property (II) implies $||C_\varphi J_g f||_{\mathcal{B}^\beta}\ge||f||_\alpha$.

Given any $f\in H^\infty_\alpha(\mathbb{D})$, then $||f||_\alpha=\lim\limits_{m\to\infty}(1-|a_m|^2)^\alpha|f(a_m)|$ for some sequence $\{a_m\}\subset D$. For any fixed m, by property (II), there is a sequence $\{z_k^m\}\subset\mathbb{D}$ such that

$$\rho(\varphi(z_k^m),a_m)\to 0 \ \text{ and } \ \frac{(1-|z_k^m|^2)^\beta\,|\varphi'(z_k^m)|}{(1-|\varphi(z_k^m)|^2)^\alpha}\,|g'(\varphi(z_k^m))|\to 1$$

as $k \to \infty$. By Lemma 2, for all m and k,

$$\left|(1-|\varphi(z_k^m)|^2)^\alpha f(\varphi(z_k^m)) - (1-|a_m|^2)^\alpha f(a_m)\right| \le C||f||_\alpha \cdot \rho(\varphi(z_k^m), a_m).$$

Consequently,

$$(1-|\varphi(z_k^m)|^2)^\alpha \, |f(\varphi(z_k^m))| \ge (1-|a_m|^2)^\alpha |f(a_m)| - C||f||_\alpha \cdot \rho(\varphi(z_k^m, a_m).$$

Therefore,

$$\begin{aligned}
||C_\varphi J_g f||_{\mathcal{B}^\beta} &= \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta \, |\varphi'(z)|}{(1-|\varphi(z)|^2)^\alpha} \, |g'(\varphi(z))| \, (1-|\varphi(z)|^2)^\alpha \, |f(\varphi(z))| \\
&\ge \limsup_{k\to\infty} \frac{(1-|z_k^m|^2)^\beta \, |\varphi'(z_k^m)|}{(1-|\varphi(z_k^m)|^2)^\alpha} \, |g'(\varphi(z_k^m))| \, (1-|\varphi(z_k^m)|^2)^\alpha \, |f(\varphi(z_k^m))| \\
&= (1-|a_m|^2)^\alpha |f(a_m)|
\end{aligned}$$

The inequality $||C_\varphi J_g f||_{\mathcal{B}^\beta} \ge ||f||_\alpha$ follows by letting $m \to \infty$.

From the above discussions, we have $\|C_\varphi J_g f\|_{\mathcal{B}^\beta} = \|f\|_\alpha^\infty$, which means that $C_\varphi J_g$ is an isometry operator from H_α^∞ to $\mathcal{B}^\beta$.

Now we turn to the necessity.

For any $a \in \mathbb{D}$, we take the test function

$$f_a(z) = \frac{(1-|a|^2)^\alpha}{(1-\bar{a}z)^{2\alpha}}. \tag{1}$$

It follows from Lemma 1 that

$$||f_a||_\alpha = \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |f_a(z)| \le \sup_{z\in\mathbb{D}} \left(1-\rho^2(a,z)\right)^\alpha \le 1. \tag{2}$$

Since $(1-|a|^2)^\alpha |f_a(a)| = 1$, we have $||f_a||_\alpha = 1$. By isometry assumption, for any $a \in \mathbb{D}$, we have

$$\begin{aligned}
1 &= ||f_{\varphi(a)}||_\alpha = ||C_\varphi J_g f_{\varphi(a)}||_{\mathcal{B}^\beta} \\
&= \sup_{z\in D} \frac{(1-|z|^2)^\beta \, |\varphi'(z)|}{(1-|\varphi(z)|^2)^\alpha} \, |g'(\varphi(z))| \, (1-|\varphi(z)|^2)^\alpha \left|f_{\varphi(a)}(\varphi(z))\right| \\
&\ge \frac{(1-|a|^2)^\beta \, |\varphi'(a)|}{(1-|\varphi(a)|^2)^\alpha} \, |g'(\varphi(a))|
\end{aligned}$$

So (I) follows by noticing a is arbitrary.

Since $||C_\varphi J_g f_a||_{\mathcal{B}^\beta} = ||f_a||_\alpha = 1$, there exists a sequence $\{z_m\} \subset \mathbb{D}$ such that

$$(1-|z_m|^2)^\beta \left|\frac{d(C_\varphi J_g f_a)}{dz}(z_m)\right| = (1-|z_m|^2)^\beta |f_a(\varphi(z_m))||\varphi'(z_m)||g'(\varphi(z_m))| \to 1 \tag{3}$$

as $m \to \infty$.

It follows from (I) that

$$\begin{aligned}
&(1-|z_m|^2)^\beta |f_a(\varphi(z_m))||\varphi'(z_m)||g'(\varphi(z_m))| \\
= \;& \frac{(1-|z_m|^2)^\beta \, |\varphi'(z_m)|}{(1-|\varphi(z_m)|^2)^\alpha} |g'(\varphi(z_m))| (1-|\varphi(z_m)|^2)^\alpha \, |f_a(\varphi(z_m))| \qquad (4) \\
\le \;& (1-|\varphi(z_m)|^2)^\alpha \, |f_a(\varphi(z_m))| \, . \qquad (5)
\end{aligned}$$

Combining (3) and (5), it follows that

$$\begin{aligned} 1 &\leq \liminf_{m\to\infty}(1-|\varphi(z_m)|^2)^\alpha\,|f_a(\varphi(z_m))| \\ &\leq \limsup_{m\to\infty}(1-|\varphi(z_m)|^2)^\alpha\,|f_a(\varphi(z_m))| \leq 1. \end{aligned}$$

The last inequality follows by (2) since $\varphi(z_m)\in\mathbb{D}$.
Consequently,

$$\lim_{m\to\infty}(1-|\varphi(z_m)|^2)^\alpha|f_a(\varphi(z_m))| = \lim_{m\to\infty}(1-\rho^2(\varphi(z_m),a))^\alpha = 1. \tag{6}$$

That is, $\lim_{m\to\infty}\rho(\varphi(z_m),a)=0$.

Combining (3), (4) and (6), we know

$$\lim_{m\to\infty}\frac{(1-|z_m|^2)^\beta\,|\varphi'(z_m)|}{(1-|\varphi(z_m)|^2)^\alpha}|g'(\varphi(z_m))| = 1.$$

This completes the proof of this theorem. □

Theorem 2. *Let φ be analytic self maps of the unit disk and $g\in H(\mathbb{D})$. Then the operator $J_gC_\varphi : H_\alpha^\infty \to \mathcal{B}^\beta$ is an isometry if and only if*

(III) $\sup_{z\in D}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha}\,|g'(z)|\leq 1$*;*

(IV) *For every $a\in\mathbb{D}$, there at least exists a sequence $\{z_n\}$ in $\mathbb{D}$, such that $\lim_{n\to\infty}\rho(\varphi(z_n),a)=0$ and $\lim_{n\to\infty}\frac{(1-|z_n|^2)^\beta}{(1-|\varphi(z_n)|^2)^\alpha}\,|g'(z_n)|=1$.*

Proof. We prove the sufficiency first.

By condition (III), for every $f\in H_\alpha^\infty(\mathbb{D})$, we have

$$\begin{aligned} ||J_gC_\varphi f||_{\mathcal{B}^\beta} &= \sup_{z\in D}(1-|z|^2)^\beta\,|f(\varphi(z))|\,|g'(z)| \\ &= \sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha}\,|g'(z)|\,(1-|\varphi(z)|^2)^\alpha\,|f(\varphi(z))| \\ &\leq \|f\|_\alpha. \end{aligned}$$

Next we show that the property (IV) implies $||J_gC_\varphi f||_{\mathcal{B}^\beta}\geq||f||_\alpha$.

In fact, for any $f\in H_\alpha^\infty(\mathbb{D})$, then $||f||_\alpha = \lim_{m\to\infty}(1-|a_m|^2)^\alpha|f(a_m)|$ for some sequence $\{a_m\}\subset\mathbb{D}$. For any fixed m, by property (IV), there is a sequence $\{z_k^m\}\subset\mathbb{D}$ such that

$$\rho(\varphi(z_k^m),a_m)\to 0 \text{ and } \frac{(1-|z_k^m|^2)^\beta}{(1-|\varphi(z_k^m)|^2)^\alpha}\,|g'(z_k^m)|\to 1$$

as $k\to\infty$. By Lemma 2, for all m and k,

$$\left|(1-|\varphi(z_k^m)|^2)^\alpha f(\varphi(z_k^m)) - (1-|a_m|^2)^\alpha f(a_m)\right| \leq C||f||_\alpha\cdot\rho(\varphi(z_k^m),a_m).$$

Hence

$$(1-|\varphi(z_k^m)|^2)^\alpha\,|f(\varphi(z_k^m))| \geq (1-|a_m|^2)^\alpha|f(a_m)| - C||f||_\alpha\cdot\rho(\varphi(z_k^m),a_m)$$

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Therefore,

$$\begin{aligned} ||J_gC_\varphi f||_{\mathcal{B}^\beta} &= \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha}\,|g'(z)|\,(1-|\varphi(z)|^2)^\alpha\,|f(\varphi(z))| \\ &\geq \limsup_{k\to\infty} \frac{(1-|z_k^m|^2)^\beta}{(1-|\varphi(z_k^m)|^2)^\alpha}\,|g'(z_k^m)|\,(1-|\varphi(z_k^m)|^2)^\alpha\,|f(\varphi(z_k^m))| \\ &= (1-|a_m|^2)^\alpha|f(a_m)|. \end{aligned}$$

The inequality $||J_gC_\varphi f||_{\mathcal{B}^\beta} \geq ||f||_\alpha$ follows by letting $m\to\infty$.

Now we turn to the necessity.

For any $a\in\mathbb{D}$, using the same test function f_a defined by (1) which satisfies $\|f_a\|_\alpha = 1$. By isometry assumption, for any $a\in\mathbb{D}$, we have

$$\begin{aligned} 1 &= ||f_{\varphi(a)}||_\alpha = ||J_gC_\varphi f_{\varphi(a)}||_{\mathcal{B}^\beta} \\ &= \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha}\,|g'(z)|\,(1-|\varphi(z)|^2)^\alpha\,\left|f_{\varphi(a)}(\varphi(z))\right| \\ &\geq \frac{(1-|a|^2)^\beta}{(1-|\varphi(a)|^2)^\alpha}\,|g'(a)| \end{aligned}$$

So (III) follows by noticing a is arbitrary.

Since $||J_gC_\varphi f_a||_{\mathcal{B}^\beta} = ||f_a||_\alpha = 1$, thus there exists a sequence $\{z_m\}\subset\mathbb{D}$ such that

$$(1-|z_m|^2)^\beta\left|\frac{d(J_gC_\varphi f_a)}{dz}(z_m)\right| = (1-|z_m|^2)^\beta|f_a(\varphi(z_m))||g'(z_m)| \to 1 \qquad (7)$$

as $m\to\infty$.

It follows from (III) that

$$\begin{aligned} & (1-|z_m|^2)^\beta|f_a(\varphi(z_m))||g'(z_m)| \\ =\; & \frac{(1-|z_m|^2)^\beta}{(1-|\varphi(z_m)|^2)^\alpha}\,|g'(z_m)|\,(1-|\varphi(z_m)|^2)^\alpha\,|f_a(\varphi(z_m))| \qquad (8) \\ \leq\; & (1-|\varphi(z_m)|^2)^\alpha\,|f_a(\varphi(z_m))|\,. \qquad (9) \end{aligned}$$

Combining (7) and (9), it follows that

$$\begin{aligned} 1 &\leq \liminf_{m\to\infty}(1-|\varphi(z_m)|^2)^\alpha\,|f_a(\varphi(z_m))| \\ &\leq \limsup_{m\to\infty}(1-|\varphi(z_m)|^2)^\alpha\,|f_a(\varphi(z_m))| \leq 1. \end{aligned}$$

The last inequality follows by (2) since $\varphi(z_m)\in\mathbb{D}$.
Consequently,

$$\lim_{m\to\infty}(1-|\varphi(z_m)|^2)^\alpha|f_a(\varphi(z_m))| = \lim_{m\to\infty}(1-\rho^2(\varphi(z_m),a))^\alpha = 1. \qquad (10)$$

That is, $\lim\limits_{m\to\infty}\rho(\varphi(z_m),a) = 0$.

Combining (7), (8) and (10), we know

$$\lim_{m\to\infty}\frac{(1-|z_m|^2)^\beta\,|\varphi'(z_m)|}{(1-|\varphi(a)|^2)^\alpha}\,|g'(z_m)| = 1,$$

the desired results follows. The proof of this theorem is completed. □

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Remark If $\alpha = 0$, $\beta = 1$ then H_α^∞ will be H^∞, and $\mathcal{B}^p$ will be $\mathcal{B}$. So the the following corollaries follow by Theorem 1 and Theorem 2.

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UMBRAL CALCULUS AND SPECIAL POLYNOMIALS

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Abstract. In this paper, we consider several special polynomials related to associated sequences of polynomials. Finally, we give some new and interesting identities of those polynomials arising from transfer formula for the associated sequences.

1. Introduction

In this paper, we assume that $\lambda \in \mathbb{C}$ with $\lambda \neq 1$. For $\alpha \in \mathbb{R}$, the Frobenius-Euler polynomials are defined by the generating function to be

$$\left(\frac{1-\lambda}{e^t-\lambda}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} H_n^{(\alpha)}(x|\lambda)\frac{t^n}{n!}, \text{ (see [1,5,13,15,20,22,23]).} \tag{1.1}$$

In the special case, $x = 0$, $H_n^{(\alpha)}(0|\lambda) = H_n^{(\alpha)}(\lambda)$ are called the *n-th Frobenius-Euler numbers* of order α. As is well known, the Bernoulli polynomials of order α are given by

$$\left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{t^n}{n!}, \text{ (see [2,3,4,6,14,15,19,21]).} \tag{1.2}$$

In the special case, $x = 0$, $B_n^{(\alpha)}(0) = B_n^{(\alpha)}$ are called the *n-th Bernoulli numbers* of order α.

For $n \geq 0$, the *Stirling numbers of the second kind* are defined by generating function to be

$$(e^t-1)^n = n!\sum_{l=n}^{\infty} S_2(l,n)\frac{t^l}{l!}, \text{ (see [8-12,17,18]),} \tag{1.3}$$

and the *Stirling numbers of the first kind* are given by

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l)x^l, \text{ (see [7,8,10,17,18]).} \tag{1.4}$$

Let $\mathcal{F}$ be the set of all formal power series in the variable t over $\mathbb{C}$ with

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!}t^k \;\middle|\; a_k \in \mathbb{C} \right\}. \tag{1.5}$$

Let $\mathbb{P}$ be the algebra of polynomials in the variable x over $\mathbb{C}$ and $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. As a notation, the action of the linear functional

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L on a polynomial $p(x)$ is denoted by $\langle L \mid p(x)\rangle$. Let $f(t)=\sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \in \mathcal{F}$. Then we define the linear functional $f(t)$ on $\mathbb{P}$ by

$$\langle f(t)|x^n\rangle = a_n,\ (n\geq 0), (\text{see [10,12,16,17,18]}). \tag{1.6}$$

From (1.6), we note that

$$\left\langle t^k|x^n\right\rangle = n!\delta_{n,k},\ (n,k\geq 0), \tag{1.7}$$

where $\delta_{n,k}$ is the Kronecker symbol (see [8, 10, 11, 17, 18]).

Let $f_L(t)=\sum_{k=0}^{\infty}\frac{\langle L|x^k\rangle}{k!}t^k$. Then, by (1.7), we get $\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle$. The map $L\mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ thought of as both a formal power series and a linear functional. We shall call $\mathcal{F}$ the *umbral algebra*. The umbral calculus is the study of umbral algebra (see [10, 16, 17, 18]).

The order $o(f(t))$ of the non-zero power series $f(t)$ is the smallest integer k for which the coefficient of t^k does not vanish (see [10, 11, 12, 17, 18]). If $o(f(t)) = 1$, then $f(t)$ is called a *delta series*, and if $o(f(t)) = 0$, then $f(t)$ is called an *invertible series*. Let $o(f(t)) = 1$ and $o(g(t)) = 0$. Then there exists a unique sequence $S_n(x)$ of polynomials such that $\left\langle g(t)f(t)^k|S_n(x)\right\rangle = n!\delta_{n,k}$ where $n,k\geq 0$. The sequence $S_n(x)$ is called *Sheffer sequence* for $(g(t), f(t))$, which is denoted by $S_n(x)\sim(g(t),f(t))$. If $S_n(x)\sim(1,f(t))$, then $S_n(x)$ is called the *associated sequence* for $f(t)$ (see [10, 16, 17, 18]). From (1.7), we note that $\langle e^{yt}|\, p(x)\rangle = p(y)$.

Let $f(t)\in\mathcal{F}$ and $p(x)\in\mathbb{P}$. Then we have

$$f(t)=\sum_{k=0}^{\infty}\frac{\left\langle f(t)|x^k\right\rangle}{k!}t^k,\ p(x)=\sum_{k=0}^{\infty}\frac{\left\langle t^k|p(x)\right\rangle}{k!}x^k,\ (\text{see [17,18]}). \tag{1.8}$$

From (1.9), we can derive the following equation:

$$p^{(k)}(0)=\left\langle t^k|p(x)\right\rangle \text{ and } \left\langle 1\left|p^{(k)}(x)\right.\right\rangle = p^{(k)}(0),\ (\text{see [10,16,17,18]}). \tag{1.9}$$

for $k\geq 0$, by (1.9), we easily see that $t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k}$.

Let $S_n(x)\sim(g(t),f(t))$. Then we see that

$$\frac{1}{g(\bar f(t))}e^{y\bar f(t)} = \sum_{k=0}^{\infty}\frac{S_k(y)}{k!}t^k,\ \text{for all } y\in\mathbb{C}, \tag{1.10}$$

where $\bar f(t)$ is the compositional inverse of $f(t)$ (see [17, 18]).

Let $p_n(x)\sim(1,f(t))$, $q_n(x)\sim(1,g(t))$. Then, the transfer formula for the associated sequence is given by

$$q_n(x)=x\left(\frac{f(t)}{g(t)}\right)^n x^{-1}p_n(x),\ (\text{see [11,12,16,17,18]}). \tag{1.11}$$

For $n\geq 0$, $b\neq 0$, the *Abel sequences* are given by

$$A_n(x;b)=x(x-bn)^{n-1}\sim\left(1,te^{bt}\right). \tag{1.12}$$

In this paper, we consider several special polynomials related to associated sequences of polynomials. Finally, we give some new and interesting identities of those polynomials arising from transfer formula for the associated sequences.

2. Umbral calculus and special polynomials

From (1.1), we note that

$$H_n^{(\alpha)}(x|\lambda) \sim \left(\left(\frac{e^t-\lambda}{1-\lambda} \right)^{\alpha}, t \right). \tag{2.1}$$

Thus, we get

$$H_n^{(\alpha)}(x|\lambda) = \left(\frac{1-\lambda}{e^t-\lambda} \right)^{\alpha} x^n. \tag{2.2}$$

Let us assume that

$$p_n(x) \sim \left(1, t(e^t-\lambda)\right), \; q_n(x) \sim \left(1, \left(\frac{e^t-\lambda}{1-\lambda}\right)^a t\right), \; (a \neq 0). \tag{2.3}$$

From $x^n \sim (1,t)$, (1.11) and (2.3), we note that

$$\begin{aligned} p_n(x) &= x \left(\frac{t}{t(e^t-\lambda)} \right)^n x^{-1} x^n = \frac{x}{(1-\lambda)^n} \left(\frac{1-\lambda}{e^t-\lambda} \right)^n x^{n-1} \\ &= \frac{1}{(1-\lambda)^n} x H_{n-1}^{(n)}(x|\lambda). \end{aligned} \tag{2.4}$$

and

$$q_n(x) = x \left(\frac{1-\lambda}{e^t-\lambda} \right)^{na} x^{-1} x^n = x H_{n-1}^{(an)}(x|\lambda). \tag{2.5}$$

From (1.11), (2.3), (2.4) and (2.5), we can derive

$$\begin{aligned} &\frac{1}{(1-\lambda)^n} x H_{n-1}^{(n)}(x|\lambda) \\ =&x \left(\frac{t\left(\frac{e^t-\lambda}{1-\lambda}\right)^a}{t(e^t-\lambda)} \right)^n x^{-1} x H_{n-1}^{(an)}(x|\lambda) \\ =&\frac{x}{(1-\lambda)^{an}} \left(e^t-\lambda\right)^{(a-1)n} H_{n-1}^{(an)}(x|\lambda) \\ =&\frac{x}{(1-\lambda)^{an}} \sum_{l=0}^{(a-1)n} \binom{(a-1)n}{l} (-\lambda)^{(a-1)n-l} e^{lt} H_{n-1}^{(an)}(x|\lambda) \\ =&\frac{x}{(1-\lambda)^{an}} \sum_{l=0}^{(a-1)n} \binom{(a-1)n}{l} (-\lambda)^{(a-1)n-l} H_{n-1}^{(an)}(x+l|\lambda), \end{aligned} \tag{2.6}$$

where $a, n \in \mathbb{N}$. Therefore, by (2.6), we obtain the following theorem.

Theorem 2.1. *For $a, n \in \mathbb{N}$, we have*

$$H_{n-1}^{(n)}(x|\lambda) = \frac{1}{(1-\lambda)^{(a-1)n}} \sum_{l=0}^{(a-1)n} \binom{(a-1)n}{l} (-\lambda)^{(a-1)n-l} H_{n-1}^{(an)}(x+l|\lambda).$$

Let us consider the following associated sequences:

$$\frac{1}{(1-\lambda)^n} x H_{n-1}^{(n)}(x|\lambda) \sim \left(1, t(e^t-\lambda)\right), \; p_n(x) \sim \left(1, \left(\frac{1-\lambda}{e^t-\lambda}\right)^a t\right), \; (a \neq 0). \tag{2.7}$$

For $x^n \sim (1,t)$, by (1.11) and (2.7), we get

$$\begin{aligned} p_n(x) &= x\left(\frac{t}{t\left(\frac{1-\lambda}{e^t-\lambda}\right)^a}\right)^n x^{-1}x^n = x\left(\frac{e^t-\lambda}{1-\lambda}\right)^{an} x^{n-1} \\ &= x\frac{1}{(1-\lambda)^{an}}\sum_{l=0}^{an}\binom{an}{l}(-\lambda)^{an-l}(x+l)^{n-1}. \end{aligned} \tag{2.8}$$

For $n \geq 1$, by (1.11) and (2.7), we get

$$\begin{aligned} p_n(x) &= x\left(\frac{t\left(e^t-\lambda\right)}{t\left(\frac{1-\lambda}{e^t-\lambda}\right)^a}\right)^n x^{-1}\frac{x}{(1-\lambda)^n}H_{n-1}^{(n)}(x|\lambda) \\ &= x\left(\frac{1}{1-\lambda}\right)^{(a+1)n}\left(e^t-\lambda\right)^{(a+1)n}H_{n-1}^{(n)}(x|\lambda). \end{aligned} \tag{2.9}$$

By (2.8) and (2.9), we get

$$\begin{aligned} &\sum_{l=0}^{an}\binom{an}{l}(-\lambda)^{an-l}(x+l)^{n-1} \\ =&\frac{1}{(1-\lambda)^n}\left(e^t-\lambda\right)^{(a+1)n}H_{n-1}^{(n)}(x|\lambda) \\ =&\frac{1}{(1-\lambda)^n}\sum_{l=0}^{(a+1)n}\binom{(a+1)n}{l}(-\lambda)^{(a+1)n-l}H_{n-1}^{(n)}(x+l|\lambda). \end{aligned} \tag{2.10}$$

Therefore, by (2.10), we obtain the following theorem.

Theorem 2.2. *For $n \geq 1$ and $a \in \mathbb{Z}_+ = \mathbb{N}\cup\{0\}$, we have*

$$\sum_{l=0}^{an}\binom{an}{l}(-\lambda)^{-l}(x+l)^{n-1} = \frac{1}{(1-\lambda)^n}\sum_{l=0}^{(a+1)n}\binom{(a+1)n}{l}(-\lambda)^{n-l}H_{n-1}^{(n)}(x+l|\lambda).$$

Let us consider the following associated sequences:

$$(x)_n \sim \left(1, e^t-1\right),\ xH_{n-1}^{(an)}(x|\lambda) \sim \left(1, t\left(\frac{e^t-\lambda}{1-\lambda}\right)^a\right),\ (a\neq 0). \tag{2.11}$$

By (1.11) and (2.11), we get

$$\begin{aligned} xH_{n-1}^{(an)}(x|\lambda) &= x\left(\frac{e^t-1}{t\left(\frac{e^t-\lambda}{1-\lambda}\right)^a}\right)^n x^{-1}(x)_n \\ &= x\left(\frac{e^t-1}{t}\right)^n\left(\frac{1-\lambda}{e^t-\lambda}\right)^{an}(x-1)_{n-1}. \end{aligned} \tag{2.12}$$

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Replacing x by $x+1$, we have

$$\begin{aligned}
H_{n-1}^{(an)}(x+1|\lambda) &= \left(\frac{e^t-1}{t}\right)^n \left(\frac{1-\lambda}{e^t-\lambda}\right)^{an} \sum_{l=0}^{n-1} S_1(n-1,l)x^l \\
&= \left(\frac{e^t-1}{t}\right)^n \sum_{l=0}^{n-1} S_1(n-1,l)H_l^{(an)}(x|\lambda) \\
&= \sum_{l=0}^{n-1}\sum_{k=0}^{l} S_1(n-1,l)\frac{n!}{(k+n)!}S_2(k+n,n)(l)_k H_{l-k}^{(an)}(x|\lambda) \\
&= \sum_{l=0}^{n-1}\sum_{k=0}^{l} S_1(n-1,l)S_2(k+n,n)\frac{\binom{l}{k}}{\binom{k+n}{n}} H_{l-k}^{(an)}(x|\lambda).
\end{aligned} \tag{2.13}$$

Therefore, by (2.13), we obtain the following theorem.

Theorem 2.3. *For $n \geq 1$, $a \in \mathbb{Z}_+$, we have*

$$H_{n-1}^{(an)}(x+1|\lambda) = \sum_{l=0}^{n-1}\sum_{k=0}^{l} S_1(n-1,l)S_2(k+n,n)\frac{\binom{l}{k}}{\binom{k+n}{n}} H_{l-k}^{(an)}(x|\lambda)$$

Let

$$\begin{gathered}
xH_{n-1}^{(an)}(x|\lambda) \sim \left(1, t\left(\tfrac{e^t-\lambda}{1-\lambda}\right)^a\right),\ (a \neq 0), \\
(x)_n \sim \left(1, e^t-1\right).
\end{gathered} \tag{2.14}$$

Then, by (1.11) and (2.14), we get

$$\begin{aligned}
(x)_n &= x\left(\frac{t\left(\frac{e^t-\lambda}{1-\lambda}\right)^a}{e^t-1}\right)^n x^{-1}xH_{n-1}^{(an)}(x|\lambda) \\
&= x\left(\frac{t}{e^t-1}\right)^n \left(\frac{e^t-\lambda}{1-\lambda}\right)^{an} H_{n-1}^{(an)}(x|\lambda) \\
&= x\left(\frac{t}{e^t-1}\right)^n x^{n-1} \\
&= xB_{n-1}^{(n)}(x).
\end{aligned} \tag{2.15}$$

and

$$(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l = x\sum_{l=0}^{n-1} S_1(n,l+1)x^l,\ (n \geq 1). \tag{2.16}$$

Therefore, by (2.15) and (2.16), we get

Theorem 2.4. *For $n \geq 1$, $0 \leq l \leq n-1$, we have*

$$S_1(n,l+1) = \binom{n-1}{l} B_{n-1-l}^{(n)}.$$

From (2.15), we note that

$$\left(\frac{e^t-1}{t}\right)^n (x-1)_{n-1} = (1-\lambda)^{-an}(e^t-\lambda)^{an}H_{n-1}^{(an)}(x|\lambda),\ (n \geq 1). \tag{2.17}$$

$$
\begin{aligned}
LHS \text{ of } (2.17) &= \left(\frac{e^t-1}{t}\right)^n \sum_{l=0}^{n-1} S_1(n-1,l)(x-1)^l \\
&= \sum_{l=0}^{n-1} S_1(n-1,l) \sum_{k=0}^{l} \frac{n!l!}{(k+n)!(l-k)!} S_2(k+n,n)(x-1)^{l-k} \\
&= \sum_{l=0}^{n-1} S_1(n-1,l) \sum_{k=0}^{l} \frac{\binom{l}{k}}{\binom{k+n}{n}} S_2(k+n,n)(x-1)^{l-k},
\end{aligned} \tag{2.18}
$$

and

$$
\begin{aligned}
RHS \text{ of } (2.17) &= (1-\lambda)^{-an} \sum_{l=0}^{an} \binom{an}{l} (-\lambda)^{an-l} e^{lt} H_{n-1}^{(an)}(x|\lambda) \\
&= (1-\lambda)^{-an} \sum_{l=0}^{an} \binom{an}{l} (-\lambda)^{an-l} H_{n-1}^{(an)}(x+l|\lambda).
\end{aligned} \tag{2.19}
$$

Therefore, by (2.17), (2.18) and (2.19), we obtain the following theorem.

Theorem 2.5. *For $n \geq 1$, $a \in \mathbb{Z}_+$, we have*

$$
\begin{aligned}
&(1-\lambda)^{-an} \sum_{l=0}^{an} \binom{an}{l} (-\lambda)^{an-l} H_{n-1}^{(an)}(x+l|\lambda) \\
=& \sum_{l=0}^{n-1} \sum_{k=0}^{l} \frac{\binom{l}{k}}{\binom{k+n}{n}} S_1(n-1,l) S_2(k+n,n)(x-1)^{l-k}.
\end{aligned}
$$

Let

$$
p_n(x) \sim \left(1, \left(\frac{1-\lambda}{e^t-\lambda}\right)^a t\right), \ (x)_n \sim \left(1, e^t-1\right), \ (a \neq 0). \tag{2.20}
$$

By (2.8), we have

$$
\begin{aligned}
p_n(x) &= \left(\frac{1}{1-\lambda}\right)^{an} x \sum_{l=0}^{an} \binom{an}{l} (-\lambda)^{an-l} (x+l)^{n-1} \\
&= \left(\frac{1}{1-\lambda}\right)^{an} x \sum_{k=0}^{an} \binom{an}{k} (-\lambda)^{an-k} \sum_{l=0}^{n-1} \binom{n-1}{l} k^{n-1-l} x^l.
\end{aligned} \tag{2.21}
$$

From (1.11) and (2.20), we have

$$\begin{aligned}
(x)_n &= x\left(\frac{t\left(\frac{1-\lambda}{e^t-\lambda}\right)^a}{e^t-1}\right)^n x^{-1}p_n(x)\\
&= x\left(\frac{t}{e^t-1}\right)^n\left(\frac{1-\lambda}{e^t-\lambda}\right)^{an}\left(\frac{1}{1-\lambda}\right)^{an}\sum_{k=0}^{an}\sum_{l=0}^{n-1}\binom{an}{k}\binom{n-1}{l}k^{n-1-l}(-\lambda)^{an-k}x^l\\
&= \left(\frac{1}{1-\lambda}\right)^{an} x\left(\frac{t}{e^t-1}\right)^n\sum_{k=0}^{an}\sum_{l=0}^{n-1}\binom{an}{k}\binom{n-1}{l}k^{n-1-l}(-\lambda)^{an-k}H_l^{(an)}(x|\lambda)\\
&= \left(\frac{1}{1-\lambda}\right)^{an} x\sum_{k=0}^{an}\sum_{l=0}^{n-1}\sum_{m=0}^{l}\binom{an}{k}\binom{n-1}{l}\binom{l}{m}k^{n-1-l}(-\lambda)^{an-k}H_{l-m}^{(an)}(\lambda)B_m^{(n)}(x)\\
&= \left(\frac{1}{1-\lambda}\right)^{an} x\sum_{k=0}^{an}\sum_{l=0}^{n-1}\sum_{m=0}^{l}\sum_{p=0}^{m}\binom{an}{k}\binom{n-1}{l}\binom{l}{m}\binom{m}{p}k^{n-1-l}(-\lambda)^{an-k}H_{l-m}^{(an)}(\lambda)B_{m-p}^{(n)}x^p\\
&= \left(\frac{1}{1-\lambda}\right)^{an} x\sum_{p=0}^{n-1}\left\{\sum_{k=0}^{an}\sum_{l=p}^{n-1}\sum_{m=p}^{l}\binom{an}{k}\binom{n-1}{l}\binom{l}{m}\binom{m}{p}k^{n-1-l}(-\lambda)^{an-k}H_{l-m}^{(an)}(\lambda)B_{m-p}^{(n)}\right\}x^p.
\end{aligned} \tag{2.22}$$

Therefore, by (2.16) and (2.22), we obtain the following theorem.

Theorem 2.6. *For $n \geq 1$, $a \in \mathbb{Z}_+$ and $0 \leq p \leq n-1$, we have*

$$S_1(n,p+1) = \left(\frac{1}{1-\lambda}\right)^{an}\sum_{k=0}^{an}\sum_{l=p}^{n-1}\sum_{m=p}^{l}\binom{an}{k}\binom{n-1}{l}\binom{l}{m}\binom{m}{p}k^{n-1-l}(-\lambda)^{an-k}H_{l-m}^{(an)}(\lambda)B_{m-p}^{(n)}.$$

Theorem 2.7. *For $n \geq 0$, we have*

$$e^{-xt}(e^t-\lambda)^n = \sum_{k=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{j=0}^{k}\binom{n}{l}(1-\lambda)^{n-l}\frac{\binom{k}{j}}{\binom{j+l}{l}}S_2(j+l,l)(-1)^{k-j}x^{k-j}\right)\frac{t^{k+l}}{k!}.$$

Proof. Note that

$$e^{-xt}(e^t-\lambda)^n = e^{-xt}(e^t-1+1-\lambda)^n = \sum_{l=0}^{n}\binom{n}{l}(1-\lambda)^{n-l}(e^t-1)^l e^{-xt}, \tag{2.23}$$

and

$$\begin{aligned}
(e^t-1)^l e^{-xt} &= \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\frac{l!k!}{(j+l)!(k-j)!}S_2(j+l,l)(-1)^{k-j}x^{k-j}\right)\frac{t^{k+l}}{k!}\\
&= \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\frac{\binom{k}{j}}{\binom{j+l}{l}}S_2(j+l,l)(-1)^{k-j}x^{k-j}\right)\frac{t^{k+l}}{k!}.
\end{aligned} \tag{2.24}$$

From (2.23) and (2.24), we can derive Theorem 2.7. □

By (1.12) and Theorem 2.7, we get

$$\begin{aligned}A_n(x;b) &= x(x-bn)^{n-1} = x\left(\frac{1}{1-\lambda}\right)^{an} e^{-nbt}(e^t-\lambda)^{an} H_{n-1}^{(an)}(x|\lambda)\\ &=\frac{x}{(1-\lambda)^{an}}\sum_{l=0}^{an}\sum_{k=0}^{n-1-l}\sum_{j=0}^{k}\frac{\binom{an}{l}\binom{k}{j}(n-1)_{k+l}}{\binom{j+l}{l}k!}(1-\lambda)^{an-l}S_2(j+l,l)(-1)^{k-j}(nb)^{k-j}H_{n-1-l-k}^{(an)}(x|\lambda).\end{aligned} \tag{2.25}$$

Therefore, by (2.25), we obtain the following theorem.

Theorem 2.8. *For $n \geq 1$, $a \in \mathbb{Z}_+$ and $b \neq 0$, we have*

$$(x-bn)^{n-1} = \sum_{l=0}^{an}\sum_{k=0}^{n-1-l}\sum_{j=0}^{k}\frac{\binom{an}{l}\binom{k}{j}(n-1)_{k+l}}{\binom{j+l}{l}(1-\lambda)^l k!}S_2(j+l,l)(-1)^{k-j}(nb)^{k-j}H_{n-1-l-k}^{(an)}(x|\lambda).$$

Let us consider the Changhee polynomials of the second kind as follows:

$$\sum_{k=0}^{\infty} C_k(x|\lambda)\frac{t^k}{k!} = \frac{1}{1+\lambda(1+t)}(1+t)^x. \tag{2.26}$$

From (1.10) and (2.26), we note that

$$C_n(x|\lambda) \sim \left(1+\lambda e^t, e^t-1\right). \tag{2.27}$$

Hence $\lambda \in \mathbb{C}$ with $\lambda \neq -1$. Thus, by (2.27), we get

$$(1+\lambda e^t)C_n(x|\lambda) = (x)_n \sim \left(1, e^t-1\right), \tag{2.28}$$

and

$$(x)_n = x\left(\frac{t}{e^t-1}\right)^n x^{-1}x^n = x\left(\frac{t}{e^t-1}\right)^n x^{n-1} = xB_{n-1}^{(n)}(x). \tag{2.29}$$

Thus, by (2.28) and (2.29), we get

$$\begin{aligned}C_n(x|\lambda) &= \frac{1}{\lambda e^t+1}xB_{n-1}^{(n)}(x) = \sum_{l=0}^{n}(-\lambda)^l e^{lt}\left(xB_{n-1}^{(n)}(x)\right)\\ &= \sum_{l=0}^{n}(-\lambda)^l(x+l)B_{n-1}^{(n)}(x+l).\end{aligned} \tag{2.30}$$

Let

$$t_n(x|\lambda) \sim \left(1, \frac{t}{1+\lambda(1+t)}\right). \tag{2.31}$$

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Then, by (1.11) and (2.31), we get

$$\begin{aligned}
t_n(x|\lambda) &= x\left(\frac{t}{\frac{t}{1+\lambda(1+t)}}\right)^n x^{-1}x^n = x(1+\lambda(1+t))^n x^{n-1}\\
&= x\sum_{l=0}^{n}\binom{n}{l}\lambda^l(1+t)^l x^{n-1} = x\sum_{a=0}^{n}\binom{n}{a}\lambda^a\sum_{b=0}^{n-1}\binom{a}{b}t^b x^{n-1}\\
&= \sum_{a=0}^{n}\sum_{b=0}^{n-1}\binom{n}{a}\binom{a}{b}\lambda^a(n-1)_b x^{n-b} = \sum_{a=0}^{n}\sum_{b=1}^{n}\lambda^a\binom{n}{a}\binom{a}{n-b}(n-1)_{n-b}x^b\\
&= \sum_{a=0}^{n}\sum_{b=1}^{n}\lambda^a\binom{n}{a}\binom{a}{n-b}\frac{(n-1)!}{(b-1)!}x^b.
\end{aligned} \tag{2.32}$$

Let us also consider the following associated sequence:

$$S_n(x|\mu) \sim \left(1, \frac{t}{(1+t)^\mu}\right), \ (\mu \in \mathbb{N}). \tag{2.33}$$

Then, by (1.11) and (2.33), we easily get

$$S_n(x|\mu) = \sum_{k=1}^{n}\binom{\mu n}{n-k}\frac{(n-1)!}{(k-1)!}x^k. \tag{2.34}$$

From (1.11), (2.32) and (2.33), we can derive

$$\begin{aligned}
S_n(x|\mu) &= x\left(\frac{\frac{t}{1+\lambda(1+t)}}{\frac{t}{(1+t)^\mu}}\right)^n x^{-1}t_n(x|\lambda) = x\left(\frac{(1+t)^\mu}{1+\lambda(1+t)}\right)^n x^{-1}t_n(x|\lambda)\\
&= x\left(\sum_{l=0}^{\infty}\frac{C_l(\mu|\lambda)}{l!}t^l\right)^n x^{-1}t_n(x|\lambda)\\
&= x\sum_{l=0}^{\infty}\left\{\sum_{l_1+\cdots+l_n=l}\binom{l}{l_1,\ldots,l_n}\left(\prod_{i=1}^{n}C_{l_i}(\mu|\lambda)\right)\right\}\frac{t^l}{l!}x^{-1}t_n(x|\lambda)\\
&= x\sum_{l=0}^{\infty}\left\{\sum_{l_1+\cdots+l_n=l}\binom{l}{l_1,\ldots,l_n}\left(\prod_{i=1}^{n}C_{l_i}(\mu|\lambda)\right)\right\}\frac{t^l}{l!}\left\{\sum_{a=0}^{n}\sum_{b=1}^{n}\lambda^a\binom{n}{a}\binom{a}{n-b}\frac{(n-1)!}{(b-1)!}x^{b-1}\right\}\\
&= x\sum_{a=0}^{n}\sum_{b=1}^{n}\sum_{l=0}^{b-1}\sum_{l_1+\cdots+l_n=l}\binom{l}{l_1,\ldots,l_n}\left(\prod_{i=1}^{n}C_{l_i}(\mu|\lambda)\right)\lambda^a\binom{n}{a}\binom{a}{n-b}\frac{(n-1)!}{(b-1)!}\frac{(b-1)_l}{l!}x^{b-1-l}\\
&= \sum_{a=0}^{n}\sum_{b=1}^{n}\sum_{l=0}^{b-1}\sum_{l_1+\cdots+l_n=l}\binom{l}{l_1,\ldots,l_n}\left(\prod_{i=1}^{n}C_{l_i}(\mu|\lambda)\right)\lambda^a\binom{n}{a}\binom{a}{n-b}\frac{(n-1)!}{(b-1)!}\binom{b-1}{l}x^{b-l}\\
&= \sum_{k=1}^{n}\left\{\sum_{a=0}^{n}\sum_{b=k}^{n}\sum_{l_1+\cdots+l_n=b-k}\binom{b-k}{l_1,\ldots,l_n}\left(\prod_{i=1}^{n}C_{l_i}(\mu|\lambda)\right)\lambda^a\binom{n}{a}\binom{a}{n-b}\binom{b-1}{k-1}\frac{(n-1)!}{(b-1)!}\right\}x^k.
\end{aligned} \tag{2.35}$$

Therefore, by (2.34) and (2.35), we obtain the following theorem.

Theorem 2.9. *For $n \geq 1$, $1 \leq k \leq n$, $b \neq 0$ and $\mu, a \in \mathbb{Z}_+$, we have*

$$\frac{\binom{\mu n}{n-k}}{(k-1)!} = \sum_{a=0}^{n}\sum_{b=k}^{n}\sum_{l_1+\cdots+l_n=b-k}\binom{n}{a}\binom{a}{n-b}\binom{b-k}{l_1,\ldots,l_n}\binom{b-1}{k-1}\lambda^a\frac{1}{(b-1)!}\left(\prod_{i=1}^{n}C_{l_i}(\mu|\lambda)\right).$$

REMARK. From (1.1), we note that

$$\begin{aligned}\frac{1-\lambda}{e^t-\lambda} &= \frac{1-\lambda}{e^t-1+1-\lambda} = \frac{1}{1+\frac{e^t-1}{1-\lambda}} = \sum_{l=0}^{\infty}(-1)^l\left(\frac{e^t-1}{1-\lambda}\right)^l \\ &= \sum_{k=0}^{\infty}\left(\sum_{l=0}^{k}(-1)^l\left(\frac{1}{1-\lambda}\right)^l l!S_2(k,l)\right)\frac{t^k}{k!},\end{aligned} \tag{2.36}$$

and

$$\frac{1-\lambda}{e^t-\lambda} = \sum_{n=0}^{\infty}H_n(\lambda)\frac{t^n}{n!}, \tag{2.37}$$

where $H_n(\lambda)$ are the Frobenius-Euler numbers. By (2.36) and (2.37), we get

$$H_k(\lambda) = \sum_{l=0}^{k}\left(\frac{1}{\lambda-1}\right)^l l!S_2(k,l). \tag{2.38}$$

Let us consider the following associated sequences:

$$p_n(x) \sim \left(1, \frac{1-\lambda}{e^t-\lambda}t\right),\ x^n \sim (1,t). \tag{2.39}$$

Then, by (1.11) and (2.39), we get

$$p_n(x) = x\left(\frac{1}{1-\lambda}\right)^n(e^t-\lambda)^n x^{n-1} = \left(\frac{1}{1-\lambda}\right)^n x\sum_{k=0}^{n}\binom{n}{k}(-\lambda)^{n-k}(x+k)^{n-1}, \tag{2.40}$$

and

$$x^n = x\left(\frac{\left(\frac{1-\lambda}{e^t-\lambda}\right)t}{t}\right)^n x^{-1}p_n(x) = x\left(\frac{1-\lambda}{e^t-\lambda}\right)^n x^{-1}p_n(x). \tag{2.41}$$

Thus, by (2.40) and (2.41), we get

$$\begin{aligned}x^{n-1} &= \sum_{l=0}^{n-1}\left\{\sum_{l_1+\cdots+l_n=l}\binom{l}{l_1,\ldots,l_n}H_{l_1}(\lambda)\cdots H_{l_n}(\lambda)\right\}\frac{t^l}{l!}\times\left\{\sum_{k=0}^{n}\binom{n}{k}(-\lambda)^{n-k}\frac{(x+k)^{n-1}}{(1-\lambda)^n}\right\} \\ &= \frac{1}{(1-\lambda)^n}\sum_{k=0}^{n}\sum_{l=0}^{n-1}\sum_{l_1+\cdots+l_n=l}\binom{l}{l_1,\ldots,l_n}\left(\prod_{i=1}^{n}H_{l_i}(\lambda)\right)\binom{n}{k}\binom{n-1}{l}(-\lambda)^{n-k}(x+k)^{n-1-l}.\end{aligned}$$

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Jackson Type Generalization of Nonlinear Integral Operators

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Abstract

In this work, we provides a global smoothness preservation result of a Jackson-type generalization of the nonlinear convolution operator defined by Angeloni and Vinti in [4]. Convergence in variation is also studied.

AMS Subject Classification: Primary 26B30, 41A35; Secondary 47G10
Key words: Nonlinear integral operators, Jackson type generalization, Convergence in variation.

1 Introduction

In [4], the Authors made use of non-linear integral operators in order to generate estimates, convergence results and rate of approximation as regards functions, which belonged to BV-spaces. They touched upon the convergence in variation by addressing the BV -spaces. Besides, both the periodic and non-periodic cases were taken into consideration in line with the classical theory of linear convolution operators (see e.g.,[8]). In [1], they obtained similar results including the case of linear integral operators. See([6], [9], [2], [10]) for Mellin-type linear and non-linear integral operators as for the results in relation to convergence in φ-variation on R+. As shown in [7], J. Musileak and W. Orlicz invented the latter concept of variaton.

In [4] Angeloni and Vinti considered the following family of nonlinear integral operators of the form

$$(T_w f)(s) = \int_{-\pi}^{\pi} K_w(t, f(s-t))dt, \quad w > 0, \quad s \in \mathbb{R}, \tag{1}$$

for $f \in BV_{2\pi}$, the space of $2\pi-$periodic functions with bounded variation, where $\{K_w(t)\}_{w>0}$ is a family of measurable functions $K_w : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ of the form $K_w(t,u) = L_w(t)H_w(u)$ for every $t,u \in \mathbb{R}$, $\{H_w\}_{w>0}$ is a family of Lipschitz kernels $H_w : \mathbb{R} \to \mathbb{R}$ such that $H_w(0) = 0$, that is for every $w > 0$, there exist $K > 0$ such that

$$|H_w(u) - H_w(v)| \leq K|u-v| \tag{2}$$

for every $u, v \in \mathbb{R}$ and $\{L_w\}_{w>0}$ is family of $2\pi-$periodic approximate identities. Moreover, the followings are satisfied:

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K$_w$.**1** $L_w : \mathbb{R} \to \mathbb{R}$ is a measurable function such that $L_w \in L^1_{2\pi}$, $\|L_w\|_1 \le A$, for some constant $A > 0$ and for every $w > 0$

$$A_w := \int_{-\pi}^{\pi} L_w(t)\,dt \to 1 \text{ as } w \to +\infty;$$

K$_w$.**2** for any fixed $\delta > 0$, $\int\limits_{\delta \le |t| \le \pi} |L_w(t)|\,dt \to 0$, as $w \to +\infty$;

K$_w$.**3** denoted by $G_w(u) := H_w(u) - u$, $u \in \mathbb{R}$, $w > 0$, there holds

$$\frac{V_J[G_w]}{m(J)} \to 0, \text{ as } w \to +\infty,$$

for every bounded interval $J \subset \mathbb{R}$.

As it is presented [4] we shall say that $\{K_w(t)\}_{w>0} \subset \kappa_w$ if (2) and $K_w.1,2$ are satistfied. Also $K_w.3$ is in [5].

As usual, for a real valued function defined on an interval of real numbers, we shall denote by $V_I[f]$ the total variation of f and by $BV(I)$ the space of all functions with bounded variation on I with the seminorm $\|f\|_{BV(I)} := V_I[f]$. If $f : \mathbb{R} \to \mathbb{R}$ is a $2\pi-$periodic function, we shall simply use the notation $V_{2\pi}[f]$ and $BV_{2\pi}$ when $I = [-\pi, \pi]$.

We shall denote by $L^1_{2\pi}$ and $AC_{2\pi}$ the classes of all $2\pi-$periodic function $f : \mathbb{R} \to \mathbb{R}$ which are integrable over $(-\pi, \pi)$ and absolutely continuous, respectively.

Bardaro and Vinti investigated convergence of family of nonlinear operators T_w defined by (1) in variation seminorm. In this work, we consider the following Jackson-type generalization of T_w.

$$(T_{w,n}f)(s) = -\sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} \int_{-\pi}^{\pi} K_w(t, f(s-kt))dt, \tag{3}$$

where $K_w(t, f(s-kt)) = L_w(t) H_w(f(s-kt))$.

The first studies on the approximation by linear Jackson type generalizations of convolution operators were made in Chapter 16 of the book [3].

Our objective here is to study the global smoothness preservation property and give approximation result for the family of nonlinear integral operators $T_{w,n}$ defined by (3). For convenience, we have to introduce the modulus of smoothness of a function f via variation seminorm. We consider the r-th order difference operators applied to f defined as

$$(\Delta_t^r f)(u) = \sum_{i=0}^{r} (-1)^i \binom{r}{i} f(u - it), \qquad t, u \in \mathbb{R}, \;\; r = 1, 2, \ldots. \tag{4}$$

Furthermore, the r-th order modulus of smoothness of any continuous f belonging to $BV_{2\pi}$ is defined as

$$\omega_r(f;\delta) = \sup_{|t| \le \delta} V_{2\pi}[\Delta_t^r f] = \sup_{|t| \le \delta} \|\Delta_t^r f\|_{BV_{2\pi}}. \tag{5}$$

From (4), for $f \in AC_{2\pi}$ we have $H_w o f \in AC_{2\pi}$ as well. Therefore, for the future correspondence, we can write

$$V_{2\pi}\left[\left(\Delta_t^{n+1} H_w o f\right)(\cdot)\right] = \int_{-\pi}^{\pi} \left| \left(\Delta_t^{n+1} H_w o f\right)'(s) \right| ds \to 0 \tag{6}$$

as t$\to 0$, by the continuity of the translation operator in $L^1_{2\pi}$ (see [4]). So, for a fixed $\varepsilon > 0$ we can choose $\delta > 0$ in such a way that $V_{2\pi}\left[\left(\Delta_t^{n+1} H_w o f\right)(\cdot)\right] \le \varepsilon$ for $|t| \le \delta$. We notice that $\omega_r(f;\delta) \to 0$ as $\delta \to 0$ for $f \in AC_{2\pi}$ by the definition (5).

The following proposition is an estimate in variation for $T_{w,n}f$, $w > 0$.

2 Estimate and Convergence

Proposition 1. *If $f \in BV_{2\pi}$, $\{L_w\}_{w>0} \subset L^1_{2\pi}$ and (2) is satisfied, then there exist a constant $D > 0$ such that*

$$V_{2\pi}[T_{w,n}] \le D(2^{n+1}-1)V_{2\pi}[f].$$

Proof. Let $\{s_0 = -\pi, ..., s_m = \pi\}$ be a partition of $[-\pi,\pi]$. Then we have

$$\sum_{i=1}^{m} |(T_{w,n}f)(s_i) - (T_{w,n}f)(s_{i-1})|$$

$$\begin{aligned}
&= \sum_{i=1}^{m} \left| \sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} \int_{-\pi}^{\pi} L_w(t) \left[H_w(f(s_i - kt)) - H_w(f(s_{i-1} - kt)) \right] dt \right| \\
&\le K \sum_{i=1}^{m} \sum_{k=1}^{n+1} \binom{n+1}{k} \int_{-\pi}^{\pi} |L_w(t)| \, |f(s_i - kt) - f(s_{i-1} - kt)| \, dt \\
&= K \sum_{k=1}^{n+1} \binom{n+1}{k} \int_{-\pi}^{\pi} |L_w(t)| \sum_{i=1}^{m} |f(s_i - kt) - f(s_{i-1} - kt)| \, dt \\
&\le AK \left(2^{n+1} - 1\right) V_{2\pi}[f],
\end{aligned}$$

where K is the constant of assumption (2) and $D = KA$. Hence, passing to the supremum over all the partitions of the interval $[-\pi,\pi]$, the thesis follows. Also notice that

$$\sum_{k=1}^{n+1} \binom{n+1}{k} = 2^{n+1} - 1.$$

□

We now show that $T_{w,n}$ maps $AC_{2\pi}$ into itself.

Proposition 2. *If $f \in AC_{2\pi}$, $\{L_w\}_{w>0} \subset L^1_{2\pi}$ and (2) is satisfied, then $T_{w,n}f \in AC_{2\pi}$.*

Proof. Being $f \in AC_{2\pi}$, for every $\varepsilon > 0$ let $\delta > 0$ be the number of the absolute continuity of f. Let $\{[\alpha_i, \beta_i]\}_{i=1,...m}$ be a finite set of nonoverlapping intervals on $[-\pi,\pi]$ such that $\sum_{i=1}^{m}(\beta_i - \alpha_i) < \delta$. Now for such a colllection of intervals we have as in Proposition 1

$$\begin{aligned}
&\sum_{i=1}^{m} |(T_{w,n}f)(\beta_i) - (T_{w,n}f)(\alpha_i)| \\
&= \sum_{i=1}^{m} \left| \sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} \int_{-\pi}^{\pi} L_w(t) \left[H_w(f(\beta_i - kt)) - H_w(f(\alpha_i - kt)) \right] dt \right| \\
&= K \sum_{k=1}^{n+1} \binom{n+1}{k} \int_{-\pi}^{\pi} |L_w(t)| \sum_{i=1}^{m} |f(\beta_i - kt) - f(\alpha_i - kt)| \, dt
\end{aligned}$$

Being $\sum_{i=1}^{m} [(\beta_i - kt) - (\alpha_i - kt)] < \delta$, since $f \in AC_{2\pi}$, we obtain that

$$\sum_{i=1}^{m} |(T_{w,n}f)(\beta_i) - (T_{w,n}f)(\alpha_i)| \le \left(2^{n+1} - 1\right) K\varepsilon \, \|L_w\|_1,$$

which proves the assertion, since $\varepsilon > 0$ is arbitrary. □

Proposition 3. *If $f \in BV_{2\pi}$, $\{L_w\}_{w>0} \subset AC_{2\pi}$ and (2) is satisfied, then $T_{w,n}f \in AC_{2\pi}$.*

Proof. We may write, putting $kt = s - z$ and being f and $\{L_w\}_{w>0}$ $2\pi-$periodic functions,

$$\begin{aligned}
(T_{w,n}f)(s) &= -\sum_{k=1}^{n+1}(-1)^k\binom{n+1}{k}\int_{-\pi}^{\pi}K_w(t,f(s-kt))\,dt \\
&= -\sum_{k=1}^{n+1}(-1)^k\binom{n+1}{k}\int_{-\pi}^{\pi}L_w(t)H_w(f(s-kt))\,dt \\
&= -\sum_{k=1}^{n+1}\frac{(-1)^k}{k}\binom{n+1}{k}\int_{s-k\pi}^{s+k\pi}L_w\left(\frac{s-z}{k}\right)H_w(f(z))\,dz \\
&= -\sum_{k=1}^{n+1}\frac{(-1)^k}{k}\binom{n+1}{k}\int_{-\pi}^{\pi}L_w\left(\frac{s-z}{k}\right)H_w(f(z))\,dz
\end{aligned}$$

Therefore, being $\{L_w\}_{w>0} \subset AC_{2\pi}$, for every $\varepsilon > 0$ let $\delta > 0$ be the number of the absolute continuity of L_w. Let $\{[\alpha_i,\beta_i]\}_{i=1,\ldots,m}$ be a finite set of nonoverlapping intervals on $[-\pi,\pi]$ such that $\sum_{i=1}^{m}(\alpha_i - \beta_i) < \delta$. Now for such a colllection of intervals we have, being $H_w(0) = 0$,

$$\begin{aligned}
&\textstyle\sum_{i=1}^{m}|(T_{w,n}f)(\beta_i) - (T_{w,n}f)(\alpha_i)| \\
&= \sum_{i=1}^{m}\left|\sum_{k=1}^{n+1}\frac{(-1)^k}{k}\binom{n+1}{k}\left[\int_{\pi}^{\pi}L_w\left(\frac{\alpha_i - z}{k}\right)H_w(f(z))\,dz - \int_{\pi}^{\pi}L_w\left(\frac{\beta_i - z}{k}\right)H_w(f(z))\,dz\right]\right| \\
&\leq \sum_{i=1}^{m}\sum_{k=1}^{n+1}\binom{n+1}{k}\int_{\pi}^{\pi}|H_w(f(z))|\left|L_w\left(\frac{\alpha_i - z}{k}\right) - L_w\left(\frac{\beta_i - z}{k}\right)\right|dz.
\end{aligned}$$

Now, by the fact that $\{L_w\}_{w>0} \subset AC_{2\pi}$ since $\sum_{i=1}^{m}\left[\left(\frac{\alpha_i - z}{k}\right) - \left(\frac{\beta_i - z}{k}\right)\right] < \delta$, we have

$$\begin{aligned}
&= K\left(2^{n+1}-1\right)\varepsilon\int_{\pi}^{\pi}|f(z)|\,dz \\
&= K\left(2^{n+1}-1\right)\varepsilon\,\|f\|_1
\end{aligned}$$

and therefore the assertion follows being $f \in BV_{2\pi}$ (which implies that $\|f\|_1 < \infty$) and by the arbitrariety of $\varepsilon > 0$. ☐

Bellow, we give the following result related to global smoothness preservation for $T_{w,n}$. We note here that similar results for some linear convolution operators can be seen in [3].

Theorem 1. *Let $f \in AC_{2\pi}$, and $T_{w,n}$ be the operator given by (3). Then we have*

$$\omega_r(T_{w,n};\delta) \leq (2^{n+1}-1)KA\,\omega_r(f;\delta),$$

where ω_r is given by (5).

Proof. Taking into account of (4) and (3) we have

$$\begin{aligned}
(\Delta_t^r T_{w,n}f)(s) &= \sum_{v=0}^{r}(-1)^{r-v}\binom{r}{v}(T_{w,n}f)(s-vt) \\
&= -\sum_{v=0}^{r}(-1)^{r-v}\binom{r}{v}\sum_{k=1}^{n+1}(-1)^k\binom{n+1}{k}\int_{-\pi}^{\pi}L_w(y)H_w(f(s-vt-ky))\,dy.
\end{aligned}$$

Let $\{s_0 = -\pi, ..., s_m = \pi\}$ be a partition of $[-\pi, \pi]$. It follows that

$$\begin{aligned}
& \sum_{i=1}^{m} \left| (\Delta_t^r T_{w,n} f)(s_i) - (\Delta_t^r T_{w,n} f)(s_{i-1}) \right| \\
= & \sum_{i=1}^{m} \left| \sum_{v=0}^{r} (-1)^{r-v} \binom{r}{v} (T_{w,n} f)(s_i - vt) - \sum_{v=0}^{r} (-1)^{r-v} \binom{r}{v} (T_{w,n} f)(s_{i-1} - vt) \right| \\
= & \sum_{i=1}^{m} \left| \sum_{v=0}^{r} (-1)^{r-v} \binom{r}{v} \sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} \int_{-\pi}^{\pi} L_w(y) H_w(f(s_i - vt - ky))\, dy \right. \\
& \left. - \sum_{v=0}^{r} (-1)^{r-v} \binom{r}{v} \sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} \int_{-\pi}^{\pi} L_w(y) H_w(f(s_{i-1} - vt - ky))\, dy \right| \\
= & \int_{-\pi}^{\pi} |L_w(y)| \sum_{i=1}^{m} \\
& \left| \sum_{v=0}^{r} (-1)^{r-v} \binom{r}{v} \sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} \{H_w(f(s_i - vt - ky)) - H_w(f(s_{i-1} - vt - ky))\} \right| dy \\
\leq & \sum_{k=1}^{n+1} \binom{n+1}{k} \int_{-\pi}^{\pi} |L_w(y)| \sum_{i=1}^{m} \left| (\Delta_t^r)[H_w(f(s_i - ky))] - (\Delta_t^r)[H_w(f(s_{i-1} - ky))] \right| dy \\
\leq & K \sum_{k=1}^{n+1} \binom{n+1}{k} \int_{-\pi}^{\pi} |L_w(y)| \sum_{i=1}^{m} \left| (\Delta_t^r) f(s_i - ky) - (\Delta_t^r) f(s_{i-1} - ky) \right| dy \\
\leq & KA(2^{n+1} - 1) V_{2\pi}[\Delta_t^r f] \\
\leq & KA(2^{n+1} - 1) \omega_r(f; \delta)
\end{aligned}$$

where we have passed to the supremums over all the partitions of $[-\pi, \pi]$ and $|t| \leq \delta$. $\square$

Theorem 2. *Let $f \in AC_{2\pi}$ and $\{K_w\}_w \subset \kappa_w$. Then*

$$\lim_{w \to \infty} V_{2\pi}[T_{w,n} f - f] = 0.$$

Proof. Let $\{s_0, ..., s_m\}$ be a partition of $[-\pi, \pi]$. Then we can write

$$\begin{aligned}
& \sum_{i=1}^{m} \left| f(s_i) - (T_{w,n} f)(s_i) - f(s_{i-1}) + (T_{w,n} f)(s_{i-1}) \right| \\
= & \sum_{i=1}^{m} \left| f(s_i) + \sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} \int_{-\pi}^{\pi} L_w(t) H_w(f(s_i - kt))\, dt \right. \\
& \left. - f(s_{i-1}) - \sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} \int_{-\pi}^{\pi} L_w(t) H_w(f(s_{i-1} - kt))\, dt \right|
\end{aligned}$$

$$\begin{aligned}
= & \sum_{i=1}^{m} \left| \sum_{k=1}^{n+1} \left[(-1)^k \binom{n+1}{k} \int_{-\pi}^{\pi} L_w(t) \left[\begin{array}{c} H_w(f(s_i - kt)) - f(s_i) \\ -H_w(f(s_{i-1} - kt)) + f(s_{i-1}) \end{array} \right] dt \right] \right. \\
& \left. + \sum_{k=1}^{n+1} \left[(-1)^k \binom{n+1}{k} A_w f(s_i) - A_w f(s_{i-1}) \right] + f(s_i) - f(s_{i-1}) \right|
\end{aligned}$$

$$
\begin{aligned}
= & \sum_{i=1}^{m}\left|\sum_{k=1}^{n+1}(-1)^k\binom{n+1}{k}\int_{-\pi}^{\pi}L_w(t)\left[\begin{array}{c} H_w(f(s_i-kt))-H_w(f(s_i)) \\ -H_w(f(s_{i-1}-kt))+H_w(f(s_{i-1}))\end{array}\right]dt\right| \\
& +\sum_{i=1}^{m}\left|\sum_{k=1}^{n+1}(-1)^k\binom{n+1}{k}\int_{-\pi}^{\pi}L_w(t)\left[H_w(f(s_i))-f(s_i)-H_w(f(s_{i-1}))+f(s_{i-1})\right]dt\right| \\
& +\sum_{i=1}^{m}\left|\sum_{k=1}^{n+1}\left[(-1)^k\binom{n+1}{k}(A_wf(s_i)-A_wf(s_{i-1}))\right]+f(s_i)-f(s_{i-1})\right| \\
\leq & \sum_{k=1}^{n+1}\binom{n+1}{k}\sum_{i=1}^{m}\int_{-\pi}^{\pi}|L_w(t)|\left|\begin{array}{c} H_w(f(s_i-kt))-H_w(f(s_{i-1}-kt)) \\ -(H_w(f(s_i))-H_w(f(s_{i-1})))\end{array}\right|dt \\
& +\sum_{k=1}^{n+1}\binom{n+1}{k}\sum_{i=1}^{m}\int_{-\pi}^{\pi}|L_w(t)|\left|H_w(f(s_i))-f(s_i)-H_w(f(s_{i-1}))+f(s_{i-1})\right|dt \\
& +\sum_{i=1}^{m}\left[\sum_{k=1}^{n+1}\binom{n+1}{k}|A_wf(s_i)-A_wf(s_{i-1})|\right]+|f(s_i)-f(s_{i-1})|:=I_1+I_2+I_3
\end{aligned}
$$

Let us first study I_1. There holds, for any fixed $\delta>0$,

$$
\begin{aligned}
I_1 = & \sum_{k=1}^{n+1}\binom{n+1}{k}\left[\sum_{i=1}^{m}\int_{0\leq|t|\leq\delta}|L_w(t)|\left|\begin{array}{c} H_w(f(s_i-kt))-H_w(f(s_{i-1}-kt)) \\ -(H_w(f(s_i))-H_w(f(s_{i-1})))\end{array}\right|dt\right. \\
& \left.+\sum_{i=1}^{m}\int_{\delta\leq|t|\leq\pi}|L_w(t)|\left|\begin{array}{c} H_w(f(s_i-kt))-H_w(f(s_{i-1}-kt)) \\ -(H_w(f(s_i))-H_w(f(s_{i-1})))\end{array}\right|dt\right] \\
: & =I_1^1+I_1^2
\end{aligned}
$$

Morever, taking into account of (6) it follows that

$$
I_1^1\leq(2^{n+1}-1)\int_{0\leq|t|\leq\delta}|L_w(t)|V_{2\pi}\left[(H_wof)(\cdot,-kt)-(H_wof)(\cdot)\right]dt
$$

since $f\in AC_{2\pi}$, so is (H_wof), then we have

$$
V_{2\pi}\left[(H_wof)(\cdot,-kt)-(H_wof)(\cdot)\right]=\int_{-\pi}^{\pi}\left|(H_wof)'(\cdot,-kt)-(H_wof)'(\cdot)\right|ds\to 0
$$

as $t\to 0$. So, for a fixed $\varepsilon>0$ we can choose a $\delta>0$ in such that a way that $V_{2\pi}\left[(H_wof)(\cdot,-kt)-(H_wof)(\cdot)\right]\leq\varepsilon$ for $0\leq|t|\leq\delta$, which implies that,

$$
I_1^1\leq(2^{n+1}-1)\varepsilon\int_{0\leq|t|\leq\delta}|L_w(t)|\,dt
$$

Therefore we get that $I_1^1<(2^{n+1}-1)A\varepsilon$. For I_1^2, we can write that

$$
\begin{aligned}
I_1^2 \leq & (2^{n+1}-1)\int\limits_{\delta\leq|t|\leq\pi}|L_w(t)|\sum_{i=1}^{m}|H_w(f(s_i-kt))-H_w(f(s_{i-1}-kt))|\,dt \\
& +(2^{n+1}-1)\int\limits_{\delta\leq|t|\leq\pi}|L_w(t)|\sum_{i=1}^{m}|H_w(f(s_i))-H_w(f(s_{i-1}))|\,dt \\
\leq & K(2^{n+1}-1)\left\{\int\limits_{\delta\leq|t|\leq\pi}|L_w(t)|\sum_{i=1}^{m}|(f(s_i-kt))-(f(s_{i-1}-kt))|\,dt\right. \\
& \left.+\int\limits_{\delta\leq|t|\leq\pi}|L_w(t)|\sum_{i=1}^{m}|(f(s_i))-(f(s_{i-1}))|\,dt\right\}
\end{aligned}
$$

$$\leq \quad 2K(2^{n+1}-1)V_{2\pi}[f] \int\limits_{\delta\leq|t|\leq\pi} |L_w(t)|\,dt$$

From $K_w.2)$, $I_1^2 \leq 2K(2^{n+1}-1)V_{2\pi}[f]\,\varepsilon$. By $K_w.3)$, about I_2, being $f \in BV_{2\pi}$, f is bounded and so there exists a bounded interval $J \subset \mathbb{R}$ such that

$$\begin{aligned} I_2 &\leq (2^{n+1}-1)\int\limits_{-\pi}^{\pi} |L_w(t)| \frac{V_J[G_w]}{m(J)}dt \\ &= (2^{n+1}-1)\,\|L_w\|_1 \frac{V_J[G_w]}{m(J)} \\ &\leq A(2^{n+1}-1)\frac{V_J[G_w]}{m(J)} \leq (2^{n+1}-1)A\varepsilon. \end{aligned}$$

Finally, for $I_{3,}$ from $K_w.1)$, we have for sufficiently large $w>0$,

$$I_3 \leq 2^{n+1}V_{2\pi}[f]\,\varepsilon$$

from the above explanations, we conclude that, for sufficiently large $w>0$,

$$I \leq \varepsilon\left((2^{n+1}-1)A + 2K(2^{n+1}-1)V_{2\pi}[f]\,. + 2^{n+1}V_{2\pi}[f]\right)$$

and therefore the thesis follows being $f \in AC_{2\pi}$ and by the arbitrariety of partition $\{s_0,...,s_m\}$ and of $\varepsilon > 0$. □

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A new estimation method for regime-switching autoregressive density functions

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Abstract

Although the regime-switching autoregressive model is a popular topic in economic literature, the estimation of this model with small sample sizes has rarely been addressed in financial areas. This model, also referred to as the autoregressive hidden Markov, and its associated estimation method can also be found in speech recognition literature. After careful examination of the estimation method in speech recognition literature, we learn that a non-ignorable approximation occurs, which can only be justified under the circumstances of extremely long data sequences in the field of speech recognition. For this reason, we develop an original parsimonious estimation method for the autoregressive linear regime-switching model in economic fields. The positive semi-definite correlation matrix issue is considered and addressed in our estimation method. Stability and accuracy is also examined.

Key words: regime-switching autoregressive model, hidden Markov, financial model, speech recognition, correlation matrix adjustment

1 Introduction

The Markov-driven regime-switching autoregressive process is a bivariate process $\{(X_n, Y_n)\}$, where $\{X_n\}$ is a Markov chain and $\{Y_n\}$ is an autoregressive process. The value of Y_n only depends on previous Y_n's and current X_n. Regime refers to X_n which is a hidden and unobserved variable, while $\{Y_n\}$ is an observable variable on which deduction and inference can be made. This general regime-switching autoregressive process can be written as

$$Y_n = f(X_n, \bar{Y}_{n-1}; e_n), \tag{1.1}$$

where $\bar{Y}_{n-1} = \{Y_{n-1}, Y_{n-2}, \dots, Y_{n-p}\}$ with p as the order of autoregression. The i.i.d. process $\{e_k\}$ denotes innovation process. Markov-switching regressions were initially introduced into economics by Goldfeld and Quandt [1973], the likelihood function for which was first correctly calculated by Cosslett and Lee [1985].

Our particular interests are regime-switching linear autoregressive models

$$Y_n = \sum_{i=1}^{p} a_i(X_n; \theta) Y_{n-i} + e_n. \tag{1.2}$$

The linear model was initially brought into economics by Hamilton [1989]. The parameter estimation is calculated as a result of an iterative algorithm similar to the Kalman filter. It is observed that this numerical estimation method is subject to computational difficulties associated with the ill-behaved likelihood surface, which include multiple local maxima, essential singularities and local increases as boundary conditions are approached. The EM algorithm is used to overcome these numerical difficulties(Karlis and Xekalaki [2003], McLachlan and Peel [2000]). Most works on EM-algorithm estimation of regime-switching models in economic fields have focused on finite mixture models, with Markov-driven autoregressive models being ignored because of the computational difficulties.

The model, also referred to as the autoregressive hidden Markov model, has been considered in speech recognition literatures such as Poritz [1982], Juang and Rabiner [1991] and Rabiner [1990]. Juang and Rabiner [1985] suggests an EM-algorithm based estimation method for autoregressive hidden Markov models and also discusses the application of this model in speech recognition.

After careful examination of this estimation method, we find that a non-ignorable approximation occurs in Juang and Rabiner [1985]'s estimation method, which is not stated in the original article. This kind of approximation can only be justified with the use of extremely long data sequences which are common in the field of speech recognition. In contrast, economic data sequences are often significantly shorter, therefore causing this kind of approximation to be non-legitimate.

We develop an original parsimonious estimation method for the autoregressive linear regime-switching model in economic fields. This method produces accurate estimation results without ignoring parts of the data sequence, making it better suited for financial and economic applications than existing methods.

We apply Gram-Schmidt orthogonalization, Frobenius norm minimization, and the EM algorithm to estimate linear autoregressive regime-switching models and develop a technique to maintain a positive semidefinite correlation matrix. Stability and accuracy of this estimation method is also examined.

The remainder of this paper is organized as follows. Section 2 presents our autoregressive hidden Markov model (HMM). It discusses the limitation of the existing p.d.f. function for the autoregressive HMM, and gives improved density functions under different assumptions. Section 3 applies the EM-algorithm to estimate the transition matrix of the autoregressive HMM . It also explores the performance of this estimation method in different settings. Section 4 provides the enhanced estimation method for autoregressive coefficients. Section 5 presents a numerical example which illustrates implementation details of our new estimation method for regime-switching autoregression processes. Empirical studies give encouraging results.

2 The Model

2.1 Regime-switching autoregressive models with Gaussian innovations

Our model assumes an observation window of length K moves along the time series data with overlapping length M. Our particular interest is $M = 1$, which denotes a natural case: moving forward one step each time. We also set length $K = 5$. To be more specific, we consider the observation vector $\vec{s}_n$ with components $(x_n, x_{n+1}, \ldots, x_{n+K-1})$. The model can be written as

$$\begin{aligned} x_n &= -\sum_{i=1}^{p} a_i(z_n) x_{n-i} + e_n \qquad n = 0,1,2,\ldots,K-1 \\ s_n &= x_n \sigma(z_n) \qquad z_n = \{1,\ldots,N\}, \end{aligned} \tag{2.1}$$

where z_n denotes a random state, N is the number of states, e_k are Gaussian i.i.d. random variables with mean 0 and variance 1, and p is the order of autoregression (Nelson et al. [2001], Meyers [2010] and Lai

and Xing [2008]). Autoregressive coefficients a_i and variance σ follows discrete Markov process. In other words, the state of the Markov chain z_n affects the values of the autoregression coefficient a_i and the scaling parameter σ_i. For example, if this is a two state Markov process and the order of autoregression $p = 2$, then $a_1 \in \{a_1(1), a_1(2)\}$, $a_2 \in \{a_2(1), a_2(2)\}$, $\sigma \in \{\sigma(1), \sigma(2)\}$. This process is driven by transition matrix $\left(\begin{smallmatrix} p_1 & 1-p_1 \\ 1-p_2 & p_2 \end{smallmatrix}\right)$. Figure (1) shows the distribution of the observation x_n depends on a subset of the previous

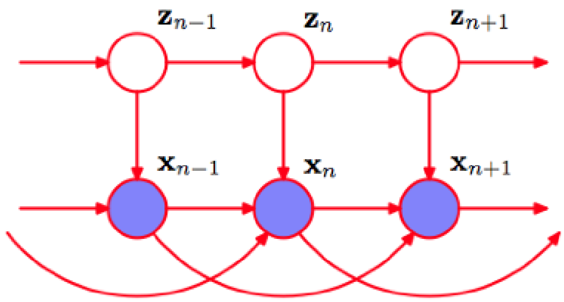


Figure 1: Illustration of autoregressive hidden Markov model.

observations as well as on the hidden state z_n. In this example, the distribution of x_n depends on the two previous observations x_{n-1} and x_{n-2}.

2.2 Analysis of existing density function of Gaussian autoregressive source

Rabiner [1990] introduces the density function of the Gaussian autoregressive source in his section on autoregressive HMMs. The density function for $\vec{s}$ is

$$f(\vec{s}) = (2\pi\sigma^2)^{-\frac{K}{2}} \exp\left(-\frac{1}{2\sigma^2}\delta(\vec{s}, a)\right), \tag{2.2}$$

where

$$\begin{aligned} \delta(\vec{s}, a) &= r_a(0)r(0) + 2\sum_{i=1}^{p} r_a(i)r(i) \\ a' &= [1, a_1, \ldots, a_p] \\ r_a(i) &= \sum_{n=0}^{p-i} a_n a_{n+i} \\ r(i) &= \sum_{n=0}^{K-i-1} x_n x_{n+i}. \end{aligned}$$

After mathematical analysis and numerical tests (See section (2.3.1) for details), we identify an approximation introduced by this density function that are not mentioned in the text. The assumption for this approximation is that the correlation of the first K observation doesn't affect the whole output. In other words, sample size T is much larger than observation window size K. In our model, observation window $K = 5$ and the length of sample size $T < 10$, thus the assumption doesn't hold for our model. We deduce a new density function for autoregressive process in following sections.

2.3 Improved density function with no structure imposed on first p sample data

We provide two improved density functions for autoregressive hidden Markov model. Both of the two density functions make no approximation, which is applicable to short sequences of financial or economic data. With regard to the first part of the sample data, the first p.d.f. assumes no particular structure while the second assumes a linear structure.

Using the Gram-Schmidt process (Golub and Van Loan [1996] and Demmel [1997]) to orthogonalize the first p samples $\{x_1 \dots x_p\}$ into $\vec{\varepsilon} = \{\varepsilon_1 \dots \varepsilon_p\}$, we may rewrite (2.1) as

$$H\vec{x} = \vec{e},$$

where $\vec{x} = \{x_1, \cdots, x_K\}$, $\vec{e} = \{\varepsilon_1, \cdots, \varepsilon_p, e_1, \cdots, e_{K-p}\}$, $\varepsilon \sim \mathbf{N}(0,1)$ and

$$H = \left(\begin{array}{cccc|cc} h_{11} & 0 & \cdots & 0 & 0 & 0 \\ h_{21} & h_{22} & \cdots & 0 & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ h_{p1} & h_{p2} & \cdots & h_{pp} & 0 & 0 \\ \hline a_p & a_{p-1} & \cdots & a_1 & 1 & 0 \\ 0 & a_p & \cdots & a_2 & a_1 & 1 \end{array}\right) = \left(\begin{array}{c|c} H_{11} & 0 \\ \hline H_{21} & H_{22} \end{array}\right). \tag{2.3}$$

To orthogonalize $\vec{x}$, without loss of generality, we assume $x_1 \sim \mathbf{N}(0,\sigma_1)$, and $x_2 \sim \mathbf{N}(0,\sigma_2)$. Since

$$\begin{aligned} &h_{11}x_1 = \varepsilon_1 \\ &h_{21}x_1 + h_{22}x_2 = \varepsilon_2, \end{aligned} \tag{2.4}$$

and $\text{var}(\varepsilon_1) = 1$, $\text{var}(\varepsilon_2) = 1$, $\text{cov}(\varepsilon_1, \varepsilon_2) = 0$, we have equations

$$\begin{aligned} &\text{Var}(h_{11}x_1) = 1 \\ &\text{Var}(h_{21}x_1 + h_{22}x_2) = 1 \\ &\text{Cov}(h_{11}x_1, h_{21}x_1 + h_{22}x_2) = 0. \end{aligned} \tag{2.5}$$

Because $x_1 \sim \mathbf{N}(0,\sigma_1)$, and $x_2 \sim \mathbf{N}(0,\sigma_2)$, equations (2.5) are equivalent to

$$\begin{aligned} &h_{11} = \frac{1}{\sigma_1} \\ &h_{21}^2\sigma_1^2 + h_{22}^2\Sigma_2^2 + 2h_{21}h_22\sigma_{12} = 1 \\ &h_{21}\sigma_1^2 + h_{22}\sigma_{12} = 0. \end{aligned}$$

The solution for this equation system is

$$\begin{aligned} h_{11} &= \frac{1}{\sigma_1^2} \\ h_{21} &= \frac{-\sigma_{12}}{\sqrt{\sigma_2^2\sigma_1^4 - \sigma_1^2\sigma_{12}^2}} \\ h_{22} &= \frac{\sigma_1}{\sqrt{\sigma_2^2\sigma_1^2 - \sigma_{12}^2}}. \end{aligned}$$

The elements of $\vec{e}$ are uncorrelated, thereby giving

$$\begin{aligned} \mathbf{I} &= \mathbf{E}\{\vec{e}\vec{e}^t\} \\ &= \mathbf{E}\{H\vec{x}\vec{x}^t H^t\} \\ &= H\mathbf{E}\{\vec{x}\vec{x}^t\}H^t \\ &=: H\Sigma_x H^t \end{aligned} \tag{2.6}$$

Equation(2.6) gives

$$\Sigma_x^{-1} = H^t H. \tag{2.7}$$

Taking the determinant of both sides, equation(2.6) leads to

$$|\Sigma_x| = |H|^{-2} = |H_{11}|^{-2}.$$

Since $\vec{x} = H^{-1}\vec{e}$ and $\vec{e}$ is Gaussian white noise , $\vec{x}$ is also multi-Gaussian. Plugging $|\Sigma_x|$ and Σ_x^{-1} into the multivariate Gaussian p.d.f. yields the p.d.f. of the autoregressive process

$$(2\pi)^{-K/2}|\Sigma_x|^{-1/2}\exp\{-\frac{1}{2}x^t\Sigma_x^{-1}x\}. \tag{2.8}$$

When unscaled $\vec{s}$ is used

$$\begin{aligned} \mathbf{I} &= \mathbf{E}\{\vec{e}\vec{e}^t\} \\ &= \mathbf{E}\{H\frac{\vec{s}}{\sigma}\frac{\vec{s}}{\sigma}^t H^t\}. \end{aligned} \tag{2.9}$$

Equation(2.6) gives

$$\Sigma_s^{-1} = \sigma^{-2}H^t H.$$

Taking determinant of both sides, equation(2.9) leads to

$$|\Sigma_s| = \sigma^{2K}|H_{11}|^{-2}.$$

2.3.1 When sample size $T \gg$ observation window size K

Equation (2.2) is the approximate density function for $\vec{s}$. We present an example with dimension equal to 3 to illustrate the difference between Equation (2.2) and Equation (2.8). We get Σ_x from Equation (2.7), then compute the p.d.f function from Equation (2.8). We only show part of computation results as

```
/*Mathematica code */
input:
H  = {{h11, 0, 0}, {h21, h22, 0}, {a2, a1, 1}};
x = {x1, x2, x3};
x.Transpose[H].H.x // Simplify
output:
a2^2 x1^2 + h11^2 x1^2 + h21^2 x1^2 + 2 h21 h22 x1 x2 +
a1^2 x2^2 + h22^2 x2^2 + 2 a1 x2 x3 + x3^2 +
2 a2 x1 (a1 x2 + x3).
```

Comparing the expansion of $x^t\Sigma_x^{-1}x$ with Equation (**??**), we can see the cross terms of a_i and x_i are missing in Equation (**??**). The assumption for this approximation is that the correlation of the first K observations doesn't affect the whole output, which is true when sample size T is much larger than observation window size K.

2.3.2 When sample size $T \gg$ observation window size K doesn't hold

Based on Equation (2.8), We can compute accurate form of density function for $\vec{s}$ as follow:

1. Compute $\Sigma_{\vec{s}} = \mathbf{E}(\vec{s}\vec{s}^T)$ with $\vec{s} = \{s_1, \ldots, s_K\}$

2. Use Eigenvalue Decomposition to get

$$B^T \Sigma_{\vec{s}} B = \beta = \begin{pmatrix} \beta_0 & & \cdots & 0 \\ & \beta_1 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & \beta_{K-1} \end{pmatrix}$$

   where B is an upper triangular matrix, the diagonal elements of which are all unity.

3. Get the probability density

$$f(x \mid \Sigma_{\vec{s}}) = (2\pi)^{-K/2} (\sigma^2)^{-(K-p)/2} \left(\prod_{i=0}^{p-1} \frac{\beta_i}{\sigma^2} \right)^{-\frac{1}{2}} \exp\{-\vec{s}^t H^t H \vec{s}/(2\sigma^2)\}. \tag{2.10}$$

2.3.3 Numerical example: testing integral of density function

This numerical test is to examine whether

$$\int_{x_1=-\infty}^{\infty} \int_{x_2=-\infty}^{\infty} \cdots \int_{x_T=-\infty}^{\infty} f(\vec{x}) == 1$$

holds, where $f(\vec{x})$ are p.d.f. functions in Equation (2.2) and (2.10). We set two sets of parameters as $a = [1, 0.5, 0.3], p = 2, K = 3, T = 3, \sigma = 0.2$ and $a = [1, 0.8, 0.4], p = 2, K = 3, T = 5, \sigma = 0.1$. The results in Table (1) confirm our argument that density function (2.2) doesn't hold when length of observation window K is not significantly smaller than data size T. Our new density function (2.10) passes the integration test.

Table 1: Integration of density function of observation $\vec{x}$ over $\{-\infty, \infty\}$.

	density function (2.2)	density function (2.10)
parameter set 1	∞	1
parameter set 2	∞	1

2.4 Improved density function with linear autoregressive structure imposed on first p sample data

To address the problem of (2.1) being undefined for the first p samples, we may introduce some ghost variables and rewrite (2.1) as

$$\begin{aligned} x_1 + a_1 x_0 + a_2 x_{-1} &= \varepsilon_1 \\ x_2 + a_1 x_1 + a_2 x_0 &= \varepsilon_2 \\ x_3 + a_1 x_2 + a_2 x_1 &= \varepsilon_3 \\ &\vdots \\ x_K + a_1 x_{K-1} + a_2 x_{K-2} &= \varepsilon_K \end{aligned}$$

where ε_i are i.i.d. $\mathbf{N}(0,1)$.

The parametres are $\{a_1, a_2, \ldots, a_p, x_0, x_{-1}, \ldots, x_{-p}, \sigma\}$. Variables x_0 and x_{-1} are ghost variables, which can be treated as scalars. We compare the degrees of freedom between (2.3) and (2.1) in Table (2), which shows the degree of freedom for the method with linear structure increases much slower than the one without linear structure as autoregressive order p increases. Small number of degree of freedom makes our model a parsimonious model which is preferred.

Table 2: Comparison of degree of freedom between with and without linear autoregressive structure imposed on first p sample data.

	Equation (2.3)	Equation (2.1)
	$\frac{1+p}{2}p + p + 1$	$2p+1$
$p=2$	6	5
$p=3$	10	7
$p=4$	15	9

We show the steps to get the density function in what follows. After normalization of Equation (2.1), we have

$$x_1 = -a_1 x_0 - a_2 x_{-1} + \varepsilon_1 \tag{2.11}$$
$$x_2 + a_1 x_1 = -a_2 x_0 + \varepsilon_2 \tag{2.12}$$
$$x_3 + a_1 x_2 + a_2 x_1 = \varepsilon_3 \tag{2.13}$$

Let $\hat{\varepsilon}$ be

$$\hat{\varepsilon} := \varepsilon + \begin{pmatrix} -a_1 x_0 - a_2 x_{-1} \\ -a_2 x_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \varepsilon + \mu,$$

therefore $\hat{\varepsilon} \sim \mathbf{N}(\mu, 1)$. We denote by H the coefficient matrix

$$\begin{pmatrix} 1 & & & & \\ a_1 & 1 & & & \\ a_2 & a_1 & 1 & & \\ 0 & a_2 & a_1 & 1 & \\ \vdots & & & & \\ 0 & 0 & a_2 & a_1 & 1 \end{pmatrix},$$

so

$$Hx = \vec{e}$$

Taking the expectation of

$$\hat{e}\hat{e}^t = (e+\mu)(e+\mu)^t = ee^t + \mu e^t + e\mu^t + \mu\mu^t,$$

yields

$$\mathbf{E}(\hat{e}\hat{e}^t) = \mathbf{E}(ee^t) + \mu\mu^t = \mathbf{I} + \mu\mu^t.$$

On the other hand, taking the expectation of (2.11) leads to

$$H\mathbf{E}(xx^t)H^t = \mathbf{E}(\hat{e}\hat{e}^t) = \mu\mu^t + \mathbf{I},$$

therefore

$$\Sigma_x = H^{-1}(\mu\mu^t + \mathbf{I})(H^t)^{-1}.$$

Since

$$\mu\mu^t + \mathbf{I} = \begin{pmatrix} \mu_1^2+1 & \mu_1\mu_2 & \dots & 0 \\ \mu_1\mu_2 & \mu_2^2+1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix},$$

we have

$$|\Sigma_x| = |H^{-1}||(\mu\mu^t + \mathbf{I})||(H^t)^{-1}| = |\mu\mu^t + \mathbf{I}| = (\mu_1^2+1)(\mu_2^2+1) - 2\mu_1\mu_2.$$

3 Estimation of transition matrix

The Baum-Welch algorithm is a particular case of a generalized expectation-maximization (EM) algorithm, which can compute transition and emission probabilities for a hidden Markov model, when only emission probabilities are given. This algorithm is introduced in Baum et al. [1970], Welch [2003], Baggenstoss [2001] and Jelinek [1998]. The modified algorithms have been applied to estimate parameters of hidden Markov model with discrete and Gaussian mixture observations (Bilmes et al. [1998]). In this section, we discuss our improved Baum-Welch algorithm for the autoregressive hidden Markov model.

3.1 Improved Baum-Welch algorithm for the autoregressive HMM model

The Markov system can be described at any time by two variables: observation o_t and state S_t. Variable o_t is observable and S_t is latent. The parameters for the HMM are $\{A, B, \pi\}$, where $A = \{a_{ij}\}$ is the transition matrix, $B = \{b_i(x_t)\}$ is emission probability and π_i is the initial distribution. Two intermediate variables need to be defined first:

$$\alpha_t(i) = P(o_1 = x_1, \dots, o_t = x_t, S_t = i)$$

and

$$\beta_t(j) = P(o_{t+1} = x_{t+1}, \dots, o_T = x_T \mid S_t = i).$$

Variable $\alpha_t(i)$ can be calculated with the forward method (Rabiner [1990], Frey [2010]) with two steps: initialization step

$$\alpha_1(i) = \pi(i) b_i(x_1)$$

and induction step

$$\alpha_{t+1}(j) = (\sum_{i=1}^{N} \alpha_t(i) a_{ij}) b_j(x_{t+1}) \qquad \text{for} \quad t = 1, \dots, T-1. \tag{3.1}$$

Variable $\beta_t(j)$ can be calculated with backward method also with two steps: initialization step

$$\beta_T(i) = 1$$

and induction step

$$\beta_t(j) = (\sum_{i=1}^{N} \beta_{t+1}(i) a_{ij}) b_j(x_{t+1}) \qquad \text{for} \quad t = T-1, \dots, 0. \tag{3.2}$$

Then with the E-step, we can get

$$\begin{aligned} \xi_t(i,j) &= P(q_t = i, q_{t+1} = j \mid X, \theta) \\ &= \frac{P(q_t = i, q_{t+1} = j, X \mid \theta)}{P(X \mid \theta)} \\ &= \frac{\alpha_t(i) \beta_{t+1}(j) a_{ij} b_j(x_{t+1})}{\sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_t(i) \beta_{t+1}(i) a_{ij} b_j(x_{t+1})}. \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \gamma_t(i) &= P(q_t = i \mid X, \theta) \\ &= \sum_{j=1}^{N} \xi_t(i,j) \\ &= \frac{\alpha_t(i) \beta_t(i)}{\sum_{j=1}^{N} \alpha_t(j) \beta_t(j)}. \end{aligned} \tag{3.4}$$

The second equation holds because

$$\beta_t(j) = (\sum_{i=1}^{N} \beta_{t+1}(i) a_{ij}) b_j(x_{t+1}).$$

After computing those intermediate variables, we estimate mean vector μ_i and covariance matrix Σ_i from sample data as

$$\mu_i = \frac{\sum_{t=1}^{T} x_t \gamma_t(i)}{\sum_{t=1}^{T} \gamma_t(i)}$$

and

$$\Sigma_i = \frac{\sum_{t=1}^{T} (x_t - \mu_i)(x_t - \mu_i)' \gamma_t(i)}{\sum_{t=1}^{T} \gamma_t(i)}, \tag{3.5}$$

then use algorithm (4) to estimate H_i from Σ_i. The density function for observation $\vec{x}$ is defined by Equation (2.8), so the emission probability is

$$b_t(i) = (2\pi)^{-K/2} |\Sigma_i|^{-1/2} \exp\{-\frac{1}{2} x^t \Sigma_i^{-1} x\},$$

where

$$\Sigma_i^{-1} = H_i^t H_i \qquad \text{and} \qquad |\Sigma_i| = |H_i|^{-2}.$$

The pseudocode below presents the algorithm of applying Balm Welch to autoregressive HMM.

Algorithm 1 HMM Forward.

1: Initialize: $t \leftarrow 0$, a_{ij}, b_j, visible sequence $\vec{o}$, $\alpha_j(0)$
2: **repeat**
3: $\quad t \leftarrow t+1$
4: $\quad \alpha_j(t) \leftarrow b_j(o_t) \sum_{i=1}^{M} \alpha_i(t-1) a_{ij}$
5: **until** $t = T$
6: **return** $\alpha_j(T)$ for the final state

Algorithm 2 HMM Backward.

1: Initialize: $t \leftarrow T$, a_{ij}, b_j, visible sequence $\vec{o}$, $\beta_j(T)$
2: **repeat**
3: $\quad t \leftarrow t-1$
4: $\quad \beta_i(t) \leftarrow \sum_{j=1}^{M} \beta_j(t+1) a_{ij} b_j o_{t+1}$
5: **until** $t = 1$
6: **return** $\beta_i(0)$ for the known initial state.

3.2 Implementation Issues

The first issue is how to select the number of states. The Bayesian Information Criterion (BIC) is the log likelihood penalized for the number of free parameters (McLachlan and Peel [2000]). The model of the smallest BIC is preferred. As BIC gives whole information of the model, we choose BIC to decide number of states.

The second issue is how to initialize and terminate the algorithm (Biernacki et al. [2003] and McKenzie and Alder [1993]). This has three sub-issues: generating an initial estimate of the parameters, terminating the algorithm, and avoiding local maxima of the log likelihood.

Algorithm 3 HMM EM algorithm.

1: Initialize: a_{ij}, b_j, training sequence $\vec{x}$, convergence criterion θ, $z \leftarrow 0$
2: **repeat**
3: $\quad z \leftarrow z+1$
4: $\quad$ Compute $\alpha_i(t)$ by forward algorithm (1).
5: $\quad$ Compute $\beta_i(t)$ by backward algorithm (1).
6: $\quad$ Compute sufficient statistics $\xi_{i,j}(t)$ from $\alpha_i(t)$, $\beta_i(t)$ and $b(z-1)$ by Eq. (3.3)
7: $\quad$ Compute sufficient statistics $\gamma_i(t)$ from $\alpha_i(t)$, $\beta_i(t)$ and $b(z-1)$ by Eq. (3.4)
8: $\quad$ Update transition matrix $a(z)$ from $a(z-1)$, $\xi_{i,j}(t)$ and $\gamma_i(t)$ by Eq. (**??**)
9: $\quad$ Estimate covariance matrix Σ_i for each state i by Eq. (3.5)
10: $\quad$ Compute H_i for each state i by Algorithm (4)
11: $\quad$ Compute emission probabilities $b(z)$ by Eq (3.1)
12: **until** $z = T$
13: **return**

The EM algorithm starts with an initial guess of the parameters. The algorithm can get stuck in a local maximum, so the choice of an initial parameter is more than just an issue of efficiency. Several approaches have been suggested. Most termination criteria do not detect convergence per se but rather lack of progress, and the likelihood function has "flat" regions that can lead to premature termination.

Therefore, it makes sense to make the termination criteria reasonably strict, and it also makes sense to start the algorithm at multiple starting points. An approach suggested in Karlis and Xekalaki [2003] is to run multiple starting points for a limited number of iterations, pick the one with the highest likelihood and then run that choice using a fairly tight termination tolerance. This is the approach taken in the demonstrations below.

The third issue is the scaling issue. Considering the forward recursion

$$\alpha_{t+1}(j) = \left(\sum_{i=1}^{N} \alpha_t(i) a_{ij} \right) b_j(x_{t+1}) \qquad \text{for} \quad t = 1, \ldots, T-1,$$

repeated multiplication by b_j at each time step can cause serious computational problems. The solution, as discussed in Rabiner [1990], is to scale the computations in a manner which will still allow one to use the same forward-backward recursions.

3.3 Numerical example: performance of Balm-Welch estimator

Assume the transition matrix is $\begin{pmatrix} p_1 & 1-p_1 \\ 1-p_2 & p_2 \end{pmatrix}$. We set up our testing cases as: $p_1 = (0.95, 0.8, 0.5)$, $p_2 = (0.8, 0.5, 0.3)$, $\mu = (0.5, -0.3)$, $\sigma = (0.5, 0.8)$ and sample size $T = 10000$.

Figure (2) shows a boxplot of Balm-Welch estimates for the hidden Markov driven Gaussian mixture model from replications 100 with sample size $T = 10000$ with parameters $\sigma_1 = 0.5$, $\sigma_2 = 0.8$, $\mu_1 = 0.5$, $\mu_2 = -0.3$, $\{p_1, p_2\} \in P$. P is a set of combinations of $\{p_1, p_2\}$. Let

$$\begin{aligned} P := &\{0.95, 0.8\}, \{0.95, 0.5\}, \{0.95, 0.3\}, \{0.8, 0.8\}, \{0.8, 0.5\}, \\ &\{0.8, 0.3\}, \{0.5, 0.8\}, \{0.5, 0.5\}, \{0.5, 0.3\}. \end{aligned}$$

We see that the estimator performs well for p_1, p_2, σ_1 and σ_2, but tends to have some bias for μ_1 and μ_2. Although this bias exists, it still delivers reasonable estimates for a number of observation as small as

100. We can also see cases with symmetric parameters, for example, $p_1 = 0.5$, $p_2 = 0.5$, have superior estimations. Figures (5a) to (5c) show number of observations = $(25, 50, 200, 3000)$ for each combination. It also demonstrates that models with symmetric parameters converge better under our algorithm.

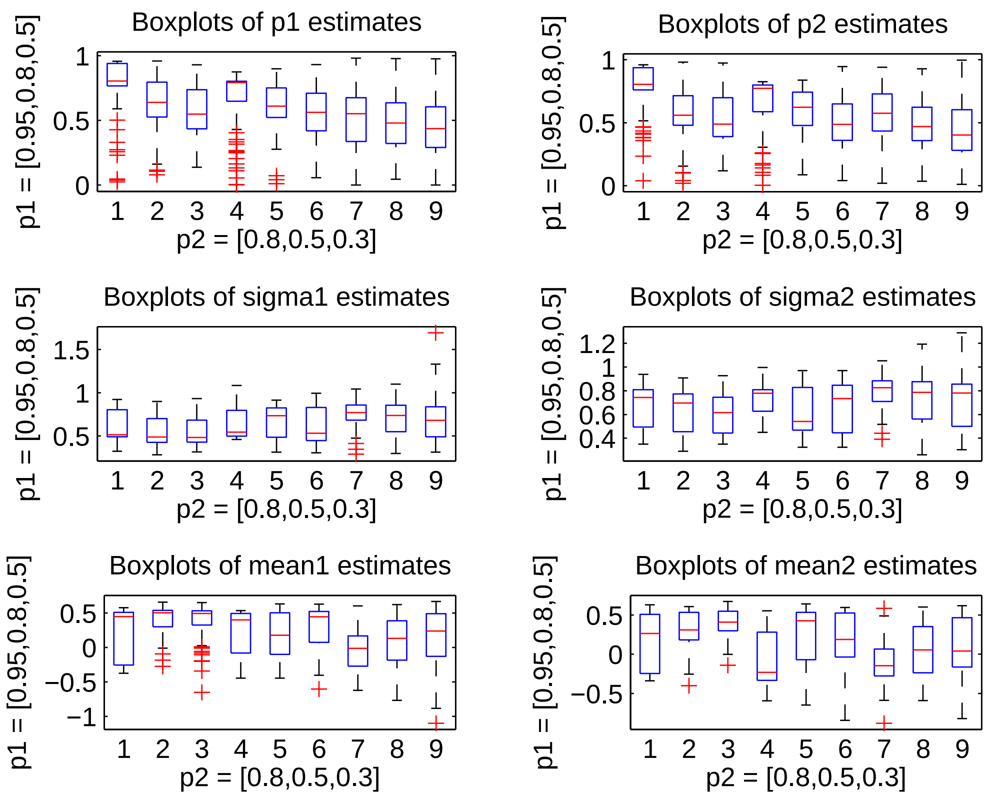
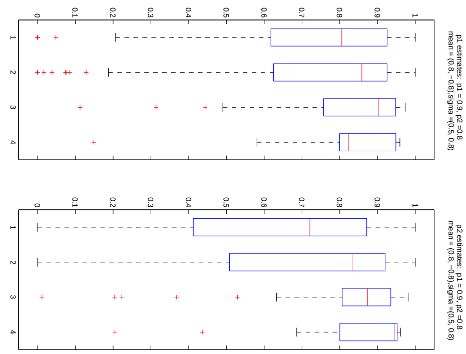


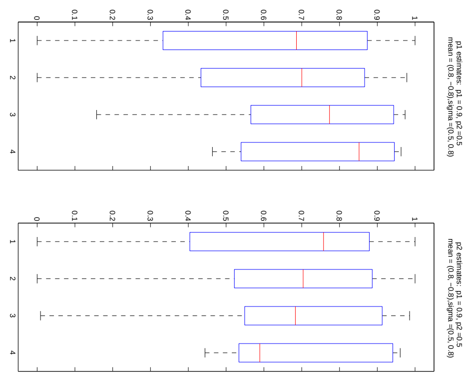
Figure 2: Boxplots of estimated p_1, p_2, σ_1, σ_2, μ_1, μ_2 for number of observations = 100 with parameters $\sigma_1 = 0.5$, $\sigma_2 = 0.8$, $\mu_1 = 0.5$, $\mu_2 = -0.3$, $\{p_1, p_2\} \in P$.

4 Estimation of autoregressive coefficients

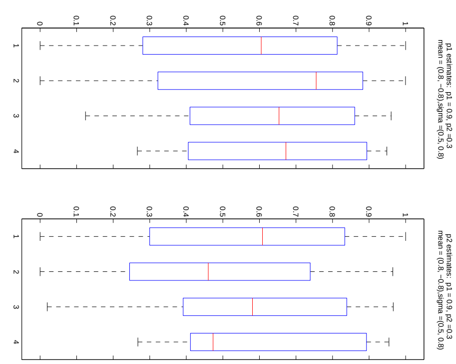
We develop three estimation methods for autoregressive coefficients. The MLE method needs approximation of the p.d.f. to some extent, thus we apply the MLE method to problems with large sample size T. The ordinary least square (OLS) method doesn't give information of distribution of data, therefore we use it to generate an initial guess. The Frobenius norm minimization method is parsimonious, stable, and shows high resolution precision according to our tests. We choose Frobenius norm minimization method as the estimation method for our model. Due to data errors, the correlation matrix we get from real data is not



(a) Estimated p_1 and p_2: $p_1 = 0.9, p_2 = 0.8, \mu = (0.8, -0.8), \sigma = (0.5, 0.8)$.

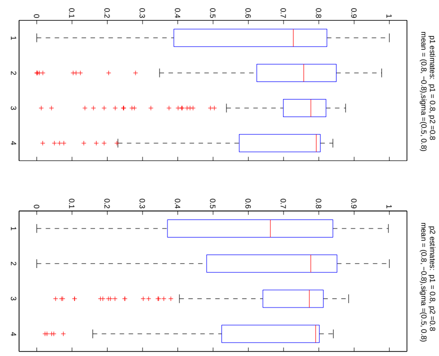


(b) Estimated p_1 and p_2: $p_1 = 0.9, p_2 = 0.5, \mu = (0.8, -0.8), \sigma = (0.5, 0.8)$.

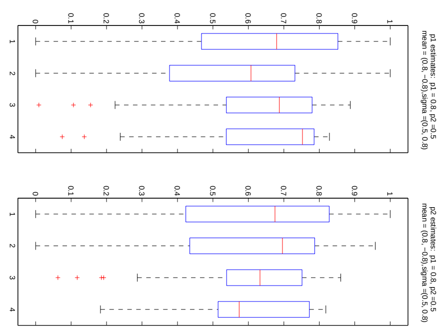


(c) Estimated p_1 and p_2: $p_1 = 0.9, p_2 = 0.3, \mu = (0.8, -0.8), \sigma = (0.5, 0.8)$.

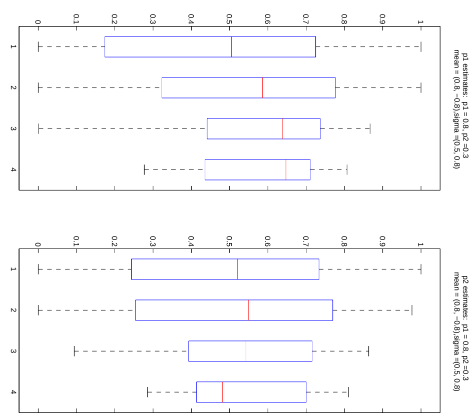
Figure 3: Boxplots of estimated p_1 and p_2 with number of observations = $(25, 50, 200, 3000)$.



(a) Estimated p_1 and p_2: $p_1 = 0.8, p_2 = 0.8, \mu = (0.8, -0.8), \sigma = (0.5, 0.8)$.

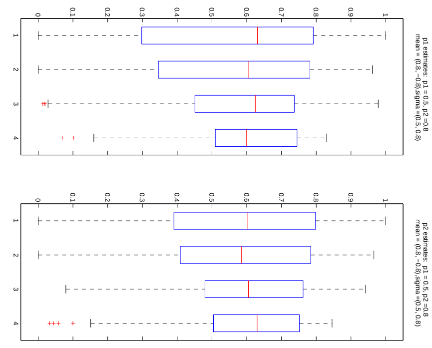


(b) Estimated p_1 and p_2: $p_1 = 0.8, p_2 = 0.5, \mu = (0.8, -0.8), \sigma = (0.5, 0.8)$.

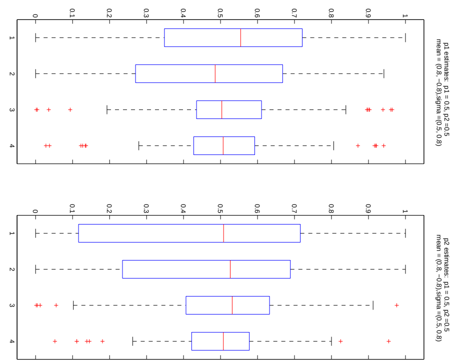


(c) Estimated p_1 and p_2: $p_1 = 0.8, p_2 = 0.3, \mu = (0.8, -0.8), \sigma = (0.5, 0.8)$.

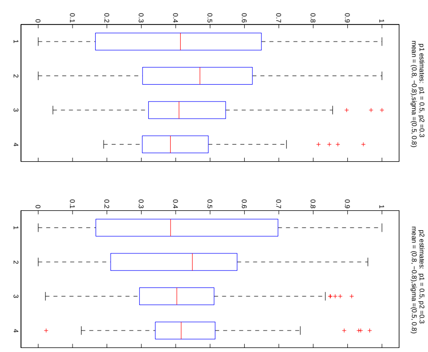
Figure 4: Boxplots of estimated p_1 and p_2 with number of observations = $(25, 50, 200, 3000)$.



(a) Estimated p_1 and p_2: $p_1 = 0.5, p_2 = 0.8, \mu = (0.8, -0.8), \sigma = (0.5, 0.8)$.



(b) Estimated p_1 and p_2: $p_1 = 0.5, p_2 = 0.5, \mu = (0.8, -0.8), \sigma = (0.5, 0.8)$.



(c) Estimated p_1 and p_2: $p_1 = 0.5, p_2 = 0.3, \mu = (0.8, -0.8), \sigma = (0.5, 0.8)$.

Figure 5: Boxplots of estimated p_1 and p_2 with number of observations = $(25, 50, 200, 3000)$.

always positive definite, thus we further provide the correlation matrix adjustment method and numerical examples.

4.1 MLE estimation with approximate expression of p.d.f.

One method to calibrate a and σ The p.d.f. function (2.2) is defined by parameter σ and a. Given a data observation vector $\vec{s} = (x_0, x_1, \ldots, x_{K-1})$, we can determine the maximum likelihood estimate of σ and a that best characterizes the observed $\vec{s}$ (Boyd and Vandenberghe [2004] and Nocedal and Wright [1999]). The log likelihood function is

$$\log f(\vec{s} \mid \sigma, a) = -\frac{K}{2}\log(2\pi\sigma^2) - \frac{\delta(\vec{s}, a)}{2\sigma^2}.$$

Instead of searching for optimal values in two dimensions, we search in one dimension first, then search in the other dimension.

$$\begin{aligned} h(\sigma, a) &:= \log f(\vec{s} \mid \sigma, a) \\ g_\sigma(a) &:= h(\sigma, a) \\ \hat{a}(\sigma) &= \arg\max_a g_\sigma(a) \\ \hat{\sigma} &= \arg\max_\sigma h(\sigma, \hat{a}). \end{aligned}$$

Therefore, the ML estimate is

$$\begin{aligned} \hat{a}_{ML} &= \arg\max_a \log f(\vec{s} \mid \sigma, a) \\ &= \arg\min_a \delta(\vec{s}, a) \\ &= \arg\min_a (a' R a) \end{aligned}$$

where $R = [r_{ij}]$ with $r_{ij} = r(|i-j|)$.

If there is no constraint for a, since R is symmetric positive definite, we have optimal result $\hat{a}^t R \hat{a} = 0$ with $\hat{a} = 0$, however, with constraint $a_0 = 1$, $\min_{\vec{a}} f(\vec{a}) = \min_{a_1, a_2, \ldots, a_{p-1}} f(\vec{a})$, which can be solved with Lagrangian multipliers.

Furthermore, we optimize object function $\log f(\vec{s} \mid \sigma, \hat{a})$ over σ

$$\min_\sigma \log f(\vec{s} \mid \sigma, \hat{a}) \Leftrightarrow \min_\sigma \left(-\frac{K}{2}\log(2\pi\sigma^2) - \frac{\delta(\vec{s}, \hat{a})}{2\sigma^2} \right)$$

Taking the derivative with respect to δ and letting $x = \sigma^2, x \geq 0$,

$$-\frac{K}{2x} + \frac{\delta}{2x^2} = 0 \Leftrightarrow -\frac{1}{2x^2}(Kx - \delta) = 0 \Leftrightarrow \hat{\sigma}^2 = \frac{\delta}{K}.$$

The estimated σ is then

$$\begin{aligned} \sigma_{ML} &= \arg\min_\sigma \log f(\vec{s} \mid \sigma, \hat{a}) \\ &= \delta(\vec{s}, \hat{a})/K. \end{aligned}$$

Alternative method to calibrate the model When $p = 2$, the objective function is

$$\begin{aligned}
&\max_{a_1,a_2} \log \prod_i^K p(e_i) \\
\Leftrightarrow &\max_{a_1,a_2} \sum_i^K \log p(e_i) \\
\Leftrightarrow &\max_{a_1,a_2} \left(K \log \frac{1}{\sqrt{2\pi}\sigma} - \frac{1}{2\sigma^2} \sum_i^K e_i^2 \right) \\
\Leftrightarrow &\min_{a_1,a_2} \sum_i^K e_i^2 .
\end{aligned}$$

This is equivalent to maximizing the probability of error term e_i appearing around mean 0. When $K = 10$,

$$\min_{a_1,a_2} \begin{pmatrix} e_1^2 \\ +e_2^2 \\ \vdots \\ +e_{10}^2 \end{pmatrix} = \begin{pmatrix} (x_2 + a_1 x_1 + a_2 x_0)^2 \\ +(x_3 + a_1 x_2 + a_2 x_1)^2 \\ \vdots \\ +(x_{10} + a_1 x_9 + a_2 x_8)^2 \end{pmatrix} =: V(a_1, a_2, x_0, \ldots, x_1 0)$$

Take $\frac{\partial V}{\partial a_1} = 0$ and $\frac{\partial V}{\partial a_2} = 0$, we can get

$$A \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = 0$$

with

$$A := \begin{pmatrix} m_1 & m_2 \\ n_1 & n_2 \end{pmatrix},$$

$$\begin{aligned}
m_1 &= x_1^2 + x_2^2 + \ldots + x_9^2 \\
&= \sum_{i=1}^9 x_i^2,
\end{aligned}$$

$$\begin{aligned}
m_2 &= x_0 x_1 + x_1 x_2 + \ldots + x_8 x_9 \\
&= \sum_{i=1}^9 x_{i-1} x_i,
\end{aligned}$$

and

$$\begin{aligned}
b_1 &= x_1 x_2 + x_2 x_3 + \ldots + x_9 x_{10} \\
&= \sum_{i=1}^9 x_i x_{i+1}
\end{aligned}$$

In similar way, we can get

$$n_1 = \sum_{i=1}^{9} x_{i-1}x_i$$
$$n_2 = \sum_{i=0}^{8} x_i^2$$
$$b_2 = \sum_{i=0}^{8} x_i x_{i+2},$$

then we have estimated

$$\begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix} = -A^{-1} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

With estimated $\{\hat{a}_1, \hat{a}_2\}$, we can get $\hat{e}$. We find the optimal σ by searching in the other direction:

$$\max_{\sigma} \left(K \log \frac{1}{\sqrt{2\pi}\sigma} - \frac{1}{2\sigma^2} \sum_{i}^{K} \hat{e}_i^2 \right)$$
$$\Leftrightarrow \min_{\sigma} \left(K \log \sqrt{2\pi}\sigma + \frac{1}{2\sigma^2} \sum_{i}^{K} \hat{e}_i^2 \right)$$
$$\Leftrightarrow \hat{\sigma} = \left(\frac{\sum_i \hat{e}_i^2}{K \log \sqrt{2\pi}}\right)^{1/3}$$

4.2 Estimation of coefficients matrix with ordinary least squares

This section discusses the application of ordinary least squares (OLS) (Lawson and Hanson [1995] and Montgomery et al. [2012]) to estimate autoregressive coefficients. For example, suppose the order of regression $p = 2$, observation window size $K = 5$, sample size $T = 10$, observations of stock returns $x_1, x_2, \ldots, x_{10}$. We wish to predict tomorrow's stock return based on today's and yesterday's return. Thus we write this problem as

$$x_3 = -a_1x_2 - a_2x_1 + e_3$$
$$x_4 = -a_1x_3 - a_2x_2 + e_4$$
$$\vdots$$
$$x_{10} = -a_1x_9 - a_2x_8 + e_{10}.$$

We want to minimize

$$e := \begin{pmatrix} e_3 \\ e_4 \\ \vdots \\ e_{10} \end{pmatrix} = \begin{pmatrix} x_2 & x_1 \\ x_3 & x_2 \\ \vdots & \\ x_9 & x_8 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} x_3 \\ x_4 \\ \vdots \\ x_{10} \end{pmatrix} := Xa - x.$$

Let $\hat{a}$ be the solution of the least square problem, which is also the estimated autoregressive coefficients. We still need to estimate σ for our model. We get estimated $\hat{\sigma} = \frac{\sigma_{imp}}{\sqrt{T}}$, where σ_{imp} is implied volatility from one year European option. The next example justifies this estimation method.

For example, s is daily return of stock, σ is standard deviation of daily returns, σ_{imp} is implied volatility of one year option, $T = 250$, $\Delta t = 1/250$. Since

$$\frac{ds}{s} = \mu dt + \sigma dW_t,$$

therefore

$$\frac{s_t - s_{t-1}}{s_{t-1}} = \mu \Delta t + \sigma_{imp}\sqrt{\Delta t} z_t,$$

mean μ is not stochastic but determinant, $\Delta t = 1/250 = 0.004$, $\sqrt{\Delta t} = 0.0632$, so the first term on the right hand side can be ignored. In other words, return of stock $r_t = \frac{s_t - s_{t-1}}{s_{t-1}} \sim z_t$ with standard deviation $\sigma_{imp}\sqrt{\Delta t}$.

The shortcoming of this estimation method is that it doesn't give information about the distribution, therefore it is used to provide an initial guess.

4.3 Estimation of coefficients matrix H by minimizing Frobenius norm

The assumption for this method is that $\vec{x}$ are correlated multivariate normal variables and $\vec{e}$ are i.i.d. $\mathbf{N}(0,1)$ random variables. We have

$$H\vec{x} = \vec{e}$$

with

$$H = \left(\begin{array}{cc|ccc} h_{11} & 0 & & 0 & \\ h_{21} & h_{22} & & & \\ \hline \vdots & \vdots & \ddots & & 0 \\ 0 & a_2 & a_1 & 1 & \\ 0 & 0 & a_2 & a_1 & 1 \end{array}\right).$$

First, we estimate covariance matrix Σ_x from sample data x.

Then, based on equation(2.6), we know

$$\Sigma_x^{-1} = H^t H,$$

so we use Cholesky decomposition to get Σ_x

$$\begin{aligned} U^t U &= \mathrm{Chol}(\Sigma_x) \\ \tilde{H}^{-1} &= U^t. \end{aligned}$$

Last, we minimize the Frobenius norm of $\tilde{H} - H$

$$\begin{aligned} \varepsilon_{ij} &= \tilde{H}_{ij} - H_{ij} \\ a^* &= \min_{a_1,a_2}\left(\sum\nolimits_{1<i,j<K} \varepsilon_{ij}^2\right) \end{aligned}$$

Algorithm 4 Estimation of autocorrelation coefficients.

1: Given $\vec{x} = x_1^{(i)}, \cdots,, x_K^{(i)}$.
2: Estimate covariance matrix $\vec{\Sigma}_x$ from $\vec{x}$.
3: $U \leftarrow \text{Chol}(\vec{\Sigma}_x)$
4: $\tilde{H} = (U^{-1})^t$
5: $\varepsilon_{ij} = \tilde{H}_{ij} - H_{ij}$
6: $a^* \leftarrow \min_{a_1,a_2}(\sum_{1<i,j<K} \varepsilon_{ij}^2)$

4.4 Numerical examples

Numerical example: accuracy test We begin with

$$H = \left(\begin{array}{cc|ccc} h_{11} & 0 & & 0 & \\ h_{21} & h_{22} & & & \\ \hline \vdots & \vdots & \ddots & & 0 \\ 0 & a_2 & a_1 & 1 & \\ 0 & 0 & a_2 & a_1 & 1 \end{array} \right).$$

After taking

$$\Sigma_x = H^{-1}H^{-t},$$

we generate scenarios $\vec{x}$ with covariance Σ_x and expectation $\mathbf{E}(\vec{x}) = 0$. Having $\vec{x}$, we use our method Algorithm (4) to find $\tilde{H}$, then make a comparison of Frobenius norms between $\tilde{H}_{ij}$ and H_{ij} to see if

$$\|\tilde{H} - H\|_F^2 \simeq 0,$$

where

$$\Delta_F := \|\tilde{H} - H\|_F^2 = \sum_{i,j} (\tilde{H}_{i,j} - H_{i,j})^2.$$

Table (3) shows estimation errors measured by Δ_F with respect to 6 sets of parameters. We can see when $\vec{a} = \{a_0, a_1, a_2\}$ with $a_0 = 1$, $|a_1| < 1$ and $|a_2| < 1$, errors Δ_F is less than 0.02. When $a_0 = 1$, $|a_1| > 1$ and $|a_2| > 1$, errors Δ_F is larger than 0.03. From the model definition, we know $a_0 = 1$, $|a_1| < 1$ and $|a_2| < 1$ is a reasonable assumption, which means yesterday's price has less impact than today's price, the day before yesterday's price has less impact than both yesterday's and today's price. Table (4) shows when $|a_1| < 1$ and $|a_2| < 1$, this method can get errors less than 0.02 with simulation number 5000.

Table 3: Errors Δ_F with respect to different parameters, with 10000 simulations.

$\vec{a}$	$\{h_{11}, h_{21}, h_{22}\}$ $\{0.02, 0.01, 0.05\}$	$\{0.2, 0.1, 0.5\}$
$\{1, 5, 3\}$	0.0548	0.0345
$\{1, 0.5, 0.3\}$	0.0015	0.0046
$\{1, 0.5, -0.3\}$	0.0030	0.0065

Table 4: Errors Δ_F with respect to different simulation numbers.

numer of simuation	$\{h_{11},h_{21},h_{22}\} = \{0.2,0.1,0.5\}$ $\vec{a} = \{1,5,3\}$	$\vec{a} = \{1,0.5,0.3\}$
100	0.4146	0.0871
1000	0.1050	0.0381
5000	0.0845	0.0197
10000	0.0319	0.0056

Numerical example: stability test We add a small value ι to zero entries in H with $|\iota_i| \leq \frac{|h_{ij}|}{100}$ to make it more realistic

$$H = \left(\begin{array}{cc|ccc} h_{11} & \iota & & \iota & \\ h_{21} & h_{22} & & & \\ \hline \vdots & \vdots & \ddots & & \\ \iota & a_2 & a_1 & 1 & \\ \iota & \iota & a_2 & a_1 & 1 \end{array}\right).$$

With new H, we apply the same algorithm as in Section (4.4). Compared with Table (4), results in Table (5) show that with perturbation, this method can also get errors less than 0.02 with simulation number 5000 when $|a_1| < 1$ and $|a_2| < 1$. To summarize, this is a stable method.

Table 5: Errors Δ_F with respect to different simulation numbers with perturbations.

numer of simuation	$\{h_{11},h_{21},h_{22}\} = \{0.2,0.1,0.5\}$ $\vec{a} = \{1,5,3\}$	$\vec{a} = \{1,0.5,0.3\}$
100	0.7477	0.1595
1000	0.1657	0.0334
5000	0.0759	0.0178
10000	0.0278	0.0083

4.5 Correlation matrix adjustment method

Given a symmetric matrix A, we construct a positive semi-definite matrix $\hat{A}$ by using eigendecomposition of A (Harville [2008]). We begin with normalizing A so that it has unit diagonal elements.

Let

$$A = DA^{(0)}D,$$

where D is diagonal matrix, $A^{(0)}$ is symmetric and has unit diagonal elements. Let

$$A^{(0)} = Q\Lambda Q^t,$$

be the symmetric eigendecomposition of $A^{(0)}$ into orthogonal matrix Q of eigenvectors and diagonal matrix Λ. Let Λ^+ be the diagonal matrix consisting of the elements $\max(\lambda_i, 0)$. Let

$$A^{(1)} = Q\Lambda^+ Q^t,$$

This matrix A^1 is positive semi-definite, but we normalize it so it has unit diagonal elements. Define a diagonal matrix S to have diagonal elements $s_{ii} = a_{ii}^{(1)^{-1/2}}$, where $a_{ii}^{(1)}$ are the diagonal elements of $A^{(1)}$. Then the matrix

$$A^{(2)} = SA^{(1)}S$$

is a positive semi-definite symmetric matrix with unit diagonal. Let

$$\hat{A} = DA^{(2)}D$$

is the desired semi-definite symmetric matrix close to our original A.

Numerical example: speed and accuracy test We perform some tests where we measured the time taken for each method along with the distance from the original A and the fixed-up matrix $\hat{A}$. To measure the distance we use the L^2-norm, i.e. we want to minimize the quantity

$$\chi^2 = \|A - \hat{A}\|_{L^2}^2 = \sum_{i,j} (a_{i,j} - \hat{a}_{i,j})^2.$$

For each matrix size N we computed around $\frac{1000}{N}$ random symmetric matrices with unit diagonal and non-diagonal elements between -1 and 1. For the ones we fixed up, we looked at the average distance χ^2. We also recorded the time taken for different methods. Table (6) also shows normalized distance χ_N^2, i.e. χ^2 divided by N^2 where N denotes the size of matrix. The times are in milliseconds.

Table 6: Speed and accuracy test for correlation matrix adjustment.

Size	Time	χ^2	χ_N^2
5	0.1	0.67	0.027
10	0.455	7.01	0.070
15	1.25	22.17	0.098
20	2.55	47.61	0.119
25	4.6	84.23	0.134
30	8.0	132.5	0.147
60	50	685.5	0.190
100	225	2174.1	0.217

5 A numerical example

To clarify our new estimation method for our model, consider a two-state autoregressive HMM model with autoregressive order $p = 2$, length of observation window $K = 5$. Coefficients matrices for each state are

$$H_1 = \begin{pmatrix} 0.1 & & & & \\ 0.9 & 0.2 & & & \\ 0.1 & 0.6 & 1 & & \\ & 0.1 & 0.6 & 1 & \\ & & 0.1 & 0.6 & 1 \end{pmatrix}$$

and

$$H_2 = \begin{pmatrix} 0.8 & & & & \\ 0.2 & 0.8 & & & \\ 0.3 & 0.5 & & & \\ & 0.3 & 0.5 & & \\ & & 0.3 & 0.5 & \end{pmatrix}$$

for each state. The transition probability matrix between two states is $\begin{pmatrix} 0.2 & 0.8 \\ 0.8 & 0.2 \end{pmatrix}$. The observation

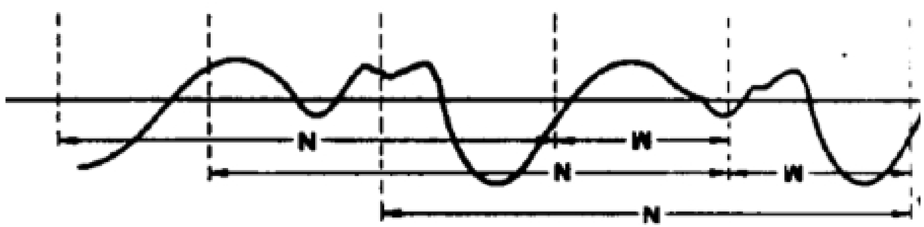


Figure 6: Illustration of overlapped observation window.

window with length 5 moves along time axis as in Figure (6). The autoregressive coefficients matrix H_t describes dependence within each observation window

$$H_t \hat{x}_t = e_t$$

with $\hat{x}_t = \{x_t, x_{t+1}, x_{t+2}, x_{t+3}, x_{t+4}\}$. Since two states of H occurs, $H_t \in \{H_1, H_2\}$, so there are two sets of dependence relationships within each observation window. Matrix H_t is driven by hidden state variable z_t. Thus distribution of observation x_t not only depends on previous observations, in this example, x_{t-1} and x_{t-2}, but also depends on hidden state z_t. We study this example through simulation.

1. Initialize the process at $t = 0$ with initial state i drawn from the distribution π;

2. Call the current state i, simulate the new state j: simulate a discrete random variable with probability distribution given by the i-th row of the transition matrix, i.e., $q_{ij}/q_i, j \neq i$;

3. Given current state i, simulate a multi-gaussian random variable with mean $H^{-1}e$, variance $H^{-1}(ee^t + \mathbf{I})H^{-t}$

Algorithm 5 Estimate parameters of Autoregressive HMM model

1. **Estimate of transition matrix** Estimate A with Balm-Welch algorithm.
2. **Fix correlation matrix** If correlation matrix is not positive semi-definite.
3. **Estimate autoregressive coefficients matrix** Estimate H with our Frobenius norm minimization method.
4. **implementation issues**
   - Initialization: Randomly initialize the parameters, use multiple restarts, and pick the best solution.
   - Termination: Set maximum iteration number $= 100$, Tolerance of convergence $= 1e-6$.

4. If t is less than a preassigned maximum time T_{max}, return to step 2.

We estimate parameters of this example with Algorithm (5), and get estimated transition matrix

$$\hat{A} = \begin{pmatrix} 0.2023 & 0.7977 \\ 0.8026 & 0.1973 \end{pmatrix}.$$

and two autoregressive coefficients matrix

$$\hat{H}_1 = \begin{pmatrix} 0.0994 & 0 & 0 & 0 & 0 \\ 0.9187 & 0.2040 & 0 & 0 & 0 \\ 0.1018 & 0.5978 & 1.0000 & 0 & 0 \\ 0 & 0.1018 & 0.5978 & 1.0000 & 0 \\ 0 & 0 & 0.1018 & 0.5978 & 1.0000 \end{pmatrix}$$

and

$$\hat{H}_2 = \begin{pmatrix} 0.8053 & 0 & 0 & 0 & 0 \\ 0.2099 & 0.7954 & 0 & 0 & 0 \\ 0.2955 & 0.5031 & 1.0000 & 0 & 0 \\ 0 & 0.2955 & 0.5031 & 1.0000 & 0 \\ 0 & 0 & 0.2955 & 0.5031 & 1.0000 \end{pmatrix}.$$

Errors are still quantified by $\Delta_F = \|H - \hat{H}\|_F^2$. We have

$$\Delta_F^{(1)} = \|H_1 - \hat{H}_1\|_F^2 = 0.0154$$

and

$$\Delta_F^{(2)} = \|H_2 - \hat{H}_2\|_F^2 = 0.0198,$$

both of it are less than 2%. We consider these to be good estimation results.

6 Conclusion

Although the regime-switching autoregressive model is a popular topic in economic literature, the estimation of this model with small sample sizes has rarely been addressed in financial areas. This kind of model, also described as an autoregressive hidden Markov model, and its associated estimation method can also be located in speech recognition literature. After careful examination of this estimation method, we learn that a non-ignorable approximation occurs, which can only be justified under the circumstances of extremely long data sequences in the field of speech recognition. For this reason, we develop an original parsimonious estimation method for autoregressive linear regime-switching models in economic fields. The positive semi-definite correlation matrix issue is considered and addressed in our estimation method. Stability and accuracy is also examined.

This paper examines existing autoregressive hidden Markov model and associated estimation methods in the speech recognition literature and discovers a significant approximation which is unstated in the original literature. This approximation can only be justified within the context of speech recognition which usually has extraordinarily long data sequences. In contrast, this estimation method is not applicable since financial data size is often limited. In a second step, we introduce a new accurate density function for this kind of model, which doesn't need any approximation and be able to function with small sample sizes. This property makes it suitable for financial and economic data. This density function passes integration check for p.d.f. functions. We also give different variations of this p.d.f function under different assumptions of first segments of data sequences, which can be used for initialization or estimation.

This paper also develops an original parsimonious estimation method to estimate autoregression coefficients matrix and transition matrix which is based on an improved Baum-Welch algorithm, Cholesky decomposition and Frobenious norm minimization. For Baum-Welch estimates of the hidden Markov driven Gaussian mixture model, our study shows that the estimator performs well for transition matrix and variance but tends to have some bias for means. Although this bias exists, it still delivers reasonable estimates for as few as 100 observations. We can also see that cases with symmetric parameters, for example, $p_1 = 0.5$, $p_2 = 0.5$, have better estimation performance. For Cholesky decomposition and Frobenious norm minimization of the autoregressive coefficients matrix, our numerical accuracy study shows this method can produces errors less than 0.02 with 5000 simulations. The stability test demonstrates it is a stable method for the reason that with perturbation, this method can achieve errors less than 0.02 with 5000 simulations. We also provide a method to fix non positive semi-definite covariance matrix by using eigendecomposition. These findings could be a basis for future research to investigate the information content provided by time series information and option prices with respect to regime-switching autoregressive models.

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The total variation model for determining the implied volatility in option pricing*

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Abstract: Volatility is a very important parameter in financial economy, it is necessary to accurately measure it in portfolio selection, asset pricing and risk management. Determining the implied volatility is a typical PDE inverse problem. In this paper, we propose the total variation regularization model for determining the implied volatility and make a rigorous mathematical analysis for this inverse problem. We not only discuss the existence and uniqueness to the solution, but also give the necessary optimality conditions for the related minimization problem. Furthermore, the stability analysis for this regularized approach are presented.
Key Words: Implied volatility; Tikhonov regularization; Total variation regularization; European option; Black-Scholes model

1 Introduction

Latest revolution of option pricing theory began in 1973, Black and Scholes [1] published their classic option pricing paper "Option Pricing and Corporate Debt", established the Black-Scholes model, obtained the partial differential equation depicting the option price change–the Black-Scholes equation:

$$\frac{\partial U}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2U}{\partial S^2}+rS\frac{\partial U}{\partial S}-rU=0,$$
$$U|_{t=T}=\begin{cases}(S-K)^+, & \text{call option},\\ (K-S)^+, & \text{put option},\end{cases}$$

here U denotes the option price, S denotes the price of underlying assets, K denotes the strike, T denotes the maturity and r denotes the riskless interest rate.

The option prices obtained from the Black-Scholes pricing framework are functions of five parameters: S, strike K, r, T and σ. Except for the volatility parameter, the other four parameters are observable quantities. In the traditional Black-Scholes formula and the emerging American options pricing formula statistical inference of the stock price volatility has a very important application in finance so that many people have been working on these problems. There exists an assumption in the Black-Scholes world that the volatility of the underlying asset is a constant, however, in the actual market volatility is changing [7, 9]. The volatility of underlying assets derived from a single option is called the implied volatility.

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The volatility of underlying assets obtained from the option prices with different strikes and different maturities is a binary function of S and T, i.e. $\sigma = \sigma(S,T)$. How to use the quotation of the options market to get the information about the volatility of the future underlying assets is a typical PDE inverse problem, called the IPOP (inverse problem of option pricing). Regularization method is a class of important methods for solving the inverse problem. The most widely applied regularization method is Tikhonov regularization method [14].

In the case $\sigma = \sigma(S)$, Isakov [10], Jiang and Tao [11, 12], Deng et al. [5] established the Tikhonov regularization model for solving the implied volatility and analyzed the existence, uniqueness and stability of the solution. Chiarella et al. [2], Crépey [3, 4] considered the case $\sigma = \sigma(S,t)$ and established the Tikhonov regularization method for determining the implied volatility. However, the traditional Tikhonov regularization method may over-smooth the solution. In image processing, these shortcomings will blur the edge of the restored image. Based on the advantage that the total variation regularization can preserve the edge of the image, Rudin et al. [13] proposed the following $TV - L^2$ model (also called the ROF model):

$$\min_{u\in\Omega} \frac{\lambda}{2}||u-f||^2_{L^2(\Omega)} + |\nabla u|_{L^1(\Omega)}.$$

Taking into account the advantages of the total variation regularization, our research is mainly focus on how to use the Black-Scholes theoretical framework to reconstruct the underlying assets price under the risk neutral probability by the information obtained from the options market, i.e., how to deduce the implied volatility of the underlying asset.

The paper is organized as follows. In the next section we transform the inverse problem of the option pricing into an optimal control problem. In Section 3, we focus on the mathematical analysis of the existence and the necessary conditions. The stability of the optimal control problem is analyzed in Section 4. In Section 5, we prove that there exists the only partial solution under appropriate assumptions and some concluding remarks are given in the last Section.

2 Total variation regularization model

In this paper, we only discuss the case of $\sigma(S,t) = \sigma(S)$ on European call options. Under the sense of the risk neutral measure, the random process of the underlying assets price evolution can be modified by

$$\frac{dS}{S} = (r-q)dt + \sigma(S)d\omega_t,$$

where q is the constant continuous dividend yield paid by the underlying asset, then the European call option $U(S,t,K,T)$ satisfies the Black-Scholes equation:

$$\begin{aligned} &\frac{\partial U}{\partial t} + \frac{1}{2}\sigma^2(S)S^2\frac{\partial^2 U}{\partial S^2} + (r-q)S\frac{\partial U}{\partial S} - rU = 0, \quad (S,t) \in R^+ \times [0,T], \\ &U(S,T) = \max(S-K,0) \equiv (S-K)^+. \end{aligned} \tag{1}$$

This inverse problem was first proposed by Dupire in [6]. He obtained a formula of the local volatility with all possible strikes and maturities, but the formula was ill-posed. It is impossible to be used in practice unless we make some appropriate amendments.

Taking into account the average price of options which includes more market information, so the above problem expressed as:

Let $U(S,t,K,T)$ be the price of a European call option which satisfies the Black-Scholes equation (1). Assume that when $S = S_0,\ t = 0$,

$$\frac{1}{T_0}\int_0^{T_0} U(S_0,0,K,T)dT = U^*(K), \quad K \in R^+,$$

is given, where $T_0 > 0$ and $U^*(K)$ is a given function of K, how to determine the function $\sigma = \sigma(S)$?

The signification of $\frac{1}{T_0}\int_0^{T_0} U(S_0,0,K,T)dT$ is the average premium corresponding with a fixed strike price and all different maturities time from 0 to T_0, i.e., the average price of options. By following Dupire's idea and using the well-known property of Green function, we reduce the above inverse problem of the option pricing to an inverse parabolic problem with the terminal observation.

Let $U = U(S,t,K,T)$ be the price of a European call option and set $G(S,t,K,T) = \frac{\partial^2 U}{\partial K^2}$, then G satisfies

$$\frac{\partial G}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 G}{\partial S^2} + (r-q)S\frac{\partial G}{\partial S} - rG = 0, \tag{2}$$
$$G(S,t) = \delta(K-S), \tag{3}$$

where $\delta(x) = \delta(-x)$. $G(S,t,K,T)$ is the basic solution of equation (2) with terminal data at $t = T$. It follows from Dupire's theory [6] that $G(S,t,\xi,\eta)$, as a function of ξ and η, is the basic solution of the adjoint equation of the Black-Scholes equation (1). Set $v(\xi,\eta) = G(S,t,\xi,\eta)$, then $v(\xi,\eta)$ satisfies

$$L^*v = -\frac{\partial v}{\partial \eta} + \frac{\sigma^2}{2}\frac{\partial^2}{\partial \xi^2}(\xi^2 v) - (r-q)\frac{\partial}{\partial \xi}(\xi v) - rv = 0,$$
$$v(\xi,\eta) = \delta(\xi - S),$$

where $0 < \xi < \infty,\ 0 < S < \infty,\ t < \eta$. The adjoint operator L^* of the operator L is defined by

$$L^*v = \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}\big(a_{ij}(x)v\big) - \sum_{i=1}^{n}\frac{\partial}{\partial x_i}\big(b_i(x)v\big) + c(x)v,$$

where

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u,$$

and both L^* and L satisfy

$$\int_\Omega (vLu - uL^*v)dx = 0, \quad \forall u,v \in C_0^\infty(\Omega).$$

Then G as a function of K and T is the basic solution of (2)'s conjugate problem

$$\frac{\partial G}{\partial T} = \frac{1}{2}\frac{\partial^2}{\partial K^2}\big(K^2\sigma^2(K)G\big) - (r-q)\frac{\partial}{\partial K}(KG) - rG, \quad (T,K) \in (t,+\infty)\times(0,+\infty), \tag{4}$$
$$G(S,t,K,T)|_{T=t} = \delta(K-S). \tag{5}$$

By integrating the equation (4)-(5) twice with respect to K, we deduce that $U(S,t,K,T)$ as a function of (K,T) satisfies the PDE

$$U_T = \frac{1}{2}K^2\sigma^2(K)U_{KK} - (r-q)KU_K - qU, \quad (T,K) \in (t,+\infty)\times(0,+\infty),$$
$$U(S,t,K,T)|_{T=t} = (S-K)^+,$$

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which implies that $U(K,T) = U(S_0, 0, K, T)$ as a function of (K,T) satisfies

$$U_T = \frac{1}{2}K^2\sigma^2(K)U_{KK} - (r-q)KU_K - qU, \quad (T,K) \in (0,+\infty)\times(0,+\infty),$$
$$U|_{T=0} = (S_0 - K)^+.$$

If the average price of options $\frac{1}{T_0}\int_0^{T_0} U(S_0,0,K,T)dT = U^*(K)$ is known, how to determine the function $\sigma = \sigma(S)$?

Set

$$y = \ln\frac{K}{S_0}, \quad \tau = T, \quad V(y,\tau) = \frac{U(S_0,0,K,T)}{S_0},$$

and $a(y) = \frac{1}{2}\sigma^2(K)$(we replace T_0 by T for convenience sake). In the actual market, the strike price of the call option is neither much lager than S nor far less than S, so in this paper, we shall assume that $y \in [A,B]$ and record this interval by Ω, then V satisfies the following PDE with the terminal observation:

$$LV \equiv V_\tau - a(y)(V_{yy} - V_y) + (r-q)V_y + qV = 0, \quad y \in \Omega, \quad \tau \in (0,T], \tag{6}$$
$$V(y,0) = (1-e^y)^+, \quad y \in \Omega, \tag{7}$$
$$\frac{1}{T}\int_0^T V(y,\tau)d\tau = V^*(y),$$

where $V^*(y) = \frac{U^*(K)}{S_0}$.

We shall assume that $V^*(y)$ satisfies the following conditions:

$$0 \le V^*(y) \le \frac{1}{qT}(1-e^{-qT}). \tag{8}$$

To reconstruct the unknown implied volatility, we hope that the average price of options $\frac{1}{T}\int_0^T V(y,\tau)d\tau$ is as close as possible to the known function $V^*(y)$. We regard V as a nonlinear operator on a.

Consider the following optimal control problem: Find $\bar{a} \in \Lambda$ such that

$$T(\bar{a}) = \min_{a\in\Lambda}\frac{1}{2}\int_\Omega \left|\frac{1}{T}\int_0^T V(y,\tau)d\tau - V^*(y)\right|^2 dy + \frac{N}{2}J(a) + \frac{\mu}{2}G(a), \tag{9}$$

where

$$J(a) = \int_\Omega |\nabla a|dy, \; G(a) = \int_\Omega |\nabla a|^2 dy,$$

N and μ are coefficients of the regular items, $\Lambda = \{a(y)|0 \le a(y) \le C, \; \nabla a \in L^2(\Omega)\}$, C is a constant, $V(y,t)$ is the solution to equations (6) and (7) for arbitrary given $a \in \Lambda$. To avoid the case $|\nabla a| \approx 0$ in the flat area, we replace $|\nabla a|$ with $|\nabla a|_\beta = \sqrt{|\nabla a|^2 + \beta^2}$, where β is a very small number. According to the definition of L^2 norm, the problem (9) is equivalent to

$$T(\bar{a}) = \min_{a\in\Lambda}\frac{1}{2}\parallel \frac{1}{T}\int_0^T V(y,\tau)d\tau - V^*(y) \parallel^2_{L^2(\Omega)} + \frac{N}{2}\int_\Omega \sqrt{|\nabla a|^2+\beta^2}dy + \frac{\mu}{2}\int_\Omega |\nabla a|^2 dy. \tag{10}$$

This problem is completely different from the traditional Tikhonov regularization, because the regular item $\int_\Omega \sqrt{|\nabla a|^2+\beta^2}dy$ is related to the total variation regularization proposed by Rudin et al. [13].

3 Existence of regularized solution

Lemma 3.1. *Under the above assumptions about the optimal control problem (10), if* $\{a_n\} \rightharpoonup a^*$*, then* $\{V(a_n, y, \tau)\} \rightharpoonup V(a^*, y, \tau)$*, where* $V(a^*, y, \tau)$ *is the solution to (6) related to* $a = a^*$*.*

The following lemma is easily obtained from the Banach-Steinhaus Theorem.

Lemma 3.2. *Let* $1 \le q \le \infty$*,* $\{f_k\}$ *weakly convergent to* f *in* $L^q(\Omega)$*, if and only if*
(1) $\{f_k\}$ *is bounded in* $L^q(\Omega)$*;*
(2) $\lim\limits_{k\to\infty} \int_E f_k dx = \int_E f dx$ *for any measurable set* $E \subset \Omega$*.*

By Lemmas 3.1 and 3.2, we easily deduce the following corollary.

Corollary 3.3. *Under the above assumptions about the optimal control problem (10), if* $\{a_n\}$ *has a subsequence* $\{a_{n_l}\}$ *which weakly converge to* a^**, then* $\{\int_0^T V_n(y,\tau)d\tau\}$ *related to* $\{a_n\}$ *also has the subsequence* $\{\int_0^T V_{n_l}(y,\tau)d\tau\}$ *which weakly converge to* $\{\int_0^T V^*(y,\tau)d\tau\}$*.*

With the previous lemmas and corollary the main result of this section can be proven:

Theorem 3.4. *Under the above assumptions, there exists at least a minimum* $\bar{a} \in \Lambda$ *in the following problem*

$$T(\bar{a}) = \min_{a\in\Lambda} \frac{1}{2}||\frac{1}{T}\int_0^T V(y,\tau)d\tau - V^*(y)||^2_{L^2(\Omega)} + \frac{N}{2}J_\beta(a) + \frac{\mu}{2}G_(a).$$

Set $W(y,\tau) = \int_0^\tau V(y,\eta)d\eta$, then the optimal control problem (10) can be transformed into

$$T(\bar{a}) = \min_{a\in\Lambda} \frac{1}{2T^2}\int_\Omega |W(y,T) - TV^*(y)|^2 dy + \frac{N}{2}\int_\Omega \sqrt{|\nabla a|^2 + \beta^2}dy + \frac{\mu}{2}\int_\Omega |\nabla a|^2 dy.$$

Set $V = ve^{-q\tau}$, then v satisfies the following system of equations:

$$\begin{aligned} &v_\tau = a(y)(v_{yy} - v_y) - (r-q)v_y, \quad y \in \Omega,\ \tau \in (0,T], \\ &v(y,0) = (1-e^y)^+. \end{aligned}$$

Similarly to the proof of Lemmas 3.1, 3.2 and Theorem 3.4 in [11], we can deduce the following results.

Lemma 3.5. *Under assumption (8), we have*

$$\int_\Omega |\frac{1}{T}\int_0^T V(y,\tau)d\tau - V^*(y)|^2 dy < \infty, \quad \forall y \in \Omega.$$

Theorem 3.6. *For any* $y \in \Omega$*, we have*

$$\int_\Omega V_y^2 dy|_{(y,\tau)} \le e^{C\tau}, \quad \int_0^\tau \int_\Omega V_{yy}^2 dy d\tau \le e^{C\tau}.$$

Noticing that $W(y,\tau) = \int_0^\tau V(y,\eta)d\eta$, the following corollary can be deduced directly from Theorem 3.6.

Corollary 3.7. *For any* $y \in \Omega$*, we have*

$$\int_\Omega W_y^2 dy|_{(y,\tau)} \le e^{C\tau}, \quad \int_0^\tau \int_\Omega W_{yy}^2 dy d\tau \le e^{C\tau}.$$

Now we turn to the necessary optimality conditions which have to be satisfied by each optimal control $\bar{a}$ for the related minimization problem.

Theorem 3.8. *Necessary optimality conditions: Let a be a solution of the optimal control problem (10). Then there exists a triple of functions (w, ϕ, a) satisfying the following equations:*

$$
\begin{aligned}
&W_\tau = a(y)(W_{yy} - W_y) - (r-q)W_y - qW + (1-e^y)^+, \quad (y,\tau) \in \Omega \times (0,T], \\
&W|_{\tau=0} = 0,
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
&-\phi_\tau - (a\phi)_{yy} - (a\phi)_y - (r-q)\phi_y + q\phi = 0, \quad (y,\tau) \in \Omega \times (0,T], \\
&\phi|_{\tau=T} = W(y,T) - TV^*(y),
\end{aligned} \tag{12}
$$

and

$$
\begin{aligned}
\frac{1}{T^2}\int_0^T \int_\Omega \phi(W_{yy} - W_y)(h-a)dyd\tau + \frac{N}{2}\int_\Omega \frac{\nabla a}{\sqrt{|\nabla a|^2 + \beta^2}} \cdot \nabla(h-a)dy \\
+ \mu \int_\Omega \nabla a \cdot \nabla(h-a)dy \geq 0
\end{aligned} \tag{13}
$$

for any $h \in \Lambda$.

Proof. The equation (11) can be deduced directly by integrating equation (6) from 0 to τ. For all $h \in \Lambda$, $\lambda \in [0,1]$, set

$$a_\lambda = (1-\lambda)a + \lambda h \in \Lambda$$

and W_λ be the solution to equation (11) with given $a = a_\lambda$. We have

$$T_\lambda \equiv T(a_\lambda) = \frac{1}{2T^2}\int_\Omega |W_\lambda - TV^*(y)|^2 dy + \frac{N}{2}\int_\Omega \sqrt{|\nabla a_\lambda|^2 + \beta^2} dy + \frac{\mu}{2}\int_\Omega |\nabla a_\lambda|^2 dy.$$

The above functional reaches the minimum at $\lambda = 0$, so

$$
\begin{aligned}
\frac{dT_\lambda}{d\lambda}|_{\lambda=0} = &\frac{1}{T^2}\int_\Omega \left[W(y,T,a) - TV^*(y)\right]\frac{\partial W_\lambda}{\partial \lambda}|_{\lambda=0} dy + \frac{N}{2}\int_\Omega \frac{\nabla a}{\sqrt{|\nabla a|^2 + \beta^2}} \cdot \nabla(h-a)dy \\
&+ \mu \int_\Omega \nabla a \cdot \nabla(h-a)dy \geq 0.
\end{aligned} \tag{14}
$$

Set $\hat{W}_\lambda = \frac{\partial W_\lambda}{\partial \lambda}$, by direct computation, we have

$$
\begin{aligned}
&\frac{\partial}{\partial \tau}(\hat{W}_\lambda) = a_\lambda\Big(\frac{\partial^2 \hat{W}_\lambda}{\partial y^2} - \frac{\partial \hat{W}_\lambda}{\partial y}\Big) - (r-q)\frac{\partial \hat{W}_\lambda}{\partial y} - q\hat{W}_\lambda + (h-a)\Big(\frac{\partial^2 W_\lambda}{\partial y^2} - \frac{\partial W_\lambda}{\partial y}\Big), \\
&\hat{W}_\lambda|_{\tau=0} = 0.
\end{aligned} \tag{15}
$$

Let $\xi = \hat{W}_\lambda|_{\lambda=0}$, then ξ satisfies

$$
\begin{aligned}
&\xi_\tau = a(y)(\xi_{yy} - \xi_y) - (r-q)\xi_y - q\xi + (h-a)(W_{yy} - W_y), \\
&\xi|_{\tau=0} = 0.
\end{aligned} \tag{16}
$$

Set $L\xi = \xi_\tau - a(y)(\xi_{yy} - \xi_y) + (r-q)\xi_y + q\xi$ and suppose ϕ is the generalized solution of the following problem:

$$\begin{aligned} &L^*\phi \equiv -\phi_\tau - (a\phi)_{yy} - (a\phi)_y - (r-q)\phi_y + q\phi = 0, \\ &\phi|_{\tau=T} = W(y,T) - TV^*(y), \end{aligned} \tag{17}$$

where L^* is the adjoint operator of the operator L. From (16), (17) and Green formula we have

$$\int_\Omega (\phi\xi)|_{\tau=T} dy = \int_0^T \int_\Omega (\phi L\xi - \xi L^*\phi) dy d\tau = \int_0^T \int_\Omega \phi(W_{yy} - W_y)(h-a) dy d\tau. \tag{18}$$

So

$$\begin{aligned} \frac{dT_\lambda}{d\lambda}|_{\lambda=0} =& \frac{1}{T^2}\int_0^T \int_\Omega \phi(W_{yy} - W_y)(h-a) dy d\tau + \frac{N}{2}\int_\Omega \frac{\nabla a}{\sqrt{|\nabla a|^2 + \beta^2}} \cdot \triangledown(h-a) dy \\ &+ \mu \int_\Omega \nabla a \cdot \nabla(h-a) dy \geq 0. \end{aligned}$$

This completes the proof. □

4 Stability

Theorem 4.1. *The minimization of (10) is stable with respect to perturbations in the data, i.e., if $\{V_k^*(y)\} \to V^*(y)$ and $\{a_k\}$ denotes the solution to the problem (10) with $V^*(y)$ replaced by $\{V_k^*(y)\}$, then $\{a_k\} \to a$, $J_\beta(a_k) \to J_\beta(a)$ and $G(a_k) \to G(a)$ in $L^2(\Omega)$.*

Proof. $\{V_k^*(y)\} \to V^*(y)$ in $L^2(\Omega)$ implies that $(a_k, \int_0^T V_k(y,\tau)d\tau)$ satisfy

$$\begin{aligned} &\frac{1}{2}||\frac{1}{T}\int_0^T V_k(y,\tau)d\tau - V_k^*(y)||^2_{L^2(\Omega)} + \frac{N}{2}J_\beta(a_k) + \frac{\mu}{2}G(a_k) \\ &\qquad \leq \frac{1}{2}||\frac{1}{T}\int_0^T V(y,\tau)d\tau - V_k^*(y)||^2_{L^2(\Omega)} + \frac{N}{2}J_\beta(a) + \frac{\mu}{2}G(a), \quad \forall a \in L^2(\Omega). \end{aligned} \tag{19}$$

Since $\{a_k\}$ is bounded in Λ, there exists a weakly convergent subsequence $\{a_m\}$ of $\{a_k\}$ such that $\{a_m\} \rightharpoonup \hat{a}$. Similarly, there exists a subsequence $\{\frac{1}{T}\int_0^T V_m(y,\tau)d\tau\}$ related to $\{a_m\}$ such that $\{\frac{1}{T}\int_0^T V_m(y,\tau)d\tau\} \rightharpoonup \frac{1}{T}\int_0^T \hat{V}(y,\tau)d\tau$. By the weak lower semi-continuity of J_β and $||\cdot||_{L^2(\Omega)}$ we get

$$J_\beta(\hat{a}) \leq \limsup J_\beta(a_m), \tag{20}$$

$$\begin{aligned} &\frac{1}{2}||\frac{1}{T}\int_0^T \hat{V}(y,\tau)d\tau - V^*(y)||^2_{L^2(\Omega)} + \frac{\mu}{2}||\nabla\hat{a}||^2_{L^2(\Omega)} \\ &\qquad \leq \limsup\Big\{\frac{1}{2}||\frac{1}{T}\int_0^T V_m(y,\tau)d\tau - V_m^*(y)||^2_{L^2(\Omega)} + \frac{\mu}{2}||\nabla a_m||^2_{L^2(\Omega)}\Big\}. \end{aligned}$$

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Combined with (19), we have

$$\begin{aligned}
&\frac{1}{2}||\frac{1}{T}\int_0^T \hat{V}(y,\tau)d\tau - V^*(y)||^2_{L^2\Omega} + \frac{N}{2}J_\beta(\hat{a}) + \frac{\mu}{2}G(\hat{a}) \\
&\qquad \le \liminf\Big\{\frac{1}{2}||\frac{1}{T}\int_0^T V_m(y,\tau)d\tau - V^*_m(y)||^2_{L^2\Omega} + \frac{N}{2}J_\beta(a_m) + \frac{\mu}{2}G(a_m)\Big\} \\
&\qquad \le \limsup\Big\{\frac{1}{2}||\frac{1}{T}\int_0^T V_m(y,\tau)d\tau - V^*_m(y)||^2_{L^2\Omega} + \frac{N}{2}J_\beta(a_m) + \frac{\mu}{2}G(a_m)\Big\} \\
&\qquad \le \lim_{m\to\infty}\Big\{\frac{1}{2}||\frac{1}{T}\int_0^T V(y,\tau)d\tau - V^*_m(y)||^2_{L^2\Omega} + \frac{N}{2}J_\beta(a) + \frac{\mu}{2}G(a)\Big\} \\
&\qquad = \frac{1}{2}||\frac{1}{T}\int_0^T V(y,\tau)d\tau - V^*(y)||^2_{L^2\Omega} + \frac{N}{2}J_\beta(a) + \frac{\mu}{2}G(a)
\end{aligned}$$

for any $a \in L^2(\Omega)$. This implies that $\hat{a}$ is a minimizer of the optimal control problem (10) and that

$$\begin{aligned}
&\lim_{m\to\infty}\Big\{\frac{1}{2}||\frac{1}{T}\int_0^T V_m(y,\tau)d\tau - V^*_m(y)||^2_{L^2(\Omega)} + \frac{N}{2}J_\beta(a_m) + \frac{\mu}{2}G(a_m)\Big\} \\
&\qquad = \frac{1}{2}||\frac{1}{T}\int_0^T \hat{V}(y,\tau)d\tau - V^*(y)||^2_{L^2(\Omega)} + \frac{N}{2}J_\beta(\hat{a}) + \frac{\mu}{2}G(\hat{a}).
\end{aligned} \tag{21}$$

If $\{a_m\}$ did not converge strongly to $\hat{a}$ with respect to $||\cdot||_{L^2(\Omega)}$, then

$$\begin{aligned}
C := \limsup&\Big\{\frac{1}{2}||\frac{1}{T}\int_0^T V_m(y,\tau)d\tau - V^*(y)||^2_{L^2(\Omega)} + \frac{\mu}{2}G(a_m)\Big\} \\
&\qquad > \frac{1}{2}||\frac{1}{T}\int_0^T \hat{V}(y,\tau)d\tau - V^*(y)||^2_{L^2\Omega} + \frac{\mu}{2}G(\hat{a}),
\end{aligned}$$

and there exists a subsequence $\{a_n\}$ of $\{a_m\}$ satisfying

$$\begin{aligned}
&a_n \rightharpoonup \hat{a},\ V_n \rightharpoonup \hat{V},\ J_\beta(\hat{a}) \le \lim_{n\to\infty} J_\beta(a_n), \\
&\frac{1}{2}||\frac{1}{T}\int_0^T V_n(y,\tau)d\tau - V^*(y)||^2_{L^2(\Omega)} + \frac{\mu}{2}G(a_n) \to C.
\end{aligned} \tag{22}$$

This combined with (21), implies

$$\frac{N}{2}\lim_{n\to\infty} J_\beta(a_n) = \frac{N}{2}J_\beta(\hat{a}) + \frac{1}{2}||\frac{1}{T}\int_0^T \hat{V}(y,\tau)d\tau - V^*(y)||^2_{L^2\Omega} + \frac{\mu}{2}G(\hat{a}) - C < \frac{N}{2}J_\beta(\hat{a}),$$

which is a contradiction to (22). This shows $a_n \to \hat{a}$, $\lim_{n\to\infty} J_\beta(a_n) = J_\beta(\hat{a})$ and $G(a_n) \to G(\hat{a})$. □

5 Uniqueness

There exists a unique minimum in (10) if its objective functional is strictly convex. Generally speaking, $\frac{1}{2}||\frac{1}{T}\int_0^T V(y,\tau)d\tau - V^*(y)||^2_{L^2(\Omega)}$ is not strictly convex or even non-convex. In a modified case we can prove that the solution to (10) is locally unique. We further assume $\nabla a \in L^\infty(\Omega)$ and change the integral measure, then the problem (10) has the only partial solution when T is sufficiently small. Suppose $\int_\Omega \cdot dy$ is replaced

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by $\int_\Omega \cdot d\omega y$, where $d\omega y = \rho(y)dy$ $(\rho(y) > 0)$ is a measure in Ω. For the following modified optimal control problem

$$T_\rho(\bar{a}) = \min_{a\in\Lambda} \frac{1}{2T^2}||W(\cdot,T) - TV^*(\cdot)||^2_{L^2(\Omega)} + \frac{N}{2}\int_\Omega \sqrt{|\nabla a|^2 + \beta^2}d\omega y + \frac{\mu}{2}\int_\Omega |\nabla a|^2 d\omega y, \tag{23}$$

it is easy to see that Theorem 3.4 and the necessary conditions (11)-(14) are still true, i.e., there at least one optimal control $\bar{a} \in \Lambda$ and any optimal control $\bar{a} \in \Lambda$ satisfies

$$\frac{1}{T^2}\int_0^T \int_\Omega \phi(W_{yy} - W_y)(h-\bar{a})d\omega y d\tau + \frac{N}{2}\int_\Omega \frac{\nabla \bar{a}}{\sqrt{|\nabla \bar{a}|^2 + \beta^2}} \cdot \nabla(h-\bar{a})d\omega y + \mu\int_\Omega \nabla\bar{a}\cdot\nabla(h-\bar{a})d\omega y \geq 0 \tag{24}$$

for any $h \in \Lambda$, where $\{W, \phi\}$ is a solution of the forward-backward parabolic system in the following:

$$\begin{aligned} &W_\tau = \bar{a}(y)(W_{yy} - W_y) - (r-q)W_y - qW + (1-e^y)^+, \quad (y,\tau) \in \Omega \times (0,T], \\ &W|_{\tau=0} = 0, \end{aligned} \tag{25}$$

and

$$\begin{aligned} &-\phi_\tau = (\bar{a}\phi)_{yy} - (\bar{a}\phi)_y + (r-q)\phi_y - q\phi, \quad (y,\tau) \in \Omega \times (0,T], \\ &\phi|_{\tau=T} = W(y,T) - TV^*(y). \end{aligned} \tag{26}$$

Set $W = we^{-q\tau}$, $\phi = e^{q\tau}\psi$, we have

$$\begin{aligned} &w_\tau = \bar{a}(y)(w_{yy} - w_y) - (r-q)w_y + e^{q\tau}(1-e^y)^+, \quad (y,\tau) \in \Omega \times (0,T], \\ &w|_{\tau=0} = 0, \\ &-\psi_\tau = (\bar{a}\psi)_{yy} - (\bar{a}\psi)_y + (r-q)\psi_y, \quad (y,\tau) \in \Omega \times (0,T], \\ &\psi|_{\tau=T} = \big[W(y,T) - TV^*(y)\big]e^{-qT}. \end{aligned}$$

The following lemma comes from [11].

Lemma 5.1. *If $\int_{-\infty}^{+\infty} \frac{dy}{\rho(y)} < \infty$, then for any bounded continuous function $f(y) \in C(\Omega)$ we have*

$$\max_\Omega |f(y)| \leq |f(x_0)| + C\Big(\int_\Omega |\nabla f|^2 d\omega y\Big)^{\frac{1}{2}},$$

where y_0 is a fixed point and $C^2 = \int_\Omega \frac{dy}{\rho(y)}$.

Suppose $a_1(y)$, $a_2(y)$ be two minimizers of the modified control problem (23) and $\{W_i, \phi_i\}(i = 1,2)$ be solutions of system (25)-(26) with $\bar{a} = a_i (i = 1,2)$. Set $\Phi = a_1\phi_1 - a_2\phi_2$, $W = W_1 - W_2$, $A = a_1 - a_2$, $\Psi = a_1\psi_1 - a_2\psi_2$ and $w = w_1 - w_2$, then w and Ψ respectively satisfy

$$\begin{aligned} &w_\tau - a_1(y)(w_{yy} - w_y) + (r-q)w_y = A(w_{2yy} - w_{2y}), \\ &w|_{\tau=0} = 0, \end{aligned} \tag{27}$$

and

$$\begin{aligned} &\frac{1}{a_1}\Psi_\tau + \Psi_{yy} + (r-q)\Big(\frac{\Psi}{a_1}\Big)_y = \Big(\frac{1}{a_2} - \frac{1}{a_1}\Big)(a_2\psi_2)_\tau + (r-q)\Big[\Big(\frac{1}{a_2} - \frac{1}{a_1}\big)(a_2\psi_2)\Big]_y, \\ &\Psi|_{\tau=T} = \big[a_1W(y,T) + A\big(W_2(y,T) - TV^*(y)\big)\big]e^{-qT}. \end{aligned} \tag{28}$$

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Lemma 5.2. *For problem (27) we have the estimate*

$$\int_0^T \int_\Omega |w_{yy} - w_y|^2 dyd\tau \leq C\big(\max|A|\big)^2 \int_0^T \int_\Omega |w_{2yy} - w_{2y}|^2 dyd\tau, \tag{29}$$

where C is a constant, independent of T.

Since $W = e^{-q\tau}w$ and T is sufficiently small, we can deduce that

$$\int_0^T \int_\Omega |W_{yy} - W_y|^2 dyd\tau \leq C\big(\max|A|\big)^2 \int_0^T \int_\Omega |W_{2yy} - W_{2y}|^2 dyd\tau, \tag{30}$$

where C is a constant, independent of T.

Setting

$$H_1 = (D_{yy} - D_y)W_1, \quad H_2 = (D_{yy} - D_y)W_2,$$

then (30) can be rewritten as

$$\int_0^T \int_\Omega |H_1 - H_2|^2 dyd\tau \leq C\big(\max|A|\big)^2 \int_0^T \int_\Omega |H_2|^2 dyd\tau. \tag{31}$$

Now set $\bar{\Phi} = \frac{\Phi}{\max|A|T}$, $\bar{\phi} = \frac{\phi}{T}$. The next lemma was proved in [5] and will play an important role in the proof of uniqueness.

Lemma 5.3. *For $\bar{\phi}$ and $\bar{\Phi}$ we have estimate*

$$\int_0^T \int_\Omega |\bar{\phi}|^2 dyd\tau \leq C, \quad \int_0^T \int_\Omega |\bar{\phi}_y|^2 dyd\tau \leq C, \tag{32}$$

$$\int_0^T \int_\Omega |(a\bar{\phi})_y|^2 dyd\tau \leq C, \quad \int_0^T \int_\Omega |(a\bar{\phi})_{yy}|^2 dyd\tau \leq C, \tag{33}$$

$$\int_0^T \int_\Omega |\bar{\phi}_\tau|^2 dyd\tau \leq C, \quad \int_0^T \int_\Omega \bar{\Phi}^2 dyd\tau \leq C \int_0^T \int_\Omega \frac{1}{T^2}|H_2|^2 dyd\tau + C, \tag{34}$$

where C is a constant, independent of T.

From lemmas 5.1, 5.2 and 5.3, it is not difficult to get the following result.

Theorem 5.4. *Suppose $a_1(y), a_2(y)$ be two minimizers of the modified optimal control problem (10). If there exists a point $y_0 \in \Omega$ such that*

$$a_1(y_0) = a_2(y_0)$$

and

$$\rho(y) \geq \rho_0 \geq 0, \quad \int_\Omega \frac{1}{\rho(y)} dy < \infty,$$

then we have $a_1(y) \equiv a_2(y)$ for sufficiently small T.

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Proof. Take $h = a_2$ with $\bar{a} = a_1$ and $h = a_1$ with $\bar{a} = a_2$ in (24), we have

$$\begin{aligned}\frac{1}{T^2}\int_0^T\int_\Omega \phi_1(W_{1yy} - W_{1y})(a_2 - a_1)d\omega y d\tau + \frac{N}{2}\int_\Omega \frac{\nabla a_1}{\sqrt{|\nabla a_1|^2 + \beta^2}} \cdot \nabla(a_2 - a_1)d\omega y \\ + \mu\int_\Omega \nabla a_1 \cdot \nabla(a_2 - a_1)dwy \geq 0,\end{aligned} \tag{35}$$

$$\begin{aligned}\frac{1}{T^2}\int_0^T\int_\Omega \phi_2(W_{2yy} - W_{2y})(a_1 - a_2)d\omega y d\tau + \frac{N}{2}\int_\Omega \frac{\nabla a_2}{\sqrt{|\nabla a_2|^2 + \beta^2}} \cdot \nabla(a_1 - a_2)d\omega y \\ + \mu\int_\Omega \nabla a_2 \cdot \nabla(a_1 - a_2)dwy \geq 0.\end{aligned} \tag{36}$$

Combining equations (35) and (36), when $\beta \to 0$ we have

$$\begin{aligned}\int_0^T\int_\Omega \phi_2(W_{2yy} - W_{2y})(a_1 - a_2)d\omega y d\tau + \int_0^T\int_\Omega \phi_1(W_{1yy} - W_{1y})(a_2 - a_1)d\omega y d\tau \\ \geq \mu T^2\int_\Omega |\nabla(a_2 - a_1)|^2 d\omega y,\end{aligned} \tag{37}$$

which yields

$$\begin{aligned}\mu T^2\int_\Omega |\nabla(a_2 - a_1)|^2 d\omega y &\leq \int_0^T\int_\Omega \big[\phi_1 H_1(a_2 - a_1) + \phi_2 H_2(a_1 - a_2)\big]d\omega y d\tau \\ &= \int_0^T\int_\Omega \big[a_2\phi_1 H_1 - a_1\phi_1 H_1 + \phi_2 H_2 a_1 - \phi_2 H_2 a_2\big]d\omega y d\tau \\ &= \int_0^T\int_\Omega \big[a_2\phi_1 H_1 - a_1\phi_1 H_1 - \frac{a_2^2}{a_1}\phi_2 H_1 + a_2\phi_2 H_1 - 2a_2\phi_2 H_1 + \frac{a_2^2}{a_1}\phi_2 H_1 \\ &\quad + a_1\phi_2 H_1 + a_1\phi_2 H_2 - a_1\phi_2 H_1 - a_2\phi_2 H_2 + a_2\phi_2 H_1\big]d\omega y d\tau \\ &= \int_0^T\int_\Omega \Big[(a_1\phi_1 - a_2\phi_2)\Big(\frac{a_2}{a_1} - 1\Big)H_1 + a_2\phi_2\Big(\frac{a_2}{a_1} - 1 + \frac{a_1}{a_2} - 1\Big)H_1 \\ &\quad + a_2\phi_2\Big(\frac{a_1}{a_2} - 1\Big)(H_2 - H_1)\Big]d\omega y d\tau \\ &= \int_0^T\int_\Omega -\Phi\Big(\frac{A}{a_1}\Big)H_1 d\omega y d\tau + \int_0^T\int_\Omega \phi_2 A(H_2 - H_1)d\omega y d\tau + \int_0^T\int_\Omega \frac{\phi_2}{a_1}A^2 H_1 d\omega y d\tau.\end{aligned} \tag{38}$$

From the assumptions in Theorem 5.4, there exists $y_0 \in \Omega$ such that

$$A(y_0) = a_1(y_0) - a_2(y_0) = 0.$$

It follows from Lemma 5.1 that

$$\max_\Omega |A| \leq C\Big(\int_\Omega |\nabla A|^2 d\omega y\Big)^{\frac{1}{2}}, \tag{39}$$

where $C^2 = \int_\Omega \frac{dy}{\rho(y)}$.

In the following we make use of the Schwarz inequality: Set $f(x), g(x)$ are integrable in $[a, b]$, then

$$\Big[\int_a^b f(x)g(x)dx\Big]^2 \leq \int_a^b f^2(x)dx \cdot \int_a^b g^2(x)dx.$$

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Combined with (38) and (39), we get

$$\begin{aligned}
T^2\int_\Omega|\nabla(a_2-a_1)|^2d\omega y &= T^2\int_\Omega|\nabla A|^2d\omega y\\
&\le C\Big\{\max|A|\sqrt{\int_0^T\int_\Omega\Phi^2dyd\tau}\cdot\sqrt{\int_0^T\int_\Omega H_1^2\rho^2(y)dyd\tau}\\
&\quad+\max|A|\sqrt{\int_0^T\int_\Omega\phi_2^2dyd\tau}\cdot\sqrt{\int_0^T\int_\Omega(H_2-H_1)^2\rho^2(y)dyd\tau}\\
&\quad+\big(\max|A|\big)^2\sqrt{\int_0^T\int_\Omega\phi_2^2dyd\tau}\cdot\sqrt{\int_0^T\int_\Omega H_1^2\rho^2(y)dyd\tau}\Big\}.
\end{aligned}$$

By (31) we have

$$\begin{aligned}
T^2\int_\Omega|\nabla A|^2d\omega y &\le C\max(A)\sqrt{\int_0^T\int_\Omega\Phi^2dyd\tau}\cdot\sqrt{\int_0^T\int_\Omega H_1^2\rho^2(y)dyd\tau}\\
&\quad+C\max(A)^2\sqrt{\int_0^T\int_\Omega\phi_2^2dyd\tau}\cdot\sqrt{\int_0^T\int_\Omega(H_1^2+H_2^2)\rho^2(y)dyd\tau},
\end{aligned}$$

dividing both sides of the above inequality by T^2 yields

$$\begin{aligned}
\int_\Omega|\nabla A|^2d\omega y &\le C\max|A|\sqrt{\int_0^T\int_\Omega\Big(\frac{\Phi}{T}\Big)^2dyd\tau}\cdot\sqrt{\int_0^T\int_\Omega\frac{H_1^2}{T^2}\rho^2(y)dyd\tau}\\
&\quad+C\big(\max|A|\big)^2\sqrt{\int_0^T\int_\Omega\Big(\frac{\phi_2}{T}\Big)^2dyd\tau}\cdot\sqrt{\int_0^T\int_\Omega\Big[\Big(\frac{H_1}{T}\Big)^2+\Big(\frac{H_2}{T}\Big)^2\Big]\rho^2(y)dyd\tau}\\
&= C\big(\max|A|\big)^2\Big\{\sqrt{\int_0^T\int_\Omega\Big(\frac{\Phi}{\max|A|T}\Big)^2dyd\tau}\cdot\sqrt{\int_0^T\int_\Omega\frac{H_1^2}{T^2}\rho^2(y)dyd\tau}\\
&\quad+\sqrt{\int_0^T\int_\Omega\Big(\frac{\phi_2}{T}\Big)^2dyd\tau}\cdot\sqrt{\int_0^T\int_\Omega\Big[\Big(\frac{H_1}{T}\Big)^2+\Big(\frac{H_2}{T}\Big)^2\Big]\rho^2(y)dyd\tau}\Big\}.
\end{aligned}\tag{40}$$

Noticing

$$\frac{H_1}{T}=\frac{1}{T}\int_0^\tau(V_{1yy}-V_{1y})d\tau,\quad \frac{H_2}{T}=\frac{1}{T}\int_0^\tau(V_{2yy}-V_{2y})d\tau,$$

by Schwarz inequality we get

$$\begin{aligned}
\Big(\frac{H_1}{T}\Big)^2&=\frac{1}{T^2}\Big(\int_0^T(V_{1yy}-V_{1y})d\tau\Big)^2\le\frac{1}{T}\int_0^T(V_{1yy}-V_{1y})^2d\tau,\\
\Big(\frac{H_2}{T}\Big)^2&=\frac{1}{T^2}\Big(\int_0^T(V_{2yy}-V_{2y})d\tau\Big)^2\le\frac{1}{T}\int_0^T(V_{2yy}-V_{2y})^2d\tau,
\end{aligned}$$

so

$$\int_0^T\int_\Omega\Big(\frac{H_1}{T}\Big)^2\rho^2(y)dyd\tau\le\int_0^T\int_\Omega(V_{1yy}-V_{1y})^2\rho^2(y)dyd\tau,\tag{41}$$

$$\int_0^T\int_\Omega\Big(\frac{H_2}{T}\Big)^2\rho^2(y)dyd\tau\le\int_0^T\int_\Omega(V_{2yy}-V_{2y})^2\rho^2(y)dyd\tau.\tag{42}$$

Take $\rho(y) = 1 + y^k, k > 1$. By substituting (32)-(34), (41) and (42) into (40), and follows from Theorem 3.6 we can deduce that

$$\int_\Omega |\nabla A|^2 d\omega y \leq C\big(\max|A|\big)^2\big[\sqrt{CT+C}\cdot\sqrt{CT}+\sqrt{C}\cdot\sqrt{CT}\big] \leq C\big(\max|A|\big)^2\sqrt{T^2+T}, \tag{43}$$

where C is independent of T. Since $\max|A| \leq C\big(\int_\Omega |\nabla A|^2 d\omega y\big)^{\frac{1}{2}}$ we have

$$\int_\Omega |\nabla A|^2 d\omega y \leq C\sqrt{T^2+T}\big(\max|A|\big)^2 \leq C\sqrt{T^2+T}\int_\Omega |\nabla A|^2 d\omega y.$$

Choosing $T \ll 1$ such that

$$C\sqrt{T^2+T} \leq \theta < 1, \quad 0 < \theta < 1,$$

so we get

$$\int_\Omega |\nabla A|^2 d\omega y = 0,$$

which implies $A(y) = a_1(y) - a_2(y) \equiv 0$. The proof is complete. □

6 Conclusion

In this paper, based on the Black-Scholes theoretical framework we studied how to recover the prices of underlying assets under the sense of risk neutral measure by using the information obtained from options market. We proposed the total variation regularization model for determining the implied volatility and gave the related necessary optimality conditions. Not only the existence and uniqueness for the solution are discussed, but also the stability is analyzed.

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Stability and superstability of J^*-homomorphisms and J^*-derivations for a generalized Cauchy-Jensen equation

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Abstract. In this paper, we prove the superstability and the Hyers-Ulam stability of J^*-homomorphisms and J^*-derivations associated with the following generalized Cauchy-Jensen functional equation

$$f\left(\frac{\sum_{i=1}^{p} x_i}{p-1}\right) + \sum_{i=1}^{p} f\left(\frac{\sum_{\substack{j=1\\ j\neq i}}^{p} x_j - x_i}{p-1}\right) = \sum_{i=1}^{p} f(x_i). \tag{1}$$

Keywords: Superstability; Functional inequality; J^*-algebra; Hyers-Ulam stability; J^*-homomorphism; Cauchy-Jensen functional equation; J^*-derivation.

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1. Introduction and preliminaries

Consider the functional equation (1). It is clear that the simplest case of (1) is $f(x+y)+f(x-y)+f(y-x) = f(x)+f(y)$. In this paper, in order to investigate the functional equation (1), we will suppose that $p \geq 2$.

The notion of J^*-algebras has been posed by Harris [15] in 1974. In general, by a J^*-algebra we mean a closed subspace A of a C^*-algebra such that $xx^*x \in A$ whenever $x \in A$ [15]. For more study about J^*-algebras, one can refer to [6, 15, 16, 17, 18]. Throughout this paper, A and B denote J^*-algebras, with norms $\|\cdot\|_A$ and $\|\cdot\|_B$.

Definition 1.1. ([9, 12, 27]) A $\mathbb{C}$-linear mapping $h : A \to B$ is called a J^**-homomorphism* if

$$h(xx^*x) = h(x)h(x)^*h(x)$$

for all $x \in A$, and a $\mathbb{C}$-linear mapping $\delta : A \to A$ is called a J^**-derivation* if

$$\delta(xx^*x) = \delta(x)x^*x + x\delta(x)^*x + xx^*\delta(x)$$

for all $x \in A$.

We say a functional equation (ξ) is stable if any function g satisfying the equation (ξ) *approximately* is near to true solution of (ξ). We say that a functional equation is *superstable* if every approximately solution is an exact solution of it [31].

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Very often instead of a functional equation, we consider a functional inequality and one can ask the following question: "when can one assert that the solutions of the inequality lie near to the solutions of the equation?"[2]. The classical problem of this kind had been formulated by Ulam [33] in 1940 for the first time. In 1941, Hyers [19] affirmatively answered to this question of Ulam for Banach spaces. Aoki [1] generalized the theorem of Hyers for approximately additive mappings in 1950, and Rassias [30] generalized that by considering the stability problem with unbounded Cauchy differences in 1978. Theorem of Rassias was generalized again by Ğavruţa [14] in 1994, by control function $\varphi(x, y)$. He proved the following:

Theorem 1.2. ([14]) *Let G be an abelian group and E a Banach space. Denote by $\varphi : G \times G \to [0, \infty)$ a function such that*

$$\phi(x, y) := \sum_{n=0}^{\infty} 2^{-(n+1)} \varphi(2^n x, 2^n y) < \infty$$

for all $x, y \in G$. Suppose $f : G \to E$ is a mapping satisfying

$$\|f(x + y) - f(x) - f(y)\| \leq \varphi(x, y)$$

for all $x, y \in G$. Then there exists a unique additive mapping $A : G \to E$ such that

$$\|f(x) - A(x)\| \leq \phi(x, x)$$

for all $x \in G$.

The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results, containing different kind of homomorphisms and derivations, concerning this problem (see [3, 4, 5, 7, 8, 9, 10, 12, 13, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 32]).

Throughout this paper, assume that n is a fixed positive integer.

2. Superstability of J^*-homomorphisms

In this section, we prove the superstability of J^*-homomorphisms associated with functional equation (1). The following lemmas will be used in the proof of the theorems.

Lemma 2.1. ([22]) *Let X and Y be linear spaces and let $f : X \to Y$ be an additive mapping such that $f(\mu x) = \mu f(x)$ for all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and all $x \in X$. Then the mapping f is $\mathbb{C}$-linear.*

Lemma 2.2. *Let $f : A \to B$ be a mapping such that*

$$\left\| nf\left(\frac{x + y}{n}\right) + f\left(\frac{x - y}{n}\right) - f(y)\right\|_B \leq \left\| f(x) - f\left(\frac{y - x}{n}\right)\right\|_B \tag{2.1}$$

for all $x, y \in A$. Then f is additive.

Proof. Letting $x = y = 0$ in (2.1), we get

$$\|nf(0)\|_B \leq \|0\|_B = 0.$$

So $f(0) = 0$. Letting $x = u + v$ and $y = (n + 1)(u + v)$ in (2.1), we get

$$f\left((n + 1)(u + v)\right) = nf\left(\frac{(n + 2)(u + v)}{n}\right) + f(-u - v)$$

for all $u, v \in A$. Letting $x = (n + 2)(u + v)$ and $y = -(u + v)$ in the above equality, we have

$$f(x + y) = nf\left(\frac{x}{n}\right) + f(y)$$

for all $x, y \in A$. Letting $y = 0$, we obtain that $f(x) = nf(x/n)$, and so $f(x+y) = f(x) + f(y)$ for all $x, y \in A$. □

Lemma 2.3. *Let $f : A \to B$ be a mapping such that*

$$\begin{aligned} &\left\| f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x+z-y}{2}\right) + f\left(\frac{x+y-z}{2}\right) - f(y) - f(z) \right\|_B \\ &\qquad \le \left\| f(x) - f\left(\frac{y+z-x}{2}\right) \right\|_B \end{aligned} \tag{2.2}$$

for all $x, y, z \in A$. Then f is additive.

Proof. Letting $z = 0$ in (2.2), we get (2.1) for $n = 2$. Applying Lemma 2.2, we get the desired result. □

Lemma 2.4. *Let p be a fixed integer with $p \ge 2$. Let $f : A \to B$ be a mapping such that*

$$\begin{aligned} &\left\| f\left(\frac{\sum_{i=1}^p x_i}{p-1}\right) + \sum_{i=2}^p f\left(\frac{\sum_{j=1, j\neq i}^p x_j - x_i}{p-1}\right) - \sum_{i=2}^p f(x_i) \right\|_B \\ &\qquad \le \left\| f(x_1) - f\left(\frac{\sum_{j=2}^p x_j - x_1}{p-1}\right) \right\|_B \end{aligned} \tag{2.3}$$

for all $x_1, \cdots, x_p \in A$. Then f is additive.

Proof. The case $p = 2$ is the case $n = 1$ in Lemma 2.2.

For the case $p \ge 3$, letting $x_1 = \cdots = x_p = 0$ in (2.3), we get $f(0) = 0$. Letting $x_3 = \cdots = x_p = 0$ in (2.3), we obtain

$$\left\| (p-1) f\left(\frac{x_1 + x_2}{p-1}\right) + f\left(\frac{x_1 - x_2}{p-1}\right) - f(x_2) \right\|_B \le \left\| f(x_1) - f\left(\frac{x_2 - x_1}{p-1}\right) \right\|_B$$

for all $x_1, x_2 \in A$, which is the case $n = p - 1 \ge 2$ in Lemma 2.2. □

Theorem 2.5. *Let $\varphi : A^p \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} b^{-3n} \varphi(b^n x, \cdots, b^n x) = 0$$

for all $x \in A$, where $b \neq 1$ is a real number. Let $f : A \to B$ be a mapping satisfying

$$\begin{aligned} &\left\| f\left(\mu \frac{\sum_{i=1}^p x_i}{p-1}\right) + \sum_{i=2}^p f\left(\mu \frac{\sum_{j=1, j\neq i}^p x_j - x_i}{p-1}\right) - \sum_{i=2}^p \mu f(x_i) \right\|_B \\ &\qquad \le \left\| f(\mu x_1) - f\left(\mu \frac{\sum_{j=2}^p x_j - x_1}{p-1}\right) \right\|_B, \end{aligned} \tag{2.4}$$

$$\|f(xx^*x) - f(x)f(x)^* f(x)\|_B \le \varphi(x, \cdots, x) \tag{2.5}$$

for all $\mu \in \mathbb{T}^1$ and all $x, x_1, \cdots, x_p \in A$. Then the mapping $f : A \to B$ is a J^*-homomorphism.

Proof. Let $\mu = 1$ in (2.4). By Lemma 2.4, the mapping $f : A \to B$ is additive. Letting $x_1 = x$ and $x_2 = px$ and $x_3 = \cdots = x_p = 0$ in (2.4), we get

$$p\|f(\mu x) - \mu f(x)\|_B = \left\| (p-1) f\left(\mu \frac{p+1}{p-1} x\right) + f(-\mu x) - \mu f(px) \right\|_B \le \|0\|_B = 0$$

for all $\mu \in \mathbb{T}^1$ and all $x \in A$. Hence $f(\mu x) = \mu f(x)$ for all $\mu \in \mathbb{T}^1$ and all $x \in A$. By Lemma 2.1, the mapping $f : A \to B$ is $\mathbb{C}$-linear.

By (2.5) and the assumption on φ, we have

$$\begin{aligned}\|f(xx^*x)-f(x)f(x)^*f(x)\|_B &= \lim_{n\to\infty}\frac{1}{b^{3n}}\|f(b^nxb^nx^*b^nx)-f(b^nx)f(b^nx)^*f(b^nx)\|_B\\ &\le \lim_{n\to\infty}\frac{1}{b^{3n}}\varphi(b^nx,\cdots,b^nx)=0\end{aligned}$$

for all $x\in A$. Therefore, the mapping $f:A\to B$ is a J^*-homomorphism. □

Corollary 2.6. *Let θ be a nonnegative real number and $q_1,\cdots,q_p\neq 3$ be positive real numbers. Let $f:A\to B$ be a mapping satisfying (2.4) and*

$$\|f(xx^*x)-f(x)f(x)^*f(x)\|_B\le\theta\left(\|x\|_A^{q_1}+\cdots+\|x\|_A^{q_p}\right)$$

for all $x\in A$. Then the mapping $f:A\to B$ is a J^*-homomorphism.

Proof. Defining $\varphi(x_1,\cdots,x_p)=\theta(\|x_1\|_A^{q_1}+\cdots+\|x_p\|_A^{q_p})$ and applying Theorem 2.5, we get the desired result. □

Corollary 2.7. *Let θ be a nonnegative real number and $q_1,\cdots,q_p$ be positive real numbers such that $q_1+\cdots+q_p\neq 3$. Let $f:A\to B$ be a mapping satisfying (2.4) and*

$$\|f(xx^*x)-f(x)f(x)^*f(x)\|_B\le\theta\|x\|_A^{q_1+\cdots+q_p}$$

for all $x\in A$. Then the mapping $f:A\to B$ is a J^*-homomorphism.

Proof. Defining $\varphi(x_1,\cdots,x_p)=\theta(\|x_1\|_A^{q_1}\cdots\|x_p\|_A^{q_p})$ and applying Theorem 2.5, we get the desired result. □

3. Hyers-Ulam stability of J^*-homomorphisms

In this section, we prove the Hyers-Ulam stability of J^*-homomorphisms associated with functional equation (1). Initially we will suppose that $p\ge 4$. The cases $p=2,3$ will be investigated separately.

For convenience, given $f:A\to B$, we define the following

$$\gamma_\mu f(x_1,\cdots,x_p):=f\left(\mu\frac{\sum_{i=1}^p x_i}{p-1}\right)+\sum_{i=1}^p f\left(\mu\frac{\sum_{\substack{j=1\\ j\neq i}}^p x_j-x_i}{p-1}\right)-\sum_{i=1}^p\mu f(x_i)$$

for all $\mu\in\mathbb{T}^1$ and all $x_1,\cdots,x_p\in A$. One can easily show that a mapping $f:A\to B$ satisfies $\gamma_\mu f(x_1,\cdots,x_p)=0$ for all $\mu\in\mathbb{T}^1$ and all $x_1,\cdots,x_p\in A$ if and only if $f(\mu x+\lambda y)=\mu f(x)+\lambda f(y)$ for all $\mu,\lambda\in\mathbb{T}^1$ and all $x,y\in A$.

Lemma 3.1. *A mapping $f:A\to B$ is a $\mathbb{C}$-linear mapping if and only if*

$$\gamma_\mu f(x_1,\cdots,x_p)=0$$

for all $\mu\in\mathbb{T}^1$ and all $x_1,\cdots,x_p\in A$.

Proof. The proof follows from Lemma 2.1 and the initial descriptions of this section. □

Theorem 3.2. *Let $\varphi:A^p\to[0,\infty)$ be a function with $\varphi(0,\cdots,0)=0$ and $p\ge 4$. Denote by ϕ a function such that*

$$\phi(x_1,\cdots,x_p):=\sum_{n=0}^\infty d^r\varphi\left(d^tx_1,\cdots,d^tx_p\right)<\infty \tag{3.1}$$

for all $x_1, \cdots, x_p \in A$*. Here* $d = \frac{p-3}{p-1} < 1$, $r = n$, $t = -(n+1)$ *or* $t = n$, $r = -(n+1)$*, and* $\lim_{n\to\infty} d^{-3n}\varphi(d^n x, \cdots, d^n x) = 0$*. Suppose that* $f : A \to B$ *is a mapping satisfying*

$$\|\gamma_\mu f(x_1, \cdots, x_p)\|_B \leq \varphi(x_1, \cdots, x_p), \tag{3.2}$$

$$\|f(xx^*x) - f(x)f(x)^* f(x)\|_B \leq \varphi(x, \cdots, x) \tag{3.3}$$

for all $\mu \in \mathbb{T}^1$ *and all* $x, x_1, \cdots, x_p \in A$*. Then there exists a unique* J^**-homomorphism* $h : A \to B$ *such that*

$$\|f(x) - h(x)\|_B \leq \frac{1}{p-1}\phi(x, \cdots, x, 0) \tag{3.4}$$

for all $x \in A$*.*

Proof. Assume that $r = n$ and $t = -(n+1)$.

Letting $x_1 = \cdots = x_p = 0$ in (3.2), we get $f(0) = 0$. Letting $\mu = 1$ and $x_1 = \cdots = x_{p-1} = x$ and $x_p = 0$ in (3.2), we obtain

$$\begin{aligned} \left\|f(x) + (p-1)f\left(\frac{p-3}{p-1}x\right) + f(x) - (p-1)f(x)\right\|_B &\leq \varphi(x, \cdots, x, 0), \\ \left\|f(x) - df\left(\frac{x}{d}\right)\right\|_B &\leq \frac{1}{p-1}\varphi\left(\frac{x}{d}, \cdots, \frac{x}{d}, 0\right) \end{aligned}$$

for all $x \in A$. Using the induction method, we get

$$\left\|f(x) - d^n f\left(\frac{x}{d^n}\right)\right\|_B \leq \frac{1}{p-1}\sum_{s=0}^{n-1} d^s \varphi\left(d^{-(s+1)}x, \cdots, d^{-(s+1)}x, 0\right) \tag{3.5}$$

for all $n \geq 1$ and all $x \in A$. Now assume that m, l are positive integers, with $m > l$. By (3.5) for $m - l > 0$ and $\frac{x}{d^l}$, we have

$$\begin{aligned} \left\|d^m f\left(\frac{x}{d^m}\right) - d^l f\left(\frac{x}{d^l}\right)\right\|_B &= d^l \left\|d^{m-l} f\left(\frac{1}{d^{m-l}}\frac{x}{d^l}\right) - f\left(\frac{x}{d^l}\right)\right\|_B \\ &\leq \frac{1}{p-1}\sum_{s=l}^{m-1} d^s \varphi\left(d^{-(s+1)}x, \cdots, d^{-(s+1)}x, 0\right) \\ &\leq \frac{1}{p-1}\sum_{s=l}^{\infty} d^s \varphi\left(d^{-(s+1)}x, \cdots, d^{-(s+1)}x, 0\right) \end{aligned}$$

for all $x \in A$. By (3.1) the right side tends to 0 as $l \to \infty$. Thus the sequence $\{d^n f(\frac{x}{d^n})\}$ is a Cauchy sequence. Since A is a complete space, the sequence $\{d^n f(\frac{x}{d^n})\}$ is a convergent sequence. Therefore, we can define, for all $x \in A$, the mapping $h : A \to B$ by

$$h(x) := \lim_{n\to\infty} d^n f\left(\frac{x}{d^n}\right).$$

Passing the limit $n \to \infty$ in (3.5) and by (3.1), we obtain (3.4).

It follows from (3.1) and (3.2) that

$$\begin{aligned} \|\gamma_\mu h(x_1, \cdots, x_p)\|_B &= \lim_{n\to\infty} d^n \left\|\gamma_\mu f\left(\frac{x_1}{d^n}, \cdots, \frac{x_p}{d^n}\right)\right\|_B \\ &\leq \lim_{n\to\infty} d^n \varphi\left(\frac{x_1}{d^n}, \cdots, \frac{x_p}{d^n}\right) = 0 \end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x_1, \cdots, x_p \in A$. Hence by Lemma 3.1, h is $\mathbb{C}$-linear.

By (3.1) and replacing x by $\frac{x}{d^n}$ in (3.3) and since $d < 1$, we obtain

$$\begin{aligned}\|h(xx^*x) - h(x)h(x)^*h(x)\|_B &= \lim_{n\to\infty} d^{3n} \left\| f\left(\frac{x}{d^n}\frac{x^*}{d^n}\frac{x}{d^n}\right) - f\left(\frac{x}{d^n}\right) f\left(\frac{x}{d^n}\right)^* f\left(\frac{x}{d^n}\right)\right\|_B \\ &\le \lim_{n\to\infty} d^{3n}\varphi\left(\frac{x}{d^n},\cdots,\frac{x}{d^n}\right) \le \lim_{n\to\infty} d^{n}\varphi\left(\frac{x}{d^n},\cdots,\frac{x}{d^n}\right) = 0\end{aligned}$$

for all $x \in A$. Hence $h(xx^*x) = h(x)h(x)^*h(x)$ for all $x \in A$.

Let $H : A \to B$ be another J^*-homomorphism satisfying (3.4). Then we have

$$\begin{aligned}\|h(x) - H(x)\|_B &\le d^n \left\| f\left(\frac{x}{d^n}\right) - h\left(\frac{x}{d^n}\right)\right\|_B + d^n \left\| f\left(\frac{x}{d^n}\right) - H\left(\frac{x}{d^n}\right)\right\|_B \\ &\le d^n \left(\frac{2}{p-1}\phi\left(\frac{x}{d^n},\cdots,\frac{x}{d^n},0\right)\right) \\ &\le \frac{2}{p-1}\sum_{s=n}^{\infty} d^s \varphi\left(d^{-(s+1)}x,\cdots,d^{-(s+1)}x,0\right)\end{aligned}$$

for all $x \in A$. By (3.1), the right side tends to 0 as $n \to \infty$. Therefore, h is unique.

Assume that $r = -(n+1)$ and $t = n$ and $\lim_{n\to\infty} d^{-3n}\varphi(d^n x,\cdots,d^n x) = 0$.

It follows from (3.2) that

$$\left\|\frac{1}{d}f(dx) - f(x)\right\|_B \le \frac{1}{p-3}\varphi(x,\cdots,x,0)$$

for all $x \in A$. Using the induction method, we get

$$\left\|\frac{1}{d^n}f(d^n x) - f(x)\right\|_B \le \frac{1}{p-1}\sum_{s=0}^{n-1} d^{-(s+1)}\varphi(d^s x,\cdots,d^s x,0)$$

for each $n \ge 1$ and all $x \in A$.

Now by the same method as in the proof of the previous part, one can obtain a $\mathbb{C}$-linear mapping $h(x) := \lim_{n\to\infty}\frac{1}{d^n}f(d^n x)$ satisfying (3.4). By (3.1) and replacing x by $d^n x$ in (3.3), we obtain

$$\begin{aligned}\|h(xx^*x) - h(x)h(x)^*h(x)\|_B &= \lim_{n\to\infty}\frac{1}{d^{3n}}\|f(d^n x d^n x^* d^n x) - f(d^n x) f(d^n x)^* f(d^n x)\|_B \\ &\le \lim_{n\to\infty}\frac{1}{d^{3n}}\varphi(d^n x,\cdots,d^n x) = 0\end{aligned}$$

for all $x \in A$. Hence $h(xx^*x) = h(x)h(x)^*h(x)$ for all $x \in A$. Thus f is a J^*-homomorphism.

The rest of the proof is similar to the proof of the last part. □

Corollary 3.3. *Let θ be a nonnegative real number and $q_1,\cdots,q_p$ positive real numbers such that $q_1,\cdots,q_p < 1$ or $q_1,\cdots,q_p > 3$. Let $f : A \to B$ be a mapping satisfying*

$$\begin{aligned}\|\gamma_\mu f(x_1,\cdots,x_p)\|_B &\le \theta\left(\|x_1\|_A^{q_1} + \cdots + \|x_p\|_A^{q_p}\right), \\ \|f(xx^*x) - f(x)f(x)^*f(x)\|_B &\le \theta\left(\|x\|_A^{q_1} + \cdots + \|x\|_A^{q_p}\right)\end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x, x_1,\cdots,x_p \in A$. Then there exists a unique J^*-homomorphism $h : A \to B$ such that

$$\|f(x) - h(x)\|_B \le \sum_{j=1}^{p-1}\frac{(p-1)^{q_j}\quad\theta\|x\|_A^{q_j}}{|(p-1)(p-3)^{q_j} - (p-1)^{q_j}(p-3)|}$$

for all $x \in A$.

Proof. Defining $\varphi(x_1,\cdots,x_p) = \theta(\|x_1\|_A^{q_1}+\cdots+\|x_p\|_A^{q_p})$ and applying Theorem 3.2 with $r = n$ and $t = -(n+1)$, for the case $q_1,\cdots,q_p < 1$, and with $r = -(n+1)$ and $t = n$ and $\lim_{n\to\infty} d^{-3n}\varphi(d^n x,\cdots,d^n x) = 0$, for the case $q_1,\cdots,q_p > 3$, we get the desired result. □

Now we investigate the cases $p = 2, 3$.

Theorem 3.4. *Let $\varphi : A^2 \to [0, \infty)$ be a function. Denote by ϕ a function such that*

$$\phi(x, y) := \sum_{n=0}^{\infty} 2^r \varphi\left(2^t x, 2^t y\right) < \infty$$

for all $x, y \in A$, with $r = n,\ t = -(n+1)$ or $t = n,\ r = -(n+1)$, $\lim_{n\to\infty} 2^{-3n}\varphi(2^n x, 2^n x) = 0$. Suppose that $f : A \to B$ is a mapping satisfying

$$\begin{aligned} \|f(\mu x + \mu y) + f(\mu x - \mu y) + f(\mu y - \mu x) - \mu[f(x) + f(y)]\|_B &\leq \varphi(x, y), \\ \|f(xx^* x) - f(x)f(x)^* f(x)\|_B &\leq \varphi(x, x) \end{aligned} \tag{3.6}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y \in A$. Then there exists a unique J^*-homomorphism $h : A \to B$ such that

$$\|f(x) - h(x)\|_B \leq \phi(x, x) \tag{3.7}$$

for all $x \in A$.

Proof. Letting $\mu = 1$ and $x = y$ in (3.6), we get $\|f(2x) - 2f(x)\| \leq \varphi(x, x)$, and so

$$\left\|\frac{1}{2}f(2x) - f(x)\right\|_B \leq \frac{1}{2}\varphi(x, x), \qquad \left\|f(x) - 2f\left(\frac{x}{2}\right)\right\|_B \leq \varphi\left(\frac{x}{2}, \frac{x}{2}\right)$$

for all $x \in A$. Using the induction method two times, we obtain

$$\begin{aligned} \left\|\frac{1}{2^n}f(2^n x) - f(x)\right\|_B &\leq \sum_{s=0}^{n-1} 2^{-(s+1)}\varphi(2^s x, 2^s x), \\ \left\|f(x) - 2^n f\left(\frac{x}{2^n}\right)\right\|_B &\leq \sum_{s=0}^{n-1} 2^s \varphi\left(2^{-(s+1)}x, 2^{-(s+1)}x\right) \end{aligned}$$

for each $n \geq 1$ and all $x \in A$. Now, by the same method as in the proof of Theorem 3.2, from the above two inequalities, one can obtain two $\mathbb{C}$-linear mapping $h(x) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ and $h(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ satisfying (3.7) and get the desired result. □

Corollary 3.5. *Let θ be a nonnegative real number and q_1, q_2 be positive real numbers such that $q_1, q_2 < 1$ or $q_1, q_2 > 3$. Let $f : A \to B$ be a mapping satisfying*

$$\begin{aligned} \|f(\mu x + \mu y) + f(\mu x - \mu y) + f(\mu y - \mu x) - \mu[f(x) + f(y)]\|_B &\leq \theta\left(\|x\|_A^{q_1} + \|y\|_A^{q_2}\right), \\ \|f(xx^* x) - f(x)f(x)^* f(x)\|_B &\leq \theta\left(\|x\|_A^{q_1} + \|x\|_A^{q_2}\right) \end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y \in A$. Then there exists a unique J^*-homomorphism $h : A \to B$ such that

$$\|f(x) - h(x)\|_B \leq \frac{\theta\|x\|_A^{q_1}}{|2^{q_1} - 2|} + \frac{\theta\|x\|_A^{q_2}}{|2^{q_2} - 2|}$$

for all $x \in A$.

Corollary 3.6. *Let θ be a nonnegative real number and q_1, q_2 be positive real numbers such that $q_1 + q_2 < 1$ or $q_1 + q_2 > 3$. Let $f : A \to B$ be a mapping satisfying*

$$\begin{aligned} \|f(\mu x + \mu y) + f(\mu x - \mu y) + f(\mu y - \mu x) - \mu[f(x) + f(y)]\|_B &\leq \theta\left(\|x\|_A^{q_1}\|y\|_A^{q_2}\right), \\ \|f(xx^* x) - f(x)f(x)^* f(x)\|_B &\leq \theta\|x\|_A^{q_1+q_2} \end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y \in A$. Then there exists a unique J^*-homomorphism $h : A \to B$ such that

$$\|f(x) - h(x)\|_B \leq \frac{\theta \|x\|_A^{q_1+q_2}}{|2^{q_1+q_2} - 2|}$$

for all $x \in A$.

Theorem 3.7. *Let $\varphi : A^3 \to [0, \infty)$ be a function. Denote by ϕ a function such that*

$$\phi(x, y, z) := \sum_{n=0}^{\infty} 2^{tn} \varphi\left(2^{-tn}x, 2^{-tn}y, 2^{-tn}z\right) < \infty$$

for all $x, y, z \in A$, with $t = -\frac{(n+1)}{n}$ or $t = 1$, $\lim_{n\to\infty} 2^{-3n}\varphi(2^n x, 2^n x, 2^n x) = 0$. Suppose that $f : A \to B$ is an odd mapping satisfying

$$\begin{aligned} &\left\| f\left(\mu \frac{x+y+z}{2}\right) + f\left(\mu \frac{y+z-x}{2}\right) + f\left(\mu \frac{x+z-y}{2}\right) + f\left(\mu \frac{x+y-z}{2}\right) \right. \\ &\qquad \left. - \mu[f(x) + f(y) + f(z)] \right\|_B \leq \varphi(x, y, z), \qquad (3.8) \\ &\|f(xx^*x) - f(x)f(x)^*f(x)\|_B \leq \varphi(x, x, x) \end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. Then there exists a unique J^*-homomorphism $h : A \to B$ such that

$$\|f(x) - h(x)\|_B \leq \phi(x, 0, 0) \tag{3.9}$$

for all $x \in A$.

Proof. Note that $f(0) = 0$ and $f(-x) = -f(x)$ for all $x \in A$, since f is an odd mapping. Letting $\mu = 1$ and $y = z = 0$ in (3.8), we have $\left\|3f\left(\frac{x}{2}\right) + f\left(-\frac{x}{2}\right) - f(x)\right\|_B \leq \varphi(x, 0, 0)$ and so

$$\left\|2f\left(\frac{x}{2}\right) - f(x)\right\|_B \leq \varphi(x, 0, 0), \qquad \left\|f(x) - \frac{1}{2}f(2x)\right\|_B \leq \frac{1}{2}\varphi(2x, 0, 0)$$

for all $x \in A$. Using the induction method, we obtain

$$\begin{aligned} \left\|2^n f\left(\frac{x}{2^n}\right) - f(x)\right\|_B &\leq \sum_{s=0}^{n-1} 2^s \varphi\left(2^{-s}x, 0, 0\right), \\ \left\|f(x) - \frac{1}{2^n} f(2^n x)\right\|_B &\leq \sum_{s=0}^{n-1} 2^{-(s+1)} \varphi(2^{(s+1)}x, 0, 0) \end{aligned}$$

for each $n \geq 1$ and all $x \in A$. By the same method as in the proof of Theorem 3.2, from the above two inequalities, we can obtain two $\mathbb{C}$-linear mappings $h(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ and $h(x) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ satisfying (3.9), which is the desired result. □

Like Corollaries 3.3, 3.5 and 3.6, that obtained from Theorems 3.2 and 3.4, we can get similar corollaries from Theorem 3.7, and also we can put $q_j = q$ in all of the corollaries of this paper to obtain better and simpler results.

The whole of the above theorems and corollaries that already are said, hold also for J^*-derivations. To clarify this point, we will just express an important theorem for J^*-derivations. The reader can also investigate the validity of the another results for J^*-derivations. Assume that A is a J^*-algebra with norm $\|\cdot\|$ and $p \geq 4$.

Theorem 3.8. *Let $\varphi : A^p \to [0, \infty)$ be a function. Denote by ϕ a function such that*

$$\phi(x_1, \cdots, x_p) := \sum_{n=0}^{\infty} d^r \varphi\left(d^t x_1, \cdots, d^t x_p\right) < \infty$$

for all $x_1, \cdots, x_p \in A$. *Here* $d = \frac{p-3}{p-1} < 1$, $r = n$, $t = -(n+1)$ *or* $t = n$, $r = -(n+1)$, *and* $\lim_{n\to\infty} d^{-3n}\varphi(d^n x, \cdots, d^n x) = 0$. *Suppose that* $f : A \to A$ *is a mapping satisfying*

$$\begin{aligned} \|\gamma_\mu f(x_1, \cdots, x_p)\| &\leq \varphi(x_1, \cdots, x_p), \\ \|f(xx^*x) - f(x)xx - xf(x)^*x - xxf(x)\| &\leq \varphi(x, \cdots, x) \end{aligned}$$

for all $\mu \in \mathbb{T}^1$ *and all* $x, x_1, \cdots, x_p \in A$. *Then there exists a unique* J^**-derivation* $\delta : A \to A$ *such that*

$$\|f(x) - \delta(x)\| \leq \frac{1}{p-1}\phi(x, \cdots, x, 0)$$

for all $x \in A$.

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NEW IDENTITIES ON GENOCCHI NUMBERS AND POLYNOMIALS

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Abstract. In [4], Kim investigated Frobenius-Euler polynomials arising from non-linear differential equations. Using Kim's idea in [4], Rim, Jeong and Park studied Euler polynomials arising from non-linear differential equations, in [8]. From these non-linear differential equations, we derive some new identities among the sums of products of Genocchi polynomials and Genocchi polynomials of higher order.

1. Introduction

As is well known, the Genocchi polynomials are defined by the generating function as follows:

$$\frac{2t}{e^t+1}e^{xt} = e^{G(x)t} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}, \text{(see [3,5,6,9]).} \tag{1}$$

with the usual convention of replacing $G^n(x)$ by $G_n(x)$.

In the special case $x = 0$, $G_n(0) = G_n$ are called the n-th Genocchi numbers. The Genocchi polynomials are also given by

$$G_n(x) = (G+x)^n = \sum_{l=0}^{n} \binom{n}{l} G_l x^{n-l}, \text{(see [3,5,6,9]).} \tag{2}$$

Thus, by (1) and (2), we get the recursive relation for G_n as follows:

$$G_0 = 0, \quad \text{and} \quad (G+1)^n + G_n = \begin{cases} 2 & \text{if } n = 1 \\ 0 & \text{if } n > 1, \end{cases} \tag{3}$$

with the usual convention of replacing G^k's in the binomial expansion of $(G+1)^n$ by G_k's.

Using the idea in [1,2] or [7] we are able to obtain a formula for the product of two Genocchi polynomials $G_n(x)$ and $G_m(x)$ as follows:

$$\begin{aligned} G_m(x)G_n(x) = {} & 2mn\sum_{r=1}^{m}\binom{m-1}{r}\frac{G_{r+1}}{r+1}\frac{B_{m+n-r-1}(x)}{m+n-r-1} \\ & + 2mn\sum_{s=1}^{n}\binom{n-1}{s}\frac{G_{s+1}}{s+1}\frac{B_{m+n-s-1}(x)}{m+n-s-1} \\ & + (-1)^{n+1}2\frac{m!n!}{(m+n)!}\frac{G_{m+n}}{m+n}, \end{aligned}$$

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where $B_m(x)$'s are the well-known Bernoulli polynomials. (See [3,5,6,9].)

For $r \in \mathbb{N}$, the Genocchi polynomials $G_n^{(r)}(x)$ of order r are defined by generating functions as follows:

$$\underbrace{\left(\frac{2t}{e^t+1}\right)\cdots\left(\frac{2t}{e^t+1}\right)}_{r\text{-times}} e^{xt} = \sum_{n=0}^{\infty} G_n^{(r)}(x)\frac{t^n}{n!}, \text{(see [3,5,6,9])}. \tag{4}$$

In the special case $x = 0$, $G_n^{(r)}(0) = G_n^{(r)}$ are called the n-th Genocchi numbers order r. (see [6,9])

In this paper we set

$$F = F(t) = \frac{1}{1+e^t}. \tag{5}$$

Then, by differentiating (5) k-times with respect to t, we get a non-linear differential equation with unknown coefficients $a_k(N)$ as follows:

$$(N-1)!F^N = \sum_{k=0}^{N-1} a_k(N)F^{(k)}, \text{(see [4, 8])}, \tag{6}$$

where $F^{(k)}(t) = \frac{d^kF(t)}{dt^k}$ and $F^N(t) = \underbrace{F(t)\times \ldots \times F(t)}_{N\text{–times}}$.

From (5) and (6), T. Kim was able to obtain an explicit formula for $a_k(N)$ as follows:

$$a_k(N) = \frac{N!}{(k+1)!}\sum_{l_1+\ldots+l_{k+1}=N}\frac{1}{l_1\ldots l_{k+1}}, \text{(see [4])}. \tag{7}$$

By (6) and (7), we get a non-linear differential equation with a solution $F(t) = \frac{1}{1+e^t}$ as follows:

$$F^N(t) = N\sum_{k=0}^{N-1}\frac{1}{(k+1)!}\sum_{l_1+\ldots+l_{k+1}=N}\frac{1}{l_1\ldots l_{k+1}}F^{(k)}(t), \text{(see [4])}. \tag{8}$$

In this paper, using the idea in [4], we derive some new identities among the sums of products of Genocchi polynomials and Genocchi polynomials of higher order.

2. Some Identities involving Genocchi polynomials

From (4), we recall that the Genocchi numbers of order r is given by

$$\underbrace{\left(\frac{2t}{e^t+1}\right)\cdots\left(\frac{2t}{e^t+1}\right)}_{r\text{–times}} = \sum_{n=0}^{\infty} G_n^{(r)}\frac{t^n}{n!}, \tag{9}$$

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From (5) and (9), we get

$$t^N F^N(t) = \left(\frac{1}{e^t+1}\right)^N t^N = \frac{1}{2^N}\left(\frac{2}{e^t+1}\right)^N t^N = \frac{1}{2^N}\sum_{n=0}^{\infty} G_n^{(N)}\frac{t^n}{n!}t^N. \tag{10}$$

By differentiating both sides of $F(t) = \frac{1}{1+e^t}$ k-times with respect to t, we get

$$t^N F^{(k)}(t) = \frac{1}{2}\sum_{n=0}^{\infty}\frac{G_{n+k+1}}{n+k+1}\frac{t^n}{n!}t^N. \tag{11}$$

From (8), (10) and (11), we get

$$\begin{aligned} t^N F^N(t) &= N\sum_{k=0}^{N-1}\frac{1}{(k+1)!}\sum_{l_1+\ldots+l_{k+1}=N}\frac{1}{l_1\ldots l_{k+1}}F^{(k)}(t)t^N \\ &= N\sum_{k=0}^{N-1}\frac{1}{(k+1)!}\sum_{l_1+\ldots+l_{k+1}=N}\frac{1}{l_1\ldots l_{k+1}}\frac{1}{2}\sum_{n=0}^{\infty}\frac{G_{n+k+1}}{n+k+1}\frac{t^n}{n!}t^N \\ &= t^N\frac{N}{2}N\sum_{n=0}^{\infty}\sum_{k=0}^{N-1}\frac{1}{(k+1)!}\sum_{l_1+\ldots+l_{k+1}=N}\frac{1}{l_1\ldots l_{k+1}}\frac{G_{n+k+1}}{n+k+1}\frac{t^n}{n!} \\ &= \frac{1}{2^N}\sum_{n=0}^{\infty}G_n^{(N)}\frac{t^n}{n!}t^N. \end{aligned} \tag{12}$$

Therefore, by (12), we obtain a new identity for Genocchi numbers of higher order.

Theorem 1 . For $N, n \in \mathbb{N}$, we have

$$G_n^{(N)} = 2^{N-1}Nt^N\sum_{k=0}^{N-1}\frac{1}{(k+1)!}\sum_{l_1+\ldots+l_{k+1}=N}\frac{1}{l_1\ldots l_{k+1}}\frac{G_{n+k+1}}{n+k+1}.$$

The following identity is well-known—we can get this identity by using the generating function for Genocchi numbers of order N.

$$\sum_{n=0}^{\infty}G_n^{(N)}\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{l_1+\ldots+l_N=n}\binom{n}{l_1, l_2, \ldots, l_N}G_{l_1}G_{l_2}\ldots G_{l_N}\right)\frac{t^n}{n!}. \tag{13}$$

From Theorem 1 and (13), we get the following identity.

Corollary 2. For $N, n \in \mathbb{N}$, we have

$$\begin{aligned} &\sum_{l_1+\ldots+l_N=n}\binom{n}{l_1, l_2, \ldots, l_N}G_{l_1}G_{l_2}\ldots G_{l_N} \\ &\qquad = 2^{N-1}Nt^N\sum_{k=0}^{N-1}\frac{1}{(k+1)!}\sum_{l_1+\ldots+l_{k+1}=N}\frac{1}{l_1\ldots l_{k+1}}\frac{G_{n+k+1}}{n+k+1}. \end{aligned}$$

We use the following notation

$$F(t,x) = \frac{1}{1+e^t}e^{xt},$$
$$F^N(t,x) = \underbrace{F(t)\times\cdots\times F(t)}_{N\text{ times}}e^{xt}. \tag{14}$$

Now by multiplying e^{tx} on both sides of (8), we have a non-linear differential equation, which has a solution $F(t,x) = \frac{1}{1+e^t}e^{xt}$ as follows:

$$t^N F^N(t,x) = N\sum_{k=0}^{N-1}\frac{1}{(k+1)!}\sum_{l_1+\ldots+l_{k+1}=N}\frac{1}{l_1\ldots l_{k+1}}F^{(k)}(t,x)t^N.$$

Using (11), we obtain the following equations:

$$\begin{aligned} t^N F^{(k)}(t)e^{xt} &= \frac{1}{2}\sum_{m=0}^{\infty}\frac{G_{m+k+1}}{m+k+1}\frac{t^m}{m!}e^{xt}t^N \\ &= \frac{1}{2}\sum_{m=0}^{\infty}\frac{G_{m+k+1}}{m+k+1}\frac{t^m}{m!}\sum_{l=0}^{\infty}\frac{t^l}{l!}x^l t^N \\ &= \frac{t^N}{2}\sum_{n=0}^{\infty}\left(\sum_{m+l=n}\frac{n!}{m!l!}\frac{t^n}{n!}\frac{G_{m+k+1}}{m+k+1}x^l\right) \\ &= \frac{t^N}{2}\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\frac{G_{m+k+1}}{m+k+1}x^{n-m}\right)\frac{t^n}{n!}. \end{aligned} \tag{15}$$

By (8) and (15), we get the value of $t^N F^N(t,x)$ as follows:

$$\begin{aligned} t^N F^N(t,x) &= N\sum_{k=0}^{N-1}\frac{1}{(k+1)!}\sum_{l_1+\ldots+l_{k+1}=N}\frac{1}{l_1\ldots l_{k+1}}F^{(k)}(t,x)t^N \\ &= \frac{1}{2}\sum_{m=0}^{\infty}\frac{G_{m+k+1}}{m+k+1}\frac{t^m}{m!}e^{xt}t^N \\ &= \frac{1}{2}\sum_{m=0}^{\infty}\frac{G_{m+k+1}}{m+k+1}\frac{t^m}{m!}\sum_{l=0}^{\infty}\frac{t^l}{l!}t^N \\ &= \frac{t^N}{2}\sum_{n=0}^{\infty}\left(\sum_{m+l=N}\frac{n!}{m!l!}\frac{t^n}{n!}\frac{G_{m+k+1}}{m+k+1}x^l\right) \\ &= \frac{t^N}{2}N\sum_{k=0}^{N-1}\frac{1}{(k+1)!}\sum_{l_1+\ldots+l_{k+1}=N}\frac{1}{l_1\ldots l_{k+1}} \\ &\qquad \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\frac{G_{m+k+1}}{m+k+1}x^{n-m}\right)\frac{t^n}{n!}. \end{aligned} \tag{16}$$

By (10), we can represent Genocchi polynomials of order N as follows:

$$2^N t^N F^N(t,x) = \sum_{n=0}^{\infty}G_n^{(N)}(x)\frac{t^n}{n!}. \tag{17}$$

From (16) and (17), we obtain a new identity for the n-th Genocchi polynomial of order N as given in the following theorem.

Theorem 3. For $N, n \in \mathbb{N}$, we have

$$G_n^{(N)}(x) = 2^{N-1} N t^N \times \left(\sum_{k=0}^{N-1} \frac{1}{(k+1)!} \sum_{l_1+\ldots+l_{k+1}=N} \frac{1}{l_1 \ldots l_{k+1}} \sum_{m=0}^{n} \binom{n}{m} \frac{G_{m+k+1}}{m+k+1} x^{n-m} \right).$$

From the generating function of Genocchi polynomials of higher order, we have the following well-known identities: For $N, n \in \mathbb{N}$,

$$G_n^{(N)}(x) = \sum_{l=0}^{n} \binom{n}{l} G_l^{(N)} x^{n-l}, \tag{18}$$

$$G_n^{(N)}(x) = \sum_{l_1+\ldots+l_N+m=n} \binom{n}{l_1, l_2, \ldots, l_N, m} G_{l_1} G_{l_2} \ldots G_{l_N} x^m. \tag{19}$$

From Theorem 3, (18) and (19) we have the following identities on Genocchi polynomials of order N with coefficients, sums of Genocchi numbers, product of sums of Genocchi numbers and Genocchi numbers of higher orders.

Corollary 4 . For $N, n \in \mathbb{N}$, we have

$$\begin{aligned} \sum_{l=0}^{n} \binom{n}{l} G_l^{(N)} x^{n-l} &= \sum_{l_1+\ldots+l_N+m=n} \binom{n}{l_1, l_2, \ldots, l_N, m} G_{l_1} G_{l_2} \ldots G_{l_N} x^m \\ &= 2^{N-1} N t^N \sum_{k=0}^{N-1} \frac{1}{(k+1)!} \\ &\qquad \sum_{l_1+\ldots+l_{k+1}=N} \frac{1}{l_1 \ldots l_{k+1}} \sum_{m=0}^{n} \binom{n}{m} \frac{G_{m+k+1}}{m+k+1} x^{n-m}. \end{aligned}$$

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SOME IDENTITIES OF HIGHER ORDER GENOCHI POLYNOMIALS ARISING FROM HIGHER ORDER GENOCCHI BASIS

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Abstract. In [9], D. Kim and T. Kim established some identities of higher order Bernoulli and Euler polynomials arising from Bernoulli and Euler basis respectively. Using the idea developed in [9], we present a study of some families of higher order Genocchi numbers an polynomials. In particular, by using the basis property of higher order Genocchi polynomials for the space of polynomials of degree less than and equal to n, we derive some interesting identities for the higher order Genocchi polynomials.

1. Introduction

As is well known, the *n-th Genocchi polynomials* of order r are defined by the generating function to be

$$\left(\frac{2t}{e^t+1}\right)^r e^{xt} = e^{G^{(r)}(x)t} = \sum_{n=0}^{\infty} G_n^{(r)}(x)\frac{t^n}{n!},\ (r \in \mathbb{R}) \tag{1.1}$$

with the usual convention about replacing $\left(G^{(r)}(x)\right)^n$ by $G_n^{(r)}(x)$ (see [1-22]). In our discussion we restrict r to be nonnegative integers. In the special case of $x=0$, $G_n^{(r)}(0) = G_n^{(r)}$ are called the *n-th Genocchi numbers* of order r.

By (1.1), we easily get

$$\begin{aligned} G_n^{(r)}(x) &= \sum_{l=0}^{n} \binom{n}{l} G_l^{(r)} x^{n-l} \\ &= \sum_{l=0}^{n} \binom{n}{l} G_{n-l}^{(r)} x^l \\ &= \sum_{n_1+n_2+\cdots+n_r+m=n} \binom{n}{n_1, n_2, \ldots, n_r, m} G_{n_1} G_{n_2} \cdots G_{n_r} x^m. \end{aligned} \tag{1.2}$$

From (1.2), we note that the leading coefficient of $G_n^{(r)}(x)$ is $\frac{n!}{m!} = \frac{n!}{(n-r)!}$ and $G_n^{(r)}(x)$ is a polynomial of degree $n-r$ with integer coefficients.

From (1.1), we have

$$G_n^{(0)}(x) = x^n \text{ and } \frac{G_n^{(r)}(x)}{dx} = (n-r)G_{n-1}^{(r)}(x).$$

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Let $\mathbb{P}_n = \{p(x) \in \mathbb{Q}[x] \mid \deg p(x) \leq n\}$ be the $(n+1)$ dimensional vector space over $\mathbb{Q}$. Probably $\{1, x, \ldots, x^n\}$ is the most natural basis for $\mathbb{P}_n$. But we consider $\left\{G_r^{(r)}(x), G_{r+1}^{(r)}(x), \ldots, G_{r+n}^{(r)}(x)\right\}$ as a basis for $\mathbb{P}_n$ for our purpose of arithmetical applications of Genocchi polynomials.

If $p(x) \in \mathbb{P}_n$, then $p(x)$ can be expressed by

$$p(x) = b_0 G_r^{(r)}(x) + b_1 G_{r+1}^{(r)}(x) + \cdots + b_n G_{r+n}^{(r)}(x).$$

In this paper, follow the idea in [10], we compute b_l from the higher order Genocchi basis for $\mathbb{P}_n$. And apply such results to arithmetically and combinatorially interesting identities involving $G_r^{(r)}(x), G_{r+1}^{(r)}(x), \ldots, G_{r+n}^{(r)}(x)$.

2. Higher order Genocchi polynomials

Let us assume that $p(x) \in \mathbb{P}_n$. Then $p(x)$ can be generated by $G_r^{(r)}(x), G_{r+1}^{(r)}(x), \ldots,$ $G_{r+n}^{(r)}(x)$ to be

$$p(x) = \sum_{k=0}^{n} b_k G_{r+k}^{(r)}(x). \tag{2.1}$$

Now we consider two linear operators $\tilde{\Delta}$ and D by

$$\tilde{\Delta} p(x) = p(x+1) + p(x) \text{ and } Dp(x) = \frac{dp(x)}{dx}. \tag{2.2}$$

Then we see that

$$\tilde{\Delta} D = D \tilde{\Delta} \text{ (see [9])}.$$

Then, by (2.1) and (2.2), we get

$$\tilde{\Delta} p(x) = \sum_{k=0}^{n} b_k \left(G_{r+k}^{(r)}(x+1) + G_{r+k}^{(r)}(x) \right). \tag{2.3}$$

From (1.1), we note that

$$\begin{aligned} &\sum_{n=0}^{\infty} \left\{ G_n^{(r)}(x+1) + G_n^{(r)}(x) \right\} \frac{t^n}{n!} \\ &= \left(\frac{2t}{e^t+1} \right)^r e^{(x+1)t} + \left(\frac{2t}{e^t+1} \right)^r e^{xt}. \end{aligned} \tag{2.4}$$

By simple calculation on (2.4), we get

$$G_n^{(r)}(x+1) + G_n^{(r)}(x) = 2n G_{n-1}^{(r-1)}(x). \tag{2.5}$$

Thus, by (2.3) and (2.5), we get

$$\begin{aligned} \tilde{\Delta} p(x) &= \sum_{k=0}^{n} b_k \tilde{\Delta} G_{r+k}^{(r)}(x) \\ &= \sum_{k=0}^{n} b_k 2(r+k) G_{r+k-1}^{(r-1)}(x) \end{aligned} \tag{2.6}$$

and

$$\tilde{\Delta}^2 p(x) = \sum_{k=0}^{n} b_k 2^2 (r+k)(r+k-1) G_{r+k-2}^{(r-2)}(x). \tag{2.7}$$

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Continuing this process, we have

$$\begin{aligned}\tilde{\Delta}^r p(x) &= \sum_{k=0}^{n} b_k 2^r (r+k)_r G_k^{(0)}(x) \\ &= \sum_{k=0}^{n} b_k 2^r \frac{(r+k)!}{r!} x^k.\end{aligned} \tag{2.8}$$

Let us take the operator D^k on (2.8), then

$$D^k \tilde{\Delta}^r p(x) = 2^r \sum_{l=k}^{n} b_l \frac{(r+l)!}{(l-k)!} x^{l-k}. \tag{2.9}$$

Let us take $x = 0$ on (2.9), we get

$$D^k \tilde{\Delta}^r p(0) = 2^r b_k (r+k)!.$$

Thus, we have

$$\begin{aligned}b_k &= \frac{1}{2^r (r+k)!} D^k \tilde{\Delta}^r p(0) \\ &= \frac{1}{2^r (r+k)!} \tilde{\Delta}^r D_k p(0) \\ &= \frac{1}{2^r (r+k)!} \sum_{j=0}^{r} \binom{r}{j} D^k p(j).\end{aligned} \tag{2.10}$$

Therefore, by (2.1) and (2.10), we obtain the following theorem.

Theorem 2.1. *For $n, r \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ and $p(x) \in \mathbb{P}_n$, we have*

$$p(x) = \frac{1}{2^r} \sum_{k=0}^{n} \left(\sum_{j=0}^{r} \frac{1}{(r+k)!} \binom{r}{j} D^k p(j) \right) G_{r+k}^{(r)}(x).$$

Let us take $p(x) = x^n \in \mathbb{P}_n$. Then we can see that $D^k x^n = \frac{n!}{(n-k)!} x^{n-k}$. Thus, by Theorem 2.1, we get

$$x^n = \frac{1}{2^r} \sum_{k=0}^{n} \sum_{j=0}^{r} \frac{k!}{(r+k)!} \binom{r}{j} \binom{n}{k} j^{n-k} G_{r+k}^{(r)}(x). \tag{2.11}$$

Let $p(x) = B_n^{(s)}(x)$, $s \in \mathbb{Z}_+$. Then, we have

$$D^k B_n^{(s)}(x) = \frac{n!}{(n-k)!} B_{n-k}^{(s)}(x). \tag{2.12}$$

By Theorem 2.1, we get

$$B_n^{(s)}(x) = \frac{1}{2^r} \sum_{k=0}^{n} \sum_{j=0}^{r} \frac{k!}{(r+k)!} \binom{r}{j} \binom{n}{k} B_{n-k}^{(s)}(j) G_{r+k}^{(r)}(x). \tag{2.13}$$

Therefore, by (2.13), we obtain the following corollary.

Corollary 2.2. *For $n, r, s \in \mathbb{Z}_+$, we have*

$$B_n^{(s)}(x) = \frac{1}{2^r} \sum_{k=0}^{n} \sum_{j=0}^{r} \frac{k!}{(r+k)!} \binom{r}{j} \binom{n}{k} B_{n-k}^{(s)}(j) G_{r+k}^{(r)}(x),$$

where $B_n^{(s)}(x)$ are the n-th Bernoulli polynomials of order s.

It is well known that

$$\frac{t}{e^t-1}e^{tx}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!},\ \frac{2}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}E_n(x)\frac{x^n}{n!}. \tag{2.14}$$

In the special case, $x=0$, let $B_n(0)=B_n$, $E_n(0)=E_n$. From (2.14), we easily derive the following identity:

$$B_n(x)=\sum_{k=0,\ k\neq 1}^{n}\binom{n}{k}B_kE_{n-k}(x)\in\mathbb{P}_n \text{ (see [8]).} \tag{2.15}$$

Let us take $p(x)=B_n(x)$. Then, we have

$$D^kB_n(x)=n(n-1)\cdots(n-k+1)B_{n-k}(x)=\frac{n!}{(n-k)!}B_{n-k}(x). \tag{2.16}$$

Therefore, by Theorem 2.1, (2.15), (2.16) and [9], we obtain the following theorem.

Theorem 2.3. *For $n,r\in\mathbb{Z}_+$, we have*

$$B_n(x)=\sum_{k=0,\ k\neq 1}^{n}\binom{n}{k}B_kE_{n-k}(x)=\frac{1}{2^r}\sum_{k=0}^{n}\sum_{j=0}^{r}\frac{k!}{(k+r)!}\binom{r}{j}\binom{n}{k}B_{n-k}(j)G_{k+r}^{(r)}(x).$$

Let us consider $p(x)=\sum_{k=0}^{n}B_k(x)B_{n-k}(x)$. Then, we have

$$D^kp(x)=\frac{(n+1)!}{(n-k+1)!}\sum_{l=k}^{n}B_{l-k}(x)B_{n-l}(x). \tag{2.17}$$

Thus, by Theorem 2.1, (2.17) and [9], we obtain the following theorem.

Theorem 2.4. *For $r,n\in\mathbb{Z}_+$, we have*

$$\begin{aligned}\sum_{k=0}^{n}B_k(x)B_{n-k}(x)&=\frac{2}{n+2}\sum_{k=0}^{n-2}\binom{n+2}{k}B_{n-k}B_k(x)+(n+1)B_n(x)\\&=\frac{1}{2^r}\sum_{k=0}^{n}\sum_{l=k}^{n}\sum_{j=0}^{r}\binom{r}{j}\binom{n+1}{k}\frac{k!}{(k+r)!}B_{l-k}(j)B_{n-l}(j)G_{k+r}^{(r)}(x).\end{aligned}$$

Let $m,n\in\mathbb{Z}_+$ with $n\geq m$, $n\geq 2$. Then, from [9], we have

$$B_m(x)B_{n-m}(x)=\sum_{l=0}^{[\frac{n}{2}]}\left\{\binom{m}{2l}(n-m)+\binom{n-m}{2l}m\right\}\frac{B_{2l}B_{n-2l}(x)}{n-2l}+(-1)^{m+1}\frac{B_n}{\binom{n}{m}}. \tag{2.18}$$

Let us take $p(x)=B_mB_{n-m}(x)\in\mathbb{P}_n$. Then, we have

$$D^kp(x)=\sum_{l=0}^{[\frac{n}{2}[}\left\{\binom{m}{2l}(n-m)+\binom{n-m}{2l}m\right\}\frac{b_{2l}}{n-2l}\times\frac{(n-2l)!}{(n-2l-k)!}B_{n-2l-k}(x), \tag{2.19}$$

for $1\leq k\leq n$.

Therefore, by Theorem 2.1 and (2.19), we obtain the following theorem.

Theorem 2.5. *For $m, n \in \mathbb{Z}_+$ with $n \geq m$, $n \geq 2$, we have*

$$B_m(x)B_{n-m}(x) = \frac{1}{2^r}\left\{\sum_{l=0}^{\left[\frac{n}{2}\right]}\sum_{j=0}^{r}\frac{k!}{(k+r)!}\binom{r}{j}\binom{n-2l}{k}\right.$$
$$\left.\times\left(\binom{m}{2l}(n-m)+\binom{n-m}{2l}m\right)\frac{B_{2l}B_{n-2l-k}(j)}{n-2l}\right\}G_{k+r}^{(r)}(x)+(-1)^{m+1}\frac{B_n}{\binom{n}{m}}.$$

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HIGHER-ORDER BERNOULLI, FROBENIUS-EULER AND EULER POLYNOMIALS

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Abstract. In this paper, we give some interesting identities of higher-order Bernoulli, Frobenius-Euler and Euler polynomials arising from umbral calculus. From our method of this paper, we can derive many interesting identities of special polynomials.

1. Introduction

For $\alpha \in \mathbb{R}$, the *Bernoulli polynomials* of order α are defined by the generating function to be

$$\left(\frac{t}{e^t-1}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{t^n}{n!}, \text{ (see [1,7,8,12,17,18]).} \tag{1.1}$$

In the special case, $x = 0$, $B_n^{(\alpha)}(0) = B_n^{(\alpha)}$ are called the *n-th Bernoulli number* of order α. By (1.1), we easily get

$$B_n^{(\alpha)}(x) = \left(B^{(\alpha)} + x\right)^n = \sum_{l=0}^{n}\binom{n}{l}B_l^{(\alpha)}x^{n-l},$$

with the usual convention about replacing $(B^{(\alpha)})^n$ by $B_n^{(\alpha)}$.

As is well known, the *Euler polynomials* of order α are also defined by the generating function to be

$$\left(\frac{2}{e^t+1}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x)\frac{t^n}{n!}, \text{ (see [11,12,13]).} \tag{1.2}$$

In the special case, $x = 0$, $E_n^{(\alpha)}(0) = E_n^{(\alpha)}$ are called the *n-th Euler numbers* of order α. From (1.2), we note that

$$E_n^{(\alpha)}(x) = \sum_{l=0}^{n}\binom{n}{l}E_n^{(\alpha)}x^{n-l}, \text{ (see [1-18]).}$$

For $\lambda \in \mathbb{C}$ with $\lambda \neq 1$, the *Frobenius-Euler polynomials* of order α are defined by the generating function to be

$$\left(\frac{1-\lambda}{e^t-\lambda}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} H_n^{(\alpha)}(x|\lambda)\frac{t^n}{n!}, \text{ (see [1,9,10,16]).} \tag{1.3}$$

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In the special case, $x = 0$, $H_n^{(\alpha)}(0|\lambda) = H_n^{(\alpha)}(\lambda)$ are called *n-th Frobenius-Euler numbers* of order α. By (1.3), we get

$$H_n^{(\alpha)}(x|\lambda) = \sum_{l=0}^{n} \binom{n}{l} H_l(\lambda) x^{n-l}, \text{ (see [1,9,10,16]).}$$

Let $\mathbb{P}$ be the algebra of polynomials in the variable x over $\mathbb{C}$ and let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. The action of the linear functional L on a polynomial $p(x)$ is denoted by $\langle L|p(x)\rangle$. We recall that the vector space on $\mathbb{P}^*$ are defined by $\langle L + M|p(x)\rangle = \langle L|p(x)\rangle + \langle M|p(x)\rangle$, $\langle cL|p(x)\rangle = c\,\langle L|p(x)\rangle$, where c is a complex constant.

Let

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \;\middle|\; a_k \in \mathbb{C} \right\}. \tag{1.4}$$

For $f(t) \in \mathcal{F}$, we define a linear functional on $\mathbb{P}$ by setting

$$\langle f(t)|x^n\rangle = a_n, \text{ for all } n \geq 0, \text{ (see [13,15]).} \tag{1.5}$$

From (1.4) and (1.5), we note that

$$\left\langle t^k|x^n\right\rangle = n!\delta_{n,k}, \ (n, k \geq 0), \text{ (see [13,15]),} \tag{1.6}$$

where $\delta_{n,k}$ is the Kronecker symbol.

For $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L|x^k\rangle}{k!} t^k$, we have $\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle$. So the map $L \mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ is thought of as both a formal power series and a linear functional. We shall call $\mathcal{F}$ the *umbral algebra*. The umbral calculus is the study of umbral algebra.

The order $o(f(t))$ of the non-zero power series $f(t)$ is the smallest integer k for which the coefficient of t^k does not vanish (see [13]). If $o(f(t)) = 1$, then $f(t)$ is called a *delta series*. If $o(f(t)) = 0$, then $f(t)$ is called an *invertible series* (see [15]). Let $o(f(t)) = 1$ and $o(g(t)) = 0$. Then there exists a unique sequence $S_n(x)$ of polynomials such that $\left\langle g(t)f(t)^k|S_n(x)\right\rangle = n!\delta_{n,k}$ $(n, k \geq 0)$. The sequence $S_n(x)$ is called *Sheffer sequence* for $(g(t), f(t))$, which is denoted by $S_n(x) \sim (g(t), f(t))$ (see [13, 15]). Let $f(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$. Then, by (1.6), we easily see that $\langle e^{yt}|\, p(x)\rangle = p(y)$, $\langle f(t)g(t)|\, p(x)\rangle = \langle f(t)|\, g(t)p(x)\rangle = \langle g(t)|\, f(t)p(x)\rangle$.

For $f(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$, we have

$$f(t) = \sum_{k=0}^{\infty} \frac{\left\langle f(t)|x^k\right\rangle}{k!} t^k, \ p(x) = \sum_{k=0}^{\infty} \frac{\left\langle t^k|p(x)\right\rangle}{k!} x^k \text{ (see [13,15]).} \tag{1.7}$$

By (1.7), we easily get

$$p^{(k)}(0) = \left\langle t^k|p(x)\right\rangle, \ \left\langle 1 \middle| p^{(k)}(x) \right\rangle = p^{(k)}(0). \tag{1.8}$$

From (1.8), we have

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k}, \ (k \geq 0), \text{ (see [13,5]).} \tag{1.9}$$

For $S_n(x) \sim (g(t), f(t))$, the generating function of Sheffer sequence $S_n(x)$ is given by

$$\frac{1}{g(\bar{f}(t))} e^{y\bar{f}(t)} = \sum_{k=0}^{\infty} \frac{S_k(y)}{k!} t^k, \text{ for all } y \in \mathbb{C}, \tag{1.10}$$

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where $\bar{f}(t)$ is the compositional inverse of $f(t)$ (see [15]). Let us assume that

$$S_n(x) \sim (1, f(t)),\ t_n(x) \sim (1, g(t)). \tag{1.11}$$

Then, we note that

$$S_n(x) = x\left(\frac{g(t)}{f(t)}\right)^n x^{-1} t_n(x),\ \text{(see [13,15])}. \tag{1.12}$$

By (1.6), we easily see that $x^n \sim (1, t)$.

In this paper, we give some interesting identities of higher-order Bernoulli, Frobenius-Euler and Euler polynomials involving multiple power and alternating sums which are derived from umbral calculus. By using our methods of this paper, we can obtain many interesting identities of special polynomials.

2. Higher-order Bernoulli, Frobenius-Euler and Euler polynomials

Let $S_n(x) \sim (g(t), f(t))$. Then we see that

$$g(t)S_n(x) \sim (1, f(t)). \tag{2.1}$$

From (1.12), (2.1) and $x^n \sim (1, t)$, we note that

$$S_n(x) = \frac{1}{g(t)} x \left(\frac{t}{f(t)}\right) x^{n-1}. \tag{2.2}$$

The equation (2.2) is important to derive our results in this paper. From (1.1), (1.2), (1.3) and (1.10), we can derive the following lemma:

Lemma 2.1. *For $n \geq 0$, $m \in \mathbb{N}$, we have*

$$m^n B_n^{(\alpha)}\left(\tfrac{x}{m}\right) \sim \left(\left(\tfrac{e^{mt}-1}{mt}\right)^\alpha, t\right),\ \tfrac{m^n}{m^\alpha} B_n^{(\alpha)}\left(\tfrac{x}{m}\right) \sim \left(\left(\tfrac{e^{mt}-1}{t}\right)^\alpha, t\right),$$
$$m^n E_n^{(\alpha)}\left(\tfrac{x}{m}\right) \sim \left(\left(\tfrac{e^{mt}+1}{2}\right)^\alpha, t\right),\ m^n H_n^{(\alpha)}\left(\tfrac{x}{m}\middle|\,\lambda\right) \sim \left(\left(\tfrac{e^{mt}-\lambda}{1-\lambda}\right)^\alpha, t\right).$$

Let us consider the following Sheffer sequences:

$$S_n(x) \sim \left(1, \frac{t^2}{e^t - 1}\right),\ t_n(x) \sim \left(1, \frac{t^2}{e^{mt} - 1}\right). \tag{2.3}$$

From (2.2), we have

$$\begin{aligned} S_n(x) =& x\left(\frac{e^t-1}{t}\right)^n x^{n-1} = x\sum_{l=0}^{\infty} \frac{n!}{(l+n)!} S_2(l+n, n) t^l x^{n-1} \\ =& x\sum_{l=0}^{n-1} \frac{\binom{n-1}{l}}{\binom{l+n}{n}} S_2(l+n, n) x^{n-1-l} \\ =& x\sum_{r=0}^{n-1} \frac{\binom{n-1}{r}}{\binom{2n-1-r}{n}} S_2(2n-1-r, n) x^r, \end{aligned} \tag{2.4}$$

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and

$$\begin{aligned}
t_n(x) =& x\left(\frac{e^{mt}-1}{t}\right)^n x^{n-1} = x\sum_{l=0}^{\infty}\frac{n!}{(l+n)!}S_2(l+n,n)m^{l+n}t^l x^{n-1}\\
=& x\sum_{l=0}^{n-1}\frac{n!}{(l+n)!}S_2(l+n,n)m^{n+l}(n-1)_l x^{n-1-l}\\
=& x\sum_{l=0}^{n-1}\frac{\binom{n-1}{l}}{\binom{l+n}{n}}S_2(l+n,n)m^{n+l}x^{n-1-l}\\
=& x\sum_{r=0}^{n-1}\frac{\binom{n-1}{r}}{\binom{2n-1-r}{n}}S_2(2n-1-r,n)m^{2n-1-r}x^r,
\end{aligned} \tag{2.5}$$

where $S_2(n,k)$ is the Stirling number of the second kind.

For $n \geq 1$, by (1.12) and (2.3), we get

$$\begin{aligned}
t_n(x) =& x\left(\frac{e^{mt}-1}{e^t-1}\right)^n x^{-1}S_n(x) = x\left(e^{-t}\sum_{l=1}^{m}e^{lt}\right)^n x^{-1}S_n(x)\\
=& xe^{-nt}\sum_{\substack{0\leq v_1,\ldots,v_m\leq n\\ v_1+\cdots+v_m=n}}\binom{n}{v_1,\ldots,v_m}e^{(v_1+2v_2+\cdots+mv_m)t}x^{-1}S_n(x)\\
=& x\sum_{s=0}^{\infty}\left\{\sum_{k=0}^{s}\binom{s}{k}(-n)^{s-k}\sum_{\substack{0\leq v_1,\ldots,v_m\leq n\\ v_1+\cdots+v_m=n}}\binom{n}{v_1,\ldots,v_m}\right.\\
&\left.\times\ (v_1+2v_2+\cdots+mv_m)^k\frac{t^s}{s!}\right\}x^{-1}S_n(x).
\end{aligned} \tag{2.6}$$

Let us define multiple power sum $S_k^{(n)}(m)$ as follows:

$$S_k^{(n)}(m) = \sum_{\substack{0\leq v_1,\ldots,v_m\leq n\\ v_1+\cdots+v_m=n}}\binom{n}{v_1,\ldots,v_m}(v_1+2v_2+\cdots+mv_m)^k. \tag{2.7}$$

By (2.5), (2.6) and (2.7), we get

$$\begin{aligned}
t_n(x) =& x\sum_{s=0}^{\infty}\sum_{k=0}^{s}\binom{s}{k}(-n)^{s-k}S_k^{(n)}(m)\frac{t^s}{s!}x^{-1}S_n(x)\\
=& x\sum_{r=0}^{n-1}\sum_{s=0}^{r}\sum_{k=0}^{s}\frac{\binom{r}{s}\binom{n-1}{r}\binom{s}{k}}{\binom{2n-1-r}{n}}(-n)^{s-k}S_2(2n-1-r,n)S_k^{(n)}(m)x^{r-s}.
\end{aligned} \tag{2.8}$$

From (1.12) and (2.3), we can also derive

$$\begin{aligned}t_n(x) =&x\left(\frac{e^{mt}-1}{e^t-1}\right)^n x^{-1}S_n(x) = x\left(\frac{e^{mt}-1}{t}\right)^n\left(\frac{t}{e^t-1}\right)^n x^{-1}S_n(x)\\ =&x\left(\frac{e^{mt}-1}{t}\right)^n\sum_{r=0}^{n-1}\frac{\binom{n-1}{r}}{\binom{2n-1-r}{n}}S_2(2n-1-r,n)B_r^{(n)}(x)\\ =&x\sum_{r=0}^{n-1}\sum_{s=0}^{r}\frac{\binom{r}{s}\binom{n-1}{r}}{\binom{s+n}{n}\binom{2n-1-r}{n}}S_2(s+n,n)S_2(2n-1-r,n)m^{n+s}B_{r-s}^{(n)}(x).\end{aligned} \tag{2.9}$$

Therefore, by (2.8) and (2.9), we obtain the following theorem.

Theorem 2.2. *For $n\geq 1$, we have*

$$\begin{aligned}&\sum_{r=0}^{n-1}\sum_{s=0}^{r}\sum_{k=0}^{s}\frac{\binom{r}{s}\binom{n-1}{r}\binom{s}{k}}{\binom{2n-1-r}{n}}(-n)^{s-k}S_2(2n-1-r,n)S_k^{(n)}(m)x^{r-s}\\ =&\sum_{r=0}^{n-1}\sum_{s=0}^{r}\frac{\binom{r}{s}\binom{n-1}{r}}{\binom{s+n}{n}\binom{2n-1-r}{n}}S_2(s+n,n)S_2(2n-1-r,n)m^{n+s}B_{r-s}^{(n)}(x).\end{aligned}$$

Let us consider the following Sheffer sequences:

$$S_n(x)\sim\left(1,e^t-1\right),\ t_n(x)\sim\left(1,e^{mt}-1\right), \tag{2.10}$$

where $m\in\mathbb{N}$ and $n\geq 0$.

For $n\geq 1$, by (2.2), we get

$$S_n(x) = x\left(\frac{t}{e^t-1}\right)^n x^{n-1} = xB_{n-1}^{(n)}(x), \tag{2.11}$$

and

$$t_n(x) = x\left(\frac{t}{e^{mt}-1}\right)^n x^{n-1}. \tag{2.12}$$

By Lemma 2.1 and (2.12), we get

$$t_n(x) = \frac{x}{m}B_{n-1}^{(n)}\left(\frac{x}{m}\right). \tag{2.13}$$

From (1.12) and (2.10), we can derive

$$\begin{aligned}S_n(x) =&x\left(\frac{e^{mt}-1}{e^t-1}\right)^n x^{-1}t_n(x) = x\left(e^{-t}\sum_{l=0}^{m}e^{lt}\right)^n x^{-1}t_n(x)\\ =&\frac{x}{m}\sum_{s=0}^{\infty}\sum_{k=0}^{s}\binom{s}{k}(-n)^{s-k}S_k^{(n)}(m)\frac{t^s}{s!}B_{n-1}^{(n)}\left(\frac{x}{m}\right)\\ =&x\sum_{s=0}^{n-1}\sum_{k=0}^{s}\binom{s}{k}\binom{n-1}{s}(-n)^{s-k}S_k^{(n)}(m)B_{n-1-s}^{(n)}\left(\frac{x}{m}\right)m^{-s-1}.\end{aligned} \tag{2.14}$$

Theorem 2.3. *For $n,m\geq 1$, we have*

$$B_{n-1}^{(n)}(x) = \sum_{r=0}^{n-1}\sum_{s=0}^{r}\sum_{k=0}^{s}\frac{\binom{r}{s}\binom{n-1}{r}\binom{s}{k}}{\binom{2n-1-r}{n}}(-n)^{s-k}S_2(2n-1-r,n)S_k^{(n)}(m)x^{r-s}.$$

Let us assume that

$$S_n(x) \sim \left(1, t\left(\frac{e^t+1}{2}\right)\right),\ t_n(x) \sim \left(1, \left(\frac{e^{mt}+1}{2}\right)t\right), \tag{2.15}$$

where $n \geq 0$ and $m \in \mathbb{N}$ with $m \equiv 1 \pmod 2$. By (2.2), we get

$$S_n(x) = x\left(\frac{2}{e^t+1}\right)^n x^{n-1} = xE_{n-1}^{(n)}(x), \tag{2.16}$$

and

$$t_n(x) = x\left(\frac{2}{e^{mt}+1}\right)^n x^{n-1}. \tag{2.17}$$

From Lemma 2.1 and (2.17), we note that

$$t_n(x) = xm^{n-1}E_{n-1}^{(n)}\left(\frac{x}{m}\right). \tag{2.18}$$

By (1.12) and (2.15), we see that

$$\begin{aligned}
&S_n(x)\\
=&x\left(\frac{e^{mt}+1}{e^t+1}\right)^n x^{-1}t_n(x) = x\left(-e^{-t}\sum_{l=1}^{m}(-e^t)^l\right)^n x^{-1}t_n(x)\\
=&(-1)^n xe^{-nt} \sum_{\substack{0\leq v_1,\dots,v_m\leq n\\ v_1+\cdots+v_m=n}} \binom{n}{v_1,\dots,v_m}\\
&\times(-1)^{v_1+2v_2+\cdots+mv_m}e^{(v_1+2v_2+\cdots+mv_m)t}x^{-1}t_n(x)\\
=&(-1)^n xe^{-nt}\sum_{k=0}^{\infty}\Bigg(\sum_{\substack{0\leq v_1,\dots,v_m\leq n\\ v_1+\cdots+v_m=n}} \binom{n}{v_1,\dots,v_m}\\
&\times(-1)^{v_1+2v_2+\cdots+mv_m}\left(v_1+2v_2+\cdots+mv_m\right)^k\Bigg)\frac{t^k}{k!}\,x^{-1}t_n(x).
\end{aligned} \tag{2.19}$$

Let us define multiple alternating power sums $T_k^{(n)}(m)$ as follows:

$$\begin{aligned}
T_k^{(n)}(m) = &\sum_{\substack{0\leq v_1,\dots,v_m\leq n\\ v_1+\cdots+v_m=n}} \binom{n}{v_1,\dots,v_m}\\
&\times(-1)^{v_1+2v_2+\cdots+mv_m}\left(v_1+2v_2+\cdots+mv_m\right)^k.
\end{aligned} \tag{2.20}$$

By (2.19) and (2.20), we get

$$\begin{aligned}
S_n(x) =&(-1)^n x\sum_{s=0}^{\infty}\sum_{k=0}^{s}\binom{s}{k}(-n)^{s-k}T_k^{(n)}(m)\frac{t^s}{s!}m^{n-1}E_{n-1}^{(n)}\left(\frac{x}{m}\right)\\
=&(-1)^n x\sum_{s=0}^{n-1}\sum_{k=0}^{s}\binom{s}{k}\binom{n-1}{s}(-n)^{s-k}T_k^{(n)}(m)m^{n-s-1}E_{n-1-s}^{(n)}\left(\frac{x}{m}\right).
\end{aligned} \tag{2.21}$$

Therefore, by (2.16) and (2.21), we obtain the following theorem.

Theorem 2.4. *For $n, m \geq 1$ with $m \equiv 1 \pmod 2$, we have*

$$E_{n-1}^{(n)}(x) = (-1)^n \sum_{s=0}^{n-1} \sum_{k=0}^{s} \binom{s}{k} \binom{n-1}{s} (-n)^{s-k} T_k^{(n)}(m) m^{n-s-1} E_{n-1-s}^{(n)}\left(\frac{x}{m}\right).$$

Let us consider the following Sheffer sequences:

$$S_n(x) \sim \left(1, t\left(\frac{e^t-\lambda}{1-\lambda}\right)\right),\ t_n(x) \sim \left(1, t\left(\frac{e^{mt}-\lambda^m}{1-\lambda^m}\right)\right), \tag{2.22}$$

where $m \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ with $\lambda \neq 0$, $\lambda^m \neq 1$. By (2.2), we get

$$S_n(x) = x\left(\frac{1-\lambda}{e^t-\lambda}\right)^n x^{n-1} = xH_{n-1}^{(n)}(x|\lambda), \tag{2.23}$$

and

$$t_n(x) = x\left(\frac{1-\lambda^m}{e^{mt}-\lambda^m}\right)^n x^{n-1}. \tag{2.24}$$

From Lemma 2.1 and (2.24), we have

$$t_n(x) = m^{n-1} x H_{n-1}^{(n)}\left(\frac{x}{m}\middle|\lambda^m\right). \tag{2.25}$$

By (1.12) and (2.22), we get

$$\begin{aligned}
&S_n(x)\\
=&x\left(\frac{1-\lambda}{1-\lambda^m}\right)^n \left(\frac{e^{mt}-\lambda^m}{e^t-\lambda}\right)^n x^{-1} t_n(x)\\
=&x\left(\frac{1-\lambda}{1-\lambda^m}\right)^n \lambda^{mn-n} \left(\frac{1-\left(\frac{e^t}{\lambda}\right)^m}{1-\frac{e^t}{\lambda}}\right)^n x^{-1} t_n(x)\\
=&x\left(\frac{1-\lambda}{1-\lambda^m}\right)^n \lambda^{mn-n} \left(\frac{\lambda}{e^t}\sum_{l=1}^{m}\left(\frac{e^t}{\lambda}\right)^l\right)^n x^{-1} t_n(x)\\
=&\left(\frac{1-\lambda}{1-\lambda^m}\right)^n \lambda^{mn} x e^{-nt} \sum_{\substack{0 \leq v_1,\ldots,v_m \leq n\\ v_1+\cdots+v_m=n}} \binom{n}{v_1,\ldots,v_m}\\
&\times \lambda^{-(v_1+2v_2+\cdots+mv_m)} e^{(v_1+2v_2+\cdots+mv_m)t} x^{-1} t_n(x)\\
=&\left(\frac{1-\lambda}{1-\lambda^m}\right)^n \lambda^{mn} x \sum_{s=0}^{\infty}\sum_{k=0}^{s}\binom{s}{k}(-n)^{s-k} \sum_{\substack{0 \leq v_1,\ldots,v_m \leq n\\ v_1+\cdots+v_m=n}} \binom{n}{v_1,\ldots,v_m}\\
&\times \lambda^{-(v_1+2v_2+\cdots+mv_m)} (v_1+2v_2+\cdots+mv_m)^k \frac{t^s}{s!} m^{n-1} H_{n-1}^{(n)}\left(\frac{x}{m}\middle|\lambda^m\right).
\end{aligned} \tag{2.26}$$

Let us define λ-analogue of multiple power sums $S_k^{(n)}(m|\lambda)$ as follows:

$$S_k^{(n)}(m|\lambda) = \sum_{\substack{0 \le v_1, \dots, v_m \le n \\ v_1 + \cdots + v_m = n}} \binom{n}{v_1, \dots, v_m} \times \lambda^{-(v_1+2v_2+\cdots+mv_m)}(v_1 + 2v_2 + \cdots + mv_m)^k. \tag{2.27}$$

From (2.26) and (2.27), we have

$$\begin{aligned} S_n(x) &= \left(\frac{1-\lambda}{1-\lambda^m}\right)^n \lambda^{mn} x \sum_{s=0}^{n-1}\sum_{k=0}^{s} \binom{s}{k}(-n)^{s-k} S_k^{(n)}(m|\lambda)\frac{t^s}{s!} m^{n-1} H_{n-1}^{(n)}\left(\frac{x}{m}\Big|\lambda^m\right) \\ &= \left(\frac{1-\lambda}{1-\lambda^m}\right)^n \lambda^{mn} x \sum_{s=0}^{n-1}\sum_{k=0}^{s} \binom{s}{k}(-n)^{s-k} S_k^{(n)}(m|\lambda) \\ &\quad \times \binom{n-1}{s} m^{n-1-s} H_{n-1-s}^{(n)}\left(\frac{x}{m}\Big|\lambda^m\right). \end{aligned} \tag{2.28}$$

Therefore, by (2.23) and (2.28), we obtain the following theorem.

Theorem 2.5. *For $m, n \ge 1$, $\lambda \in \mathbb{C}$ with $\lambda \ne 0$, $\lambda^m \ne 1$, we have*

$$\begin{aligned} H_{n-1}^{(n)}(x|\lambda) = \left(\frac{1-\lambda}{1-\lambda^m}\right)^n \lambda^{mn} \sum_{s=0}^{n-1}\sum_{k=0}^{s} \binom{s}{k}\binom{n-1}{s}(-n)^{s-k} m^{n-1-s} \\ \times S_k^{(n)}(m|\lambda) H_{n-1-s}^{(n)}\left(\frac{x}{m}\Big|\lambda^m\right). \end{aligned}$$

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On lacunary statistical convergence of double sequences in locally solid Riesz spaces

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Abstract. In this paper, we introduce the concepts of lacunary statistical τ-convergence, lacunary statistical τ-bounded and lacunary statistical τ-Cauchy for double sequences by using the double lacunary density in the framework of locally solid Riesz spaces. We also introduce the notion of $S^{*}_{\bar{\theta}}(\tau)$-convergence of double sequences in this setup which seems to be a quite new and interesting idea and prove some interesting results related to these notions.

Keywords and phrases: Double sequences; statistical convergence; statistical Cauchy; lacunary sequence; locally solid Riesz space.
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1. Background, notations and preliminaries

The concept of lacunary statistical convergence as a generalization of statistical convergence, and any concept involving statistical convergence plays a vital role not only in pure mathematics but also in other branches of science involving mathematics, especially in information theory, computer science, biological science, dynamical systems, geographic information systems, population modelling, and motion planning in robotics.

The idea of statistical convergence was formerly given under the name almost convergence by Zygmund in the first edition of his celebrated monograph published in Warsaw in 1935 [51]. The concept was formally introduced by Steinhaus [48] and Fast [13] and later was introduced by Schoenberg [47], and also independently by Buck [4]. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Later on it was further investigated from various points of view. For example, statistical convergence has been investigated in summability theory by (Çakalli and Khan [7], Fridy [15], Prullage [40], Šalát [43]), topological groups (Çakalli [5], [6]), topological spaces (Di Maio and Kočinac [22]), function spaces (Caserta and Kočinac [10], Caserta et al. [9]), locally convex spaces (Maddox [21]), locally solid Riesz spaces (Albayrak and Pehlivan [1], Mohiuddine et al. [24, 28]), intuitionistic fuzzy normed spaces (Mohiuddine and Lohani [30]), fuzzy 2-normed spaces (Mohiuddine et al., [33]). Mursaleen and Edely [34] extended the above idea from single to double sequences by using two dimensional analogue of natural density and established

relations between statistical convergence and strongly Cesáro summable double sequences. Mursaleen and Mohiuddine [35] defined this notions for double sequences intuitionistic fuzzy normed spaces. Recently, Mohiuddine et al. [27] introduced this notions for double sequences locally solid Riesz spaces and proved some interesting results. Fridy and Orhan [16] introduced the concept of lacunary statistical convergence for real sequences. Savaş and Patterson ([45, 46]) extended the notion of lacunary statistical convergence from single to double sequences of real numbers and proved some interesting results. For more details, related concepts and applications, we refer to ([3, 8, 11, 16, 17, 19, 23, 25, 26, 29, 32, 36, 37, 38, 44, 49].)

Now we recall some of the basic concepts related to statistical convergence and lacunary sequence.

Let $E \subseteq \mathbb{N}$. Then the natural density of E is denoted by $\delta(E)$ and is defined by

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} |\{k \in E : k \leq n\}| \text{ exists,}$$

where the vertical bar denotes the cardinality of the respective set.

The number sequence $x = (x_k)$ is said to be *statistically convergent* to the number ℓ if for each $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |x_k - \ell| \geq \epsilon\}| = 0.$$

In this case, we write $st - \lim x_k = \ell$.

By a lacunary sequence $\theta = (k_r)$, where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $h_r : k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be defined by q_r (see [14]).

Let θ be a lacunary sequence and $I_r = \{k : k_{r-1} < k \leq k_r\}$. Let $K \subseteq \mathbb{N}$. The number

$$\delta_\theta(K) = \lim_r \frac{1}{h_r} |\{i \in I_r : i \in K\}|$$

is said to be *lacunary density* i.e. *θ-density* of K, provided the limit exists.

The idea of statistical convergence for single sequences in topological spaces has been introduced in ([22]). Now we introduced the concept of lacunary statistical convergence in topological spaces as follows:

Let θ be a lacunary sequence. A sequence $x = (x_k)$ in a topological space X is said to be *lacunary statistical convergent* or *S_θ-convergent* to ℓ provided that for each neighborhood V of ℓ, the set

$$K(V) = \{k \in \mathbb{N} : x_k \notin V\}$$

has θ-density zero. In this case we write $S_\theta\text{-}\lim x = \ell$ or $(x_k) \xrightarrow{S_\theta} \ell$.

By the convergence of a double sequence we mean the convergence in the Pringsheim's sense [39]. A double sequence $x = (x_{k,l})$ has a *Pringsheim* limit L (denoted by $P - \lim x = L$) provided that given an $\varepsilon > 0$ there exists an $n \in \mathbb{N}$ such that $|x_{k,l} - L| < \varepsilon$ whenever $k, l > n$. We shall describe such an $x = (x_{k,l})$ more briefly as "$P - convergent$".

Let $K \subset \mathbb{N} \times \mathbb{N}$ and $K(m, n)$ denotes the number of (i, j) in K such that $i \leq m$ and $j \leq n$, (see [34]). Then the lower natural density of K is defined by $\underline{\delta}_2(K) = \liminf_{m,n\to\infty} \frac{|K(m,n)|}{mn}$. In case, the sequence $(\frac{K(m,n)}{mn})$ has a limit in Pringsheim's

sense, then we say that K has a *double natural density* and is defined by $P-\lim_{m,n\to\infty}\frac{|K(m,n)|}{mn}=\delta_2(K)$.

The double sequence $\overline{\theta}=\theta_{r,s}=\{(k_r,l_s)\}$ is called *double lacunary sequence* if there exist two increasing sequences of integers such that (see [45])

$$k_o=0,\ h_r=k_r-k_{r-1}\to\infty \ \text{ as } r\to\infty$$

and

$$l_o=0,\ \overline{h_s}=l_s-l_{s-1}\to\infty \ \text{ as } s\to\infty.$$

Notations: $k_{r,s}=k_rl_s$, $h_{r,s}=h_r\overline{h_s}$ and $\theta_{r,s}$ is determined by

$$I_{r,s}=\{(k,l):\ k_{r-1}<k\le k_r \text{ and } l_{s-1}<l\le l_s\},$$

$$q_r=\frac{k_r}{k_{r-1}},\overline{q_s}=\frac{l_s}{l_{s-1}} \ \textit{and}\ q_{r,s}=q_r\overline{q_s}.$$

Let $\overline{\theta}=\{(k_r,l_s)\}$ be a double lacunary sequence. Let $K\subseteq\mathbb{N}\times\mathbb{N}$. The number

$$\delta_{\overline{\theta}}(K)=P-\lim_{r,s}\frac{1}{h_{r,s}}|\{(i,j)\in I_{r,s}:(i,j)\in K\}|$$

is said to be *double lacunary density* [8] i.e. $\overline{\theta}$-*density* of K, provided the limit exists.

2. Locally solid Riesz spaces

On the other hand, a Riesz space is an ordered vector space which is a lattice at the same time. It was first introduced by Riesz [42] in 1928 and we refer to ([1, 2, 18, 20, 50]) for more details. Now, we recall some basic definitions and notions related to the concept of locally solid Riesz spaces. Let X be a real vector space and let $\le$ be a partial order on this space. Then X is said to be an *ordered vector space* if it satisfies the following properties:

(i) if $x,y\in X$ and $y\le x$, then $y+z\le x+z$ for each $z\in X$.
(ii) if $x,y\in X$ and $y\le x$, then $ay\le ax$ for each $a\ge 0$.

If, in addition, X is a lattice with respect to the partial order, then X is said to be a *Riesz space* (or a *vector lattice*)(see [50]), if for each pair of elements $x,y\in X$ the supremum and infimum of the set $\{x,y\}$ both exist in X. We shall write

$$x\vee y=\sup\{x,y\} \text{ and } x\wedge y=\inf\{x,y\}.$$

For an element x of a Risez space X, the *positive part* of x is defined by $x^+=x\vee\overline{0}=\sup\{x,\overline{0}\}$, the *negative part* of x by $x^-=(-x)\vee\overline{0}$ and the *absolute value* of x by $|x|=x\vee(-x)$, where $\overline{0}$ is the zero element of X.

A subset S of a Riesz space X is said to be solid if $y\in S$ and $|y|\le|x|$ implies $x\in S$.

A *topological vector space* (X,τ) is a vector space X which has a topology (linear) τ, such that the algebraic operations of addition and scalar multiplication in X are continuous. Continuity of addition means that the function $f:X\times X\to X$ defined by $f(x,y)=x+y$ is continuous on $X\times X$, and continuity of scalar multiplication means that the function $f:\mathbb{R}\times X\to X$ defined by $f(a,x)=ax$ is continuous on $\mathbb{R}\times X$.

Every linear topology τ on a vector space X has a base N for the neighborhoods of $\overline{\theta}$ satisfying the following properties:

(1) Each $Y \in N$ is a *balanced set*, that is, $ax \in Y$ holds for all $x \in Y$ and for every $a \in \mathbb{R}$ with $|a| \leq 1$.
(2) Each $Y \in N$ is an *absorbing set* , that is , for every $x \in X$, there exists $a > 0$ such that $ax \in Y$.
(3) For each $Y \in N$ there exists some $E \in N$ with $E + E \subseteq Y$.

A linear topology τ on a Riesz space X is said to be *locally solid* (see[41]) if τ has a base at zero consisting of solid sets. A *locally solid Riesz space* (X, τ) is a Riesz space equipped with a locally solid topology τ.

Recall that a topological space is first countable if each point has a countable (decreasing) local base.

Throughout the article, the symbol N_{sol} we will denote any base at zero consisting of solid sets and satisfying the conditions (1), (2) and (3) in a locally solid topology. Also we assume $\overline{\theta}$ is a double lacunary sequence.

3. Double lacunary statiatical convergence in LSR-spaces

Throughout the article X will denote the Hausdorff locally solid Riesz space, which satisfies the first axiom of countability. For our convenience, here and in what follows, we shall write an LSR-space instead of a locally solid Riesz space. Quite recently, Mohiuddine and Alghamdi [24] introduced the concept of lacunary statistical convergence in locally solid Riesz spaces as follows.

Definition 3.1 [24]. Let (X, τ) be a LSR-space and let θ be a lacunary sequence. A sequence $x = (x_k)$ in X is said to be *lacunary statistical τ-convergent* or *$S_\theta(\tau)$-convergent* to the element $\ell \in X$ if for every τ-neighborhood V of zero, the set

$$K(V) = \{k \in \mathbb{N} : x_k - \ell \notin V\}$$

has θ-density zero. i.e. $\delta_\theta(K(V)) = 0$, or

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : x_k - \ell \notin V\}| = 0.$$

In this case we write $S_\theta(\tau)$-$\lim x = \ell$ or $(x_k) \overset{S_\theta(\tau)}{\longrightarrow} \ell$.

Definition 3.2 [31]. Let (X, τ) be a LSR-space. Then, a double sequence $(x_{k,l})$ of points in X is said to be *double lacunary statistical τ-convergent* or *$S_{\overline{\theta}}(\tau)$-convergent* to an element x_0 of X if for each τ-neighborhood V of zero,

$$\delta_{\overline{\theta}}(\{(k, l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_0 \notin V\}) = 0$$

i.e.,

$$P - \lim_{r,s} \frac{1}{h_{r,s}} |\{(k, l) \in I_{r,s} : x_{k,l} - x_0 \notin V\}| = 0.$$

In this case, we write $S_{\overline{\theta}}(\tau) - \lim_{k,l\to\infty} x_{k,l} = x_0$ or $(x_{k,l}) \overset{S_{\overline{\theta}}(\tau)}{\longrightarrow} x_0$.

Now we define:

Definition 3.3. Let (X, τ) be a LSR-space. Then, a double sequence $(x_{k,l})$ of points in X is said to be *double lacunary statistical τ-bounded* or *$S_{\overline{\theta}}(\tau)$-bounded* in X if for each τ-neighborhood V of zero, there is some $a > 0$,

$$\delta_{\overline{\theta}}(\{(k, l) \in \mathbb{N} \times \mathbb{N} : ax_{k,l} \notin V\}) = 0.$$

Theorem 3.1. Let (X, τ) be a LSR-space and $x = (x_{k,l})$ be a double sequence in X. Then, every $S_{\overline{\theta}}(\tau)$-convergent sequences in X has only one limit.

Proof. Assume that $x = (x_{k,l})$ is a double sequence in X such that $S_{\overline{\theta}}(\tau)$-$\lim_{k,l} x_{k,l} = x_0$ and $S_{\overline{\theta}}(\tau)$-$\lim_{k,l} x_{k,l} = y_0$.

Let V be any τ-neighborhood of zero. Also for each τ-neighborhood V of zero there exists $Y \in N_{sol}$ such that $Y \subseteq V$. Choose any $W \in N_{sol}$ such that $W + W \subseteq Y$. We define the following sets:

$$A_1 = \{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_0 \in W\}$$

$$A_2 = \{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - y_0 \in W\}.$$

Since $S_{\overline{\theta}}(\tau) - \lim x_{k,l} = x_0$ and $S_{\overline{\theta}}(\tau) - \lim x_{k,l} = y_0$, we get $\delta_{\overline{\theta}}(A_1) = 1$ and $\delta_{\overline{\theta}}(A_2) = 1$.
Now, let $A = A_1 \cap A_2$. Then, we have

$$x_0 - y_0 = x_0 - x_{k,l} + x_{k,l} - y_0 \in W + W \subseteq Y \subseteq V,$$

for every $(k,l) \in A$. Hence for each τ-neighborhood V of zero we have $x_0 - y_0 \in V$. Since (X, τ) is Hausdorff, the intersection of all τ-neighborhoods V of zero is the singleton set $\{\overline{0}\}$. Thus, we get $x_0 - y_0 = \overline{0}$, i.e., $x_0 = y_0$. $\square$

The following theorem establish an algebraic characterization of double lacunary statistical convergence in locally solid Riesz spaces.

Theorem 3.2. Let (X, τ) be a LSR-space and $(x_{k,l})$ and $(y_{k,l})$ be two double sequences in X. Then, we have the following:

(i) If $S_{\overline{\theta}}(\tau)$-$\lim_{k,l} x_{k,l} = x_0$ and $S_{\overline{\theta}}(\tau)$-$\lim_{k,l} y_{k,l} = y_0$ then $S_{\overline{\theta}}(\tau)$-$\lim_{k,l}(x_{k,l}+y_{k,l}) = x_0 + y_0$.

(ii) If $S_{\overline{\theta}}(\tau)$-$\lim_{k,l} x_{k,l} = x_0$ then $S_{\theta}(\tau)$-$\lim_{k,l} ax_{k,l} = ax_0$ for $a \in \mathbb{R}$.

Proof. (i) Assume that V is an arbitrary τ-neighborhood of zero. Then there exists $Y \in N_{sol}$ such that $Y \subseteq V$. Choose $W \in N_{sol}$ such that $W + W \subseteq Y$. Since $S_{\overline{\theta}}(\tau)$-$\lim_{k,l} x_{k,l} = x_0$ and $S_{\overline{\theta}}(\tau)$-$\lim_{k,l} y_{k,l} = y_0$. We write

$$B_1 = \{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_0 \in W\}$$

$$B_2 = \{(k,l) \in \mathbb{N} \times \mathbb{N} : y_{k,l} - y_0 \in W\}.$$

Then we have $\delta_{\overline{\theta}}(B_1) = 1 = \delta_{\overline{\theta}}(B_2)$.

Let $B = B_1 \cap B_2$. Hence we have $\delta_{\overline{\theta}}(B) = 1$ and

$$(x_{k,l} + y_{k,l}) - (x_0 + y_0) = (x_{k,l} - x_0) + (y_{k,l} - y_0) \in W + W \subseteq Y \subseteq V,$$

for every $(k,l) \in B$. Therefore

$$P - \lim_{r,s \to \infty} \frac{1}{h_{r,s}} |\{(k,l) \in I_{r,s} : (x_{k,l} + y_{k,l}) - (x_0 + y_0) \in V\}| = 1.$$

Since V is arbitrary, we have $S_{\overline{\theta}}(\tau) - \lim(x_{k,l} + y_{k,l}) = x_0 + y_0$.

(ii) Assume that V is an arbitrary τ-neighborhood of zero and $S_{\overline{\theta}}(\tau)$-$\lim_{k,l} x_{k,l} = x_0$. Then there exists $Y \in N_{sol}$ such that $Y \subseteq V$ and we have

$$P-\lim_{r,s\to\infty}\frac{1}{h_{r,s}}|\{(k,l)\in I_{r,s}: x_{k,l}-x_0\in Y\}|=1.$$

Since Y is balanced, $x_{k,l}-x_0 \in Y$ implies that $a(x_{k,l}-x_0)\in Y$ for every $a\in\mathbb{R}$ with $|a|\leq 1$. Hence

$$\{(k,l)\in\mathbb{N}\times\mathbb{N}: x_{k,l}-x_0\in Y\}$$
$$\subseteq\{(k,l)\in\mathbb{N}\times\mathbb{N}: ax_{k,l}-ax_0\in Y\}\subseteq\{(k,l)\in\mathbb{N}\times\mathbb{N}: x_{k,l}-x_0\in V\}.$$

Thus, we have

$$P-\lim_{r,s\to\infty}\frac{1}{h_{r,s}}|\{(k,l)\in I_{r,s}: x_{k,l}-x_0\in V\}|=1$$

for each τ-neighborhood V of zero. Now, let $|a|>1$ and $[|a|]$ be the smallest integer greater than or equal to $|a|$. Then there exists $W\in N_{sol}$ such that $[|a|]W\subseteq Y$. Since $S_{\overline{\theta}}(\tau)$-$\lim_{k,l} x_{k,l}=x_0$ we have the set

$$K=\{(k,l)\in\mathbb{N}\times\mathbb{N}: x_{k,l}-x_0\in W\}$$

has $\overline{\theta}$-density 1. Therefore

$$|ax_{k,l}-ax_0|=|a||x_{k,l}-x_0|\leq[|a|]|x_{k,l}-x_0|\in[|a|]W\subseteq Y\subseteq V,$$

for every $(k,l)\in K$. Since Y is solid, we have $ax_{k,l}-ax_0\in Y$ for every $(k,l)\in K$. This implies that $ax_{k,l}-ax_0\in V$ for every $(k,l)\in K$. Thus,

$$P-\lim_{r,s\to\infty}\frac{1}{h_{r,s}}|\{(k,l)\in I_{r,s}: ax_{k,l}-ax_0\in V\}|=1,$$

for each τ-neighborhood V of zero. Hence $S_{\overline{\theta}}(\tau)$-$\lim_{k,l} ax_{k,l}=ax_0$. □

Theorem 3.3. Let (X,τ) be a LSR-space. If a double sequence $(x_{k,l})$ in X is $S_{\overline{\theta}}(\tau)$-convergent, then it is $S_{\overline{\theta}}(\tau)$-bounded.

Proof. Assume that $(x_{k,l})$ is $S_{\overline{\theta}}(\tau)$-convergent to a point x_0 in X and V is an arbitrary τ-neighborhood of zero. Suppose that there exists $Y\in N_{sol}$ such that $Y\subseteq V$. Choosing $W\in N_{sol}$ be such that $W+W\subseteq Y$. Since $S_{\overline{\theta}}(\tau)$-$\lim_{k,l\to\infty} x_{k,l}=x_0$, the set

$$A=\{(k,l)\in\mathbb{N}\times\mathbb{N}: x_{k,l}-x_0\notin W\}$$

has $\overline{\theta}$-density zero. Since W is absorbing, there exists $a>0$ such that $ax_0\in W$. Let b be such that $|b|\leq 1$ and $b\leq a$. Since W is solid and $|bx_0|\leq|ax_0|$, we have $bx_0\in W$. Also, since W is balanced, $x_{k,l}-x_0\in W$ implies $b(x_{k,l}-x_0)\in W$. Then, we have

$$bx_k=b(x_{k,l}-x_0)+bx_0\in W+W\subseteq V,\ \text{for each } (k,l)\in\mathbb{N}\times\mathbb{N}-A.$$

Thus

$$P-\lim_{r,s\to\infty}\frac{1}{h_{r,s}}|\{(k,l)\in I_{r,s}: bx_{k,l}\notin W\}|=0.$$

Hence $(x_{k,l})$ is $S_{\overline{\theta}}(\tau)$-bounded. □

Theorem 3.4. Let (X,τ) be a LSR-space and $(x_{k,l})$, $(y_{k,l})$ and $(z_{k,l})$ be three double sequences of points in X such that

(i) $x_{k,l} \leq y_{k,l} \leq z_{k,l}$ for all $k,l \in \mathbb{N}$,

(ii) $S_{\overline{\theta}}(\tau)$-$\lim_{k,l} x_{k,l} = x_0 = S_{\overline{\theta}}(\tau)$-$\lim_{k,l} z_{k,l}$.

Then $S_{\overline{\theta}}(\tau)$-$\lim_{k,l} y_{k,l} = x_0$.

Proof. Let V be an arbitrary τ-neighborhood of zero, there exists $Y \in N_{sol}$ such that $Y \subseteq V$. We choose $W \in N_{sol}$ such that $W + W \subseteq Y$. From given condition (ii), we have $\delta_{\overline{\theta}}(A) = 1 = \delta_{\overline{\theta}}(B)$, where

$$A = \{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_0 \in W\}$$

and

$$B = \{(k,l) \in \mathbb{N} \times \mathbb{N} : z_{k,l} - x_0 \in W\}.$$

Also from the given condition (i), we have

$$x_{k,l} - x_0 \leq y_{k,l} - x_0 \leq z_{k,l} - x_0$$
$$\Rightarrow |y_{k,l} - x_0| \leq |x_{k,l} - x_0| + |z_{k,l} - x_0| \in W + W \subseteq Y,$$

for every $(k,l) \in A \cap B$. Since Y is solid, we have $y_{k,l} - x_0 \in Y \subseteq V$. Thus,

$$P - \lim_{r,s\to\infty} \frac{1}{h_{r,s}} |\{(k,l) \in I_{r,s} : y_{k,l} - x_0 \in V\}| = 1,$$

for each τ-neighborhood V of zero. Hence $S_{\overline{\theta}}(\tau)$-$\lim_{k,l} y_{k,l} = x_0$. $\square$

4. Double lacunary statistically τ-Cauchy and double lacunary $S^*(\tau)$-convergence

In this section we define the notions of lacunary statistically τ-Cauchy sequence and a new type of convergence, that is, $S^*_{\overline{\theta}}(\tau)$-convergence in the framework of LSR-space and determine prove some related results.

Definition 4.1. Let (X,τ) be a LSR-space. Then, $(x_{k,l})$ of points in X is said to be *double lacunary statistically τ-Cauchy sequence* or $S_{\overline{\theta}}(\tau)$-*Cauchy* in X if for each τ-neighborhood V of zero, there are an integers $m,n \in \mathbb{N}$,

$$\delta_{\overline{\theta}}(\{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} - x_{m,n} \notin V\}) = 0.$$

Theorem 4.1. Let (X,τ) be a LSR-space. If a double sequence $(x_{k,l})$ is $S_{\overline{\theta}}(\tau)$-convergent to x_0 in X then it is double lacunary statistically-τ-Cauchy.

Proof. Let $x = (x_{k,l})$ be a sequence in X such that $S_\theta(\tau) - \lim_{k,l\to\infty} x_{k,l} = x_0$. Let V be an arbitrary τ-neighborhood of zero. Then there exists $Y \in N_{sol}$ such that $Y \subseteq V$. Choose $W \in N_{sol}$ such that $W + W \subseteq Y$. Then

$$P - \lim_{r,s} \frac{1}{h_{r,s}} |\{(k,l) \in I_{r,s} : x_{k,l} - x_0 \notin W\}| = 0.$$

Also, we have

$x_{k,l} - x_{m,n} = x_{k,l} - x_0 + x_0 + x_{m,n} \in W + W \subseteq Y \subseteq V$. for all $(k,l),(m,n) \in \mathbb{N}\times\mathbb{N} - K$,

where

$$K = \{(k,l) \in \mathbb{N}\times\mathbb{N} : x_{k,l} - x_0 \notin W\}.$$

Therefore, we have

$$\{(k,l) \in \mathbb{N}\times\mathbb{N} : x_{k,l} - x_{m,n} \notin V\} \subseteq K.$$

For every τ-neighborhood V of zero, there exists $m, n \in \mathbb{N}$ such that

$$P - \lim_{r,s} \frac{1}{h_{r,s}} |\{(k,l) \in \mathbb{N}\times\mathbb{N} : x_{k,l} - x_{m,n} \notin V\}| = 0.$$

Hence $(x_{k,l})$ is a double lacunary statistically τ-Cauchy. □

Now, we define $S^*_{\overline{\theta}}(\tau)$-convergence in locally solid Riesz spaces.

Definition 4.2. Let (X,τ) be a LSR-space. Then, a double sequence $x = (x_{k,l})$ in X is said to be $S^*_{\overline{\theta}}(\tau)$-convergent to x_0 if there exists a set $K = \{(k,l)\} \subseteq \mathbb{N}\times\mathbb{N}$, $k, l = 1, 2, \ldots$, with $\delta_{\overline{\theta}}(K) = 1$ such that $\lim_{k,l\to\infty} x_{k,l} = x_0$. In this case we write $S^*_{\overline{\theta}}(\tau)$-$\lim x = x_0$.

Theorem 4.2. Let (X,τ) be a LSR-space. If a double sequence $x = (x_{k,l})$ in X is $S^*_{\overline{\theta}}(\tau)$-convergent to x_0, then it is $S_{\overline{\theta}}(\tau)$-convergent to the same limit.

Proof. Assume that $S^*_{\overline{\theta}}(\tau)$-$\lim_{k,l} x_{k,l} = x_0$. Let V be an arbitrary τ-neighborhood V of x_0. Since $S^*_{\overline{\theta}}(\tau)$-$\lim_{k,l} x_{k,l} = x_0$, there is a set $K = \{(k,l)\} \subseteq \mathbb{N}\times\mathbb{N}$, $k, l = 1, 2, 3, \cdots$ with $\delta_{\overline{\theta}}(K) = 1$ and $n_0 = n_0(V), m_0 = m_0(V)$ such that $k \geq n_0, l \geq m_0$ and $(k,l) \in K$ imply $x_{k,l} - x_0 \in V$. Then

$$K_V = \{(k,l) \in \mathbb{N}\times\mathbb{N} : x_{k,l} - x_0 \notin V\} \subseteq \mathbb{N}\times\mathbb{N} - \{(k_{n_0+1}, l_{m_0+1}), (k_{n_0+2}, l_{m_0+2}), \cdots\}.$$

Therefore

$$\delta_{\overline{\theta}}(K_V) \leq 1 - 1 = 0.$$

Hence x is $S_{\overline{\theta}}(\tau)$-convergent to x_0. □

We remark that the converse holds for a first countable space.

Theorem 4.3. Let (X,τ) be a first countable LSR-space and $x = (x_{k,l})$ be a double sequence in X. Then, $x = (x_{k,l})$ is $S^*_{\overline{\theta}}(\tau)$-convergent to x_0 if it is $S_{\overline{\theta}}(\tau)$-convergent to x_0.

Proof. Let x be double lacunary statistically τ-convergent to a number x_0. Fix a countable local base $V_1 \supset V_2 \supset V_3 \supset \cdots$ at x_0. For each $i \in \mathbb{N}$, put

$$K_i = \big\{(k,l) \in \mathbb{N}\times\mathbb{N} : x_{k,l} - x_0 \notin V_i\big\}.$$

By hypothesis, $\delta_{\overline{\theta}}(K_i) = 0$ for each i. Since the ideal $\mathcal{I}$ of all subsets of $\mathbb{N}\times\mathbb{N}$ having $\overline{\theta}$-density zero is a P-ideal (see for instance [12]), then there exists a sequence of sets $(J_i)_i$ such that the symmetric difference $K_i \Delta J_i$ is a finite set for any $i \in \mathbb{N}$ and $J := \cup_{i=1}^{\infty} J_i \in \mathcal{I}$.

Let $K = \mathbb{N}\times\mathbb{N} \setminus J$, then $\delta_{\overline{\theta}}(K) = 1$. In order to prove the theorem, it is enough to check that $\lim_{(k,l)\in K} x_{k,l} = x_0$.

Let $i \in \mathbb{N}$. Since $K_i \Delta J_i$ is a finite, there is $(k_i, l_i) \in \mathbb{N} \times \mathbb{N}$, without loss of generality with $(k_i, l_i) \in K$, $k_i, l_i > i$, such that

$$(\mathbb{N}\times\mathbb{N}\backslash J_i)\cap\{(k,l) \in \mathbb{N}\times\mathbb{N} : k \geq k_i, l \geq l_i\} = (\mathbb{N}\times\mathbb{N}\backslash K_i)\cap\{(k,l) \in \mathbb{N}\times\mathbb{N} : k \geq k_i, l \geq l_i\}. \tag{4.1}$$

If $(k,l) \in K$ and $k \geq k_i, l \geq l_i$, then $(k,l) \notin J_i$, and by (4.1) $(k,l) \notin K_i$. Thus $x_{k,l} - x_0 \in V_i$. So we have proved that for all $i \in \mathbb{N}$ there is $(k_i, l_i) \in K$, $k_i, l_i > i$, with $x_{k,l} - x_0 \in V_i$ for every $k \geq k_i, l \geq l_i$: without loss of generality, we can suppose $k_{i+1} > k_i$ and $l_{i+1} > l_i$ for every $i \in \mathbb{N}$. The assertion follows taking into account that the V_i's form a countable local base at x_0. □

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ON THE STABILITY OF STATES IN EFFECT ALGEBRAS

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ABSTRACT. In this paper, the $\mathbb{R}$-effect algebra is defined. We prove the Hyers-Ulam stability of states in effect algebras.

1. INTRODUCTION AND PRELIMINARIES

The stability problem of functional equations originated from a question of Ulam [14] in 1940, concerning the stability of group homomorphisms. Let $(G_1, .)$ be a group and let $(G_2, *)$ be a metric group with the metric $d(.,.)$. Given $\epsilon > 0$, does there exist a $\delta 0$, such that if a mapping $h : G_1 \longrightarrow G_2$ satisfies the inequality $d(h(x.y), h(x) * h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$? In the other words, Under what condition does there exists a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, Hyers [7] gave the first affirmative answer to the question of Ulam for Banach spaces. Let $f : E \to E'$ be a mapping between Banach spaces such that

$$\|f(x+y) - f(x) - f(y)\| \le \delta$$

for all $x, y \in E$, and some $\delta > 0$. Then there exists a unique additive mapping $T : E \to E'$ such that

$$\|f(x) - T(x)\| \le \delta$$

for all $x \in E$. In 1978, Th.M. Rassias [11] proved the following theorem.

Theorem 1.1. *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then there exists a unique additive mapping $T : E \to E'$ such that

$$\|f(x) - T(x)\| \le \frac{2\epsilon}{2 - 2^p}\|x\|^p \tag{1.2}$$

for all $x \in E$. If $p < 0$ then inequality (1.1) holds for all $x, y \neq 0$, and (1.2) for $x \neq 0$. Also, if the function $t \mapsto f(tx)$ from $\mathbb{R}$ into E' is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, then T is $\mathbb{R}$-linear.

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In 1991, Gajda [6] answered the question for the case $p > 1$, which was raised by Th.M. Rassias. More generalizations and applications of the Hyers-Ulam stability to a number of functional equations and mappings can be found in [1]–[4], [8]–[10] and [12, 13].

We recall some basic facts concerning effect algebra. A structure $(L, \oplus, 0, 1)$ is called an *effect algebra* if $0, 1$ are two distinguished elements and $\oplus$ is a partially defined operation on L which satisfies the following conditions for any $a, b, c \in L$ ([5]):

(1) $b \oplus a = a \oplus b$ if $a \oplus b$ is defined (it is said that a and b are orthogonal elements).
(2) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ if one side is defined.
(3) For every $a \in L$ there exists a unique $b \in L$ such that $a \oplus b = 1$ (we denote b by a').
(4) If $1 \oplus a$ is defined then $a = 0$.

Let $F = \{a_i : 1 \leq i \leq n\}$ be a finite subset of L. If $a_1 \oplus a_2$, $(a_1 \oplus a_2) \oplus a_3, \ldots (a_1 \oplus a_2 \oplus \cdots \oplus a_{n-1}) \oplus a_n$ are defined, we say that F is *orthogonal* and we denote $\bigoplus F = (a_1 \oplus a_2 \oplus \cdots \oplus a_{n-1}) \oplus a_n$. Let L be an effect algebra and let $p \in L$. We define $0p = 0$ and $1p = p$. More generally, if n is a positive integer and $(n-1)p$ is defined, we say that np is defined if and only if $(n-1)p \oplus p$ is defined, in which case $np := (n-1)p \oplus p$.

If G is an arbitrary subset of L, we will say that G is *orthogonal* if each finite subset $F \subseteq G$ is orthogonal.

If G is orthogonal and the supremum $\bigvee\{\bigoplus F \ : \ F \subseteq G, \ F \,\text{finite}\}$ exists, then $\bigoplus G = \bigvee\{\bigoplus F : F \subset G, \ F \,\text{finite}\}$ is called the $\oplus$*-sum* of G.

L is said to be *complete* if $\bigoplus G$ exists for each orthogonal subset $G \subseteq L$.

L is σ*-complete* if $\bigoplus G$ exists for each countable orthogonal subset $G \subseteq L$.

We say that an effect algebras L is σ*-orthocomplete (resp. orthocomplete)* if $\bigoplus_{i \in I} a_i$ exists for any countable (resp. arbitrary) orthogonal system $\{a_i : i \in I\}$ of elements of L. We recall the an effect algebra is σ-orthocomplete if and only if for every nondecreasing sequence $\{a_i\}_{i \in \mathbb{N}}$ there is a supremum $a = \bigvee_{i \in \mathbb{N}} a_i$.

A function $s : L \to [0, 1]$ from L to unit interval $[0, 1]$ of real numbers is a *state* on L if (i) $s(1) = 1$, (ii) $s(a \oplus b) = s(a) + s(b)$ whenever $a \oplus b$ exists in L. It is clear that $s(0) = 0$. A state $s : L \to [0, 1]$ is said to be σ*-additive,* or *completely additive* if the equality

$$s\left(\bigoplus_{i \in I} a_i\right) = \sum_{i \in I} s(a_i)$$

holds for any countable, or arbitrary index set I, respectively, such that $\bigoplus_{i \in I} a_i$ exists in L.

In this paper, we investigate the following the problem:

Let L be an effect algebra, and L be completely additive. Given $\varepsilon >$, does there exist a $\delta > 0$, such that if a function $s : L \to [0, 1]$ (resp. $\mathbb{R}$) satisfies the inequality $d(f(a \oplus b), f(a) + f(b)) < \delta$ for all $a, b \in L$ and $a \oplus b$, then there exists a state $s : L \to [0, 1]$ (resp. $\mathbb{R}$) with $d(h(a), s(a)) < \varepsilon$ for all $a \in L$?

2. Hyers-Ulam stability of states

Theorem 2.1. *Let $f : L \to [0, 1]$ be a function for which there exists a function $\varphi : L \times L \to [0, \infty)$ such that*

$$\widetilde{\varphi}(a, b) := \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi\left(2^j a, 2^j b\right) < \infty,$$

$$|f(a \oplus b) - f(a) - f(b)| \le \varphi(a, b) \tag{2.1}$$

for all $a, b \in L$ *with* $a \oplus b \in L$. *Then there exists a unique state* $s : L \to [0,1]$ *such that*

$$|f(a) - s(a)| \le \frac{1}{2}\widetilde{\varphi}(a, a) \tag{2.2}$$

for all $a \in L$.

Proof. Letting $b = a$ in (2.1), we get

$$|f(a \oplus a) - 2f(a)| \le \varphi(a, a)$$

for all $a \in L$. So

$$\left| f(a) - \frac{1}{2} f(a \oplus a) \right| \le \frac{1}{2} \varphi(a, a)$$

for all $a \in L$. Hence one may have the following formula, for positive integers m, l with $m > l$,

$$\left| \frac{1}{2^l} f(2^l a) - \frac{1}{2^m} f(2^m a) \right| \le \frac{1}{2} \sum_{i=l}^{m-1} \frac{1}{2^i} \varphi\left(2^i a, 2^i a\right). \tag{2.3}$$

It follows from (2.3) that the sequence $\left\{\frac{1}{2^k} f\left(2^k a\right)\right\}$ is a Cauchy sequence for all $a \in L$. Since $[0,1]$ is a bounded closed set, the sequence $\left\{\frac{1}{2^k} f\left(2^k a\right)\right\}$ converges. So one may define the function $s : L \to [0,1]$ by

$$s(a) := \lim_{k\to\infty} \frac{1}{2^k} f\left(2^k a\right), \quad \forall a \in L.$$

By (2.1),

$$\begin{aligned} |s(a \oplus b) - s(a) - s(b)| &\le \lim_{k\to\infty} \frac{1}{2^k} |f(2^k(a \oplus b)) - f(2^k a) - f(2^k b)| \\ &\le \lim_{k\to\infty} \frac{1}{2^k} \varphi(2^k a, 2^k b) = 0 \end{aligned}$$

for all $a, b \in L$ with $a \oplus b \in L$. So

$$s(a \oplus b) = s(a) + s(b)$$

for all $a, b \in L$ with $a \oplus b \in L$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.3), we get (2.2).

Now, we show that the uniqueness of s. Let $T : L \to [0,1]$ be another state satisfying (2.2). Then one have

$$\begin{aligned} |s(a) - T(a)| &= \left| \frac{1}{2^k} s\left(2^k a\right) - \frac{1}{2^k} T\left(2^k a\right) \right| \\ &\le \frac{1}{2^k} \left(\left| s\left(2^k a\right) - f\left(2^k a\right) \right| + \left| T\left(2^k a\right) - f\left(2^k a\right) \right| \right) \\ &\le \frac{1}{2^k} \widetilde{\varphi}\left(2^k a, 2^k a\right), \end{aligned}$$

which tends to zero as $k \to \infty$ for all $a \in L$. So we can conclude that $s(a) = T(a)$ for all $a \in L$. □

Corollary 2.2. *Let $f : L \to [0, 1]$ be a function*

$$|f(a \oplus b) - f(a) - f(b)| \leq \varepsilon$$

for all $a, b \in L$ with $a \oplus b \in L$. Then there exists a unique state $s : L \to [0, 1]$ such that

$$|f(a) - s(a)| \leq \varepsilon$$

for all $a \in L$.

3. Hyers-Ulam stability of states in $\mathbb{R}$

In this section, we introduce the abstract unit product on L as follows.

Let $\diamond$ be a binary operation on $\mathbb{R} \times L$, i.e., $\diamond : \mathbb{R} \times L \to L$, if it satisfies the following.

1. For any $\alpha, \beta \in \mathbb{R}$, if $a, b \in L$ with $a \oplus b$ is defined, then $\alpha \diamond a \oplus \beta \diamond b$.
2. For any $\alpha \in \mathbb{R}$, if $a, b \in L$ with $a \oplus b$ is defined, then $\alpha \diamond (a \oplus b) = \alpha \diamond a \oplus \alpha \diamond b$.
3. For any $\alpha, \beta \in \mathbb{R}$, if $\alpha \diamond a$ is defined, then $\beta \diamond (\alpha \diamond a) = \alpha\beta \diamond a$.

If L has a binary operation $\diamond$ satisfying the conditions (1) and (2), then $(L, 0, 1, \oplus, \diamond)$ is called an $\mathbb{R}$-effect algebra. For the sake of simplicity, $\alpha \diamond a$ has been simplified αa.

Next, we discuss that the Hyers-Ulam stability of states in $\mathbb{R}$. A function $s : L \to \mathbb{R}$ from L to real numbers is an $\mathbb{R}$-*state* on L if (i) $s(1) = 1$, (ii) $s(a \oplus b) = s(a) + s(b)$ whenever $a \oplus b$ exists in L.

Theorem 3.1. *Let $f : L \to \mathbb{R}$ be a function for which there exists a function $\varphi : L \times L \to [0, \infty)$ such that*

$$\widetilde{\varphi}(a, b) := \sum_{j=0}^{\infty} \frac{1}{|\alpha|^j} \varphi\left(\alpha^j a, \alpha^j b\right) < \infty,$$

$$|f(\alpha a \oplus \beta b) - \alpha f(a) - \beta f(b)| \leq \varphi(a, b) \tag{3.1}$$

for all $a, b \in L$ with $a \oplus b \in L$. Then there exists a unique state $s : L \to \mathbb{R}$ such that

$$|f(a) - s(a)| \leq \frac{1}{|\alpha|} \widetilde{\varphi}(a, 0) \tag{3.2}$$

for all $a \in L$.

Proof. Letting $b = 0$ in (3.1), we obtain

$$|f(\alpha a) - \alpha f(a)| \leq \varphi(a, 0)$$

for all $a \in L$. So

$$\left| f(a) - \frac{1}{\alpha} f(\alpha a) \right| \leq \left| \frac{1}{\alpha} \right| \varphi(a, 0)$$

for all $a \in L$. Similarly, we may get

$$\left| f(b) - \frac{1}{\beta} f(\beta b) \right| \leq \left| \frac{1}{\beta} \right| \varphi(0, b)$$

for all $b \in L$. Hence

$$\left| \frac{1}{\alpha^l} f(\alpha^l a) - \frac{1}{\alpha^m} f(\alpha^m a) \right| \leq \sum_{j=l}^{m-1} \left| \frac{1}{\alpha} \right|^{j+1} \varphi(\alpha^j a, 0) \tag{3.3}$$

for all nonnegative integers m and l with $m > l$ and all $a \in L$. It follows from (3.1) and (3.3) that the sequence $\{\frac{1}{\alpha^n} f(\alpha^n a)\}$ is a Cauchy sequence for all $a \in L$. Since $\mathbb{R}$ is

complete with $|\cdot|$, the sequence $\{\frac{1}{\alpha^n}f(\alpha^n a)\}$ converges. So one can define the function $s : L \to \mathbb{R}$ by

$$s(a) := \lim_{n\to\infty} \frac{1}{\alpha^n} f(\alpha^n a)$$

for all $a \in L$.

Now, by (3.1), we prove that s is an $\mathbb{R}$-state.

$$\begin{aligned}
&|s(a\oplus b) - s(a) - s(b)| \le \lim_{n\to\infty} \left|\frac{1}{\alpha^n}\right| |f(\alpha^n a \oplus \alpha^n b) - f(\alpha^n a) - f(\alpha^n b)| \\
&\le \lim_{n\to\infty} \left|\frac{1}{\alpha^n}\right| \left\{ \left| f\left(\alpha\alpha^{n-1}a \oplus \beta\frac{\alpha^n b}{\beta}\right) - \alpha f(\alpha^{n-1}a) - \beta f\left(\frac{\alpha^n b}{\beta}\right)\right| \right. \\
&\left. + |\alpha f(\alpha^{n-1}a) - f(\alpha^n a)| + \left|\beta f\left(\frac{\alpha^n b}{\beta}\right) - f(\alpha^n b)\right| \right\} \\
&\le \lim_{n\to\infty} \left|\frac{1}{\alpha^n}\right| \left\{ \varphi\left(\alpha^{n-1}a, \frac{\alpha^n}{\beta} b\right) + \varphi(\alpha^{n-1}a, 0) + \varphi\left(0, \frac{\alpha^n b}{\beta}\right)\right\} \\
&= 0
\end{aligned}$$

for all $a, b \in L$ with $a \oplus b \in L$. So

$$s(a\oplus b) = s(a) + s(b)$$

for all $a, b \in L$ with $a \oplus b$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.3), we get (3.2).

Now, let $T : L \to \mathbb{R}$ be another state satisfying (3.2). Then we have

$$\begin{aligned}
&|s(a) - T(a)| \le \left|\frac{1}{\alpha}\right|^n |s(\alpha^n a) - T(\alpha^n a)| \\
&\le \left|\frac{1}{\alpha}\right|^n \{|s(\alpha^n a) - f(\alpha^n a)| + |T(\alpha^n a) - f(\alpha^n a)|\} \\
&\le 2\frac{1}{|\alpha|^n}\frac{1}{|\alpha|}\widetilde{\varphi}(\alpha^n a, 0),
\end{aligned}$$

which tends to zero as $n \to \infty$ for all $a \in L$. So we can conclude that $s(a) = T(a)$ for all $a \in L$. This proves the uniqueness of s. ☐

Corollary 3.2. *Let $f : L \to \mathbb{R}$ be a function which satisfies*

$$|f(\alpha a \oplus \beta b) - \alpha f(a) - \beta f(b)| \le \varepsilon$$

for all $a, b \in L$ with $a \oplus b \in L$ and all α with $|\alpha| > 1$. Then there exists a unique state $s : L \to \mathbb{R}$ such that

$$|f(a) - s(a)| \le \frac{|\alpha|}{|\alpha| - 1}\varepsilon$$

for all $a \in L$.

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The Form of The Solution and Dynamics of a Rational Recursive Sequence

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ABSTRACT

We discuss in this paper the form of the solutions of the following recursive sequences

$$x_{n+1} = \frac{x_{n-3}x_{n-4}}{x_n(\pm 1 \pm x_{n-3}x_{n-4})}, \quad n = 0, 1, ...,$$

where the initial conditions are arbitrary real numbers. Moreover, we study the dynamics and behavior of the solutions.

Keywords: difference equations, recursive sequences, stability, periodic solution.
Mathematics Subject Classification: 39A10

1 Introduction

In this paper, we obtain the form of the solutions of the following recursive sequences

$$x_{n+1} = \frac{x_{n-3}x_{n-4}}{x_n(\pm 1 \pm x_{n-3}x_{n-4})}, \quad n = 0, 1, ..., \tag{1}$$

where the initial conditions are arbitrary real numbers. Moreover, we study the dynamics and behavior of the solutions.
Here, we recall some notations and results, which will be useful in our investigation.
Let I be some interval of real numbers and let

$$f : I^{k+1} \to I,$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, ..., x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, ..., \tag{2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$ [32].
Definition 1. (Equilibrium Point)

A point $\overline{x} \in I$ is called an equilibrium point of Eq.(2) if

$$\overline{x} = f(\overline{x}, \overline{x}, ..., \overline{x}).$$

That is, $x_n = \overline{x}$ for $n \geq 0,$ is a solution of Eq.(2), or equivalently, $\overline{x}$ is a fixed point of f.

Definition 2. (Stability)

(i) The equilibrium point $\overline{x}$ of Eq.(2) is locally stable if for every $\epsilon > 0,$ there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \epsilon \quad \text{for all} \quad n \geq -k.$$

(ii) The equilibrium point $\overline{x}$ of Eq.(2) is locally asymptotically stable if $\overline{x}$ is locally stable solution of Eq.(2) and there exists $\gamma > 0,$ such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \gamma,$$

such that

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iii) The equilibrium point $\overline{x}$ of Eq.(2) is global attractor if for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I,$ we require

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iv) The equilibrium point $\overline{x}$ of Eq.(2) is globally asymptotically stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of Eq.(2).

(v) The equilibrium point $\overline{x}$ of Eq.(2) is unstable if $\overline{x}$ is not locally stable.

The linearized equation of Eq.(2) about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f(\overline{x}, \overline{x}, ..., \overline{x})}{\partial x_{n-i}} y_{n-i}.$$

Theorem A [33]: Assume that $p_i \in R,\ i = 1, 2, ..., k$ and k is non-negative integer. Then

$$\sum_{i=1}^{k} |p_i| < 1,$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+k} + p_1 x_{n+k-1} + ... + p_k x_n = 0, \quad n = 0, 1, ...$$

Definition 3. (Periodicity)

A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be a periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k.$

The study of asymptotic stability and oscillatory properties of solutions of difference equations is extremely useful in studying the behavior of mathematical models of various biological systems and other applications. This is due to the fact that difference equations are appropriate models for describing situations where the variable is assumed to be a discrete set of values.

These values arise frequently in studying biological models, formulation and analysis of discrete time systems, the numerical integration of differential equations, deterministic chaos, and $etc.$ For example, Ladas [34] discussed the oscillation of positive solutions about the positive steady state N in the delay logistic difference equation

$$N_{n+1} = N_n \ \exp\left[r\left(1-\sum_{j=0}^{m} p_j N_{n-j}\right)\right],$$

where $r,\ p_m \in (0,\infty), p_0, p_1, ..., p_{m-1} \in [0,\infty)$ and $m+r \neq 1$, which describes situations where population growth is not continuous but seasonal with non-overlapping generations. This leads to the study of oscillations about zero of a linear difference equation of the form

$$x_{n+1} - x_n + \sum_{i=0}^{m} p_i x_{n-k_i} = 0, \quad n = 0, 1, ...$$

Furthermore, difference equations are appropriate models for describing situations, where population growth is not continuous but seasonal with overlapping generations.
El-Metwally et al. [16] investigated the asymptotic behavior of the population model

$$x_{n+1} = \alpha + \beta x_{n-1} e^{-x_n},$$

where α is the immigration rate and β is the population growth rate.
Ding et al. [10] studied the following discrete delay mosquito population equation

$$x_{n+1} = (\alpha x_n + \beta x_{n-1})\, e^{-x_n}.$$

The generalized Beverton-Holt stock recruitment model had been investigated in [7,9]:

$$x_{n+1} = a x_n + \tfrac{b x_{n-1}}{1 + c x_{n-1} + d x_n}.$$

See also [29, 32]. The long term behavior of solutions of nonlinear difference equations of order greater than one has been extensively studied during the last decade. For example, various results about boundedness, stability and periodic character of solutions of second-order nonlinear difference equation, see [1-20].
Many researchers have investigated the behavior of the solution of difference equations. For instance, Agarwal et al. [2-3] investigated the global stability, and periodicity character and then gave solutions of some special cases of the difference equations

$$x_{n+1} = a + \tfrac{d x_{n-l} x_{n-k}}{b - c x_{n-s}}, \quad x_{n+1} = a x_n + \tfrac{b x_n x_{n-3}}{c x_{n-2} + d x_{n-3}}.$$

Aloqeili [4] has obtained solutions of the difference equation

$$x_{n+1} = \tfrac{x_{n-1}}{a - x_n x_{n-1}}.$$

While, Cinar [8] investigated solutions of the following difference equation

$$x_{n+1} = \tfrac{a x_{n-1}}{1 + b x_n x_{n-1}}.$$

Later, Elabbasy et al. [14] investigated the global attractivity of equilibrium point and asymptotic behavior of solutions of the following difference equation

$$x_{n+1} = \frac{ax_{n-l}x_{n-k}}{bx_{n-p}-cx_{n-q}}.$$

what is more, they also gave the solution of some special cases of the difference equation.
In [22], Elsayed dealt with the dynamics and found the solution of the following rational recursive sequences

$$x_{n+1} = \frac{x_{n-3}}{\pm 1 \pm x_{n-1}x_{n-3}}.$$

Karatas et al., [31] got the form of the solution of the difference equation

$$x_{n+1} = \frac{x_{n-5}}{1+x_{n-2}x_{n-5}}.$$

Other related results on rational difference equations can be found in references [21-39].

2 On the Recursive Sequence $x_{n+1} = \frac{x_{n-3}x_{n-4}}{x_n(1+x_{n-3}x_{n-4})}$

In this section, we give a specific form of the solution of the equation in the form

$$x_{n+1} = \frac{x_{n-3}x_{n-4}}{x_n(1+x_{n-3}x_{n-4})}, \quad n = 0, 1, ..., \tag{3}$$

where the initial values are arbitrary positive real numbers.

Theorem 2.1. *Suppose that the sequence* $\{x_n\}_{n=-4}^{\infty}$ *be a solution of Eq.(3). Then for* $n = 0, 1, ...$

$$\begin{aligned}
x_{4n-3} &= \frac{x_{-3}x_{-4}^n}{x_0^n}\prod_{i=0}^{n-1}\left(\frac{(1+ix_{-2}x_{-3})(1+ix_{-1}x_0)}{(1+(i+1)x_{-3}x_{-4})(1+ix_{-1}x_{-2})}\right),\\
x_{4n-2} &= \frac{x_{-2}x_0^n}{x_{-4}^n}\prod_{i=0}^{n-1}\left(\frac{(1+(i+1)x_{-3}x_{-4})(1+ix_{-1}x_{-2})}{(1+(i+1)x_{-2}x_{-3})(1+ix_{-1}x_0)}\right),\\
x_{4n-1} &= \frac{x_{-1}x_{-4}^n}{x_0^n}\prod_{i=0}^{n-1}\left(\frac{(1+(i+1)x_{-2}x_{-3})(1+ix_{-1}x_0)}{(1+(i+1)x_{-3}x_{-4})(1+(i+1)x_{-1}x_{-2})}\right),\\
x_{4n} &= \frac{x_0^{n+1}}{x_{-4}^n}\prod_{i=0}^{n}\left(\frac{(1+ix_{-3}x_{-4})(1+ix_{-1}x_{-2})}{(1+ix_{-2}x_{-3})(1+ix_{-1}x_0)}\right).
\end{aligned}$$

Proof: For $n = 0$ the result holds. Now, suppose that $n > 0$ and that our assumption holds for $n-1$. That is;

$$\begin{aligned}
x_{4n-7} &= \frac{x_{-3}x_{-4}^{n-1}}{x_0^{n-1}}\prod_{i=0}^{n-2}\left(\frac{(1+ix_{-2}x_{-3})(1+ix_{-1}x_0)}{(1+(i+1)x_{-3}x_{-4})(1+ix_{-1}x_{-2})}\right),\\
x_{4n-6} &= \frac{x_{-2}x_0^{n-1}}{x_{-4}^{n-1}}\prod_{i=0}^{n-2}\left(\frac{(1+(i+1)x_{-3}x_{-4})(1+ix_{-1}x_{-2})}{(1+(i+1)x_{-2}x_{-3})(1+ix_{-1}x_0)}\right),\\
x_{4n-5} &= \frac{x_{-1}x_{-4}^{n-1}}{x_0^{n-1}}\prod_{i=0}^{n-2}\left(\frac{(1+(i+1)x_{-2}x_{-3})(1+ix_{-1}x_0)}{(1+(i+1)x_{-3}x_{-4})(1+(i+1)x_{-1}x_{-2})}\right),
\end{aligned}$$

$$\begin{aligned}
x_{4n-4} &= \frac{x_0^n}{x_{-4}^{n-1}} \prod_{i=0}^{n-1} \left(\frac{(1+ix_{-3}x_{-4})(1+ix_{-1}x_{-2})}{(1+ix_{-2}x_{-3})(1+ix_{-1}x_0)} \right), \\
x_{4n-8} &= \frac{x_0^{n-1}}{x_{-4}^{n-2}} \prod_{i=0}^{n-2} \left(\frac{(1+ix_{-3}x_{-4})(1+ix_{-1}x_{-2})}{(1+ix_{-2}x_{-3})(1+ix_{-1}x_0)} \right).
\end{aligned}$$

Now, it follows from Eq.(3) that

$$x_{4n-3} = \frac{x_{4n-7}x_{4n-8}}{x_{4n-4}(1+x_{4n-7}x_{4n-8})}$$

$$\begin{aligned}
&= \frac{\frac{x_{-3}x_{-4}^{n-1}}{x_0^{n-1}} \prod_{i=0}^{n-2} \left(\frac{(1+ix_{-2}x_{-3})(1+ix_{-1}x_0)}{(1+(i+1)x_{-3}x_{-4})(1+ix_{-1}x_{-2})} \right) \frac{x_0^{n-1}}{x_{-4}^{n-2}} \prod_{i=0}^{n-2} \left(\frac{(1+ix_{-3}x_{-4})(1+ix_{-1}x_{-2})}{(1+ix_{-2}x_{-3})(1+ix_{-1}x_0)} \right)}{\frac{x_0^n}{x_{-4}^{n-1}} \prod_{i=0}^{n-1} \left(\frac{(1+ix_{-3}x_{-4})(1+ix_{-1}x_{-2})}{(1+ix_{-2}x_{-3})(1+ix_{-1}x_0)} \right)} \\
&\times \frac{1}{\left(1 + \frac{x_{-3}x_{-4}^{n-1}}{x_0^{n-1}} \prod_{i=0}^{n-2} \left(\frac{(1+ix_{-2}x_{-3})(1+ix_{-1}x_0)}{(1+(i+1)x_{-3}x_{-4})(1+ix_{-1}x_{-2})} \right) \frac{x_0^{n-1}}{x_{-4}^{n-2}} \prod_{i=0}^{n-2} \left(\frac{(1+ix_{-3}x_{-4})(1+ix_{-1}x_{-2})}{(1+ix_{-2}x_{-3})(1+ix_{-1}x_0)} \right) \right)} \\
&= \frac{x_{-3}x_{-4}^n \prod_{i=0}^{n-2} \left(\frac{(1+ix_{-3}x_{-4})}{(1+(i+1)x_{-3}x_{-4})} \right)}{x_0^n \prod_{i=0}^{n-1} \left(\frac{(1+ix_{-3}x_{-4})(1+ix_{-1}x_{-2})}{(1+ix_{-2}x_{-3})(1+ix_{-1}x_0)} \right) \left(1+x_{-3}x_{-4} \prod_{i=0}^{n-2} \left(\frac{(1+ix_{-3}x_{-4})}{(1+(i+1)x_{-3}x_{-4})} \right) \right)} \\
&= \prod_{i=0}^{n-1} \left(\frac{(1+ix_{-2}x_{-3})(1+ix_{-1}x_0)}{(1+ix_{-3}x_{-4})(1+ix_{-1}x_{-2})} \right) \frac{x_{-3}x_{-4}^n}{x_0^n (1+(n-1)x_{-3}x_{-4}) \left(1 + \left(\frac{x_{-3}x_{-4}}{(1+(n-1)x_{-3}x_{-4})} \right) \right)} \\
&= \frac{x_{-3}x_{-4}^n}{x_0^n} \prod_{i=0}^{n-1} \left(\frac{(1+ix_{-2}x_{-3})(1+ix_{-1}x_0)}{(1+ix_{-3}x_{-4})(1+ix_{-1}x_{-2})} \right) \frac{1}{(1+(n-1)x_{-3}x_{-4}+x_{-3}x_{-4})} \\
&= \frac{x_{-3}x_{-4}^n}{x_0^n} \prod_{i=0}^{n-1} \left(\frac{(1+ix_{-2}x_{-3})(1+ix_{-1}x_0)}{(1+ix_{-3}x_{-4})(1+ix_{-1}x_{-2})} \right) \frac{1}{(1+nx_{-3}x_{-4})}.
\end{aligned}$$

Hence, we have

$$x_{4n-3} = \frac{x_{-3}x_{-4}^n}{x_0^n} \prod_{i=0}^{n-1} \left(\frac{(1+ix_{-2}x_{-3})(1+ix_{-1}x_0)}{(1+(i+1)x_{-3}x_{-4})(1+ix_{-1}x_{-2})} \right).$$

Similarly,

$$\begin{aligned}
x_{4n-2} &= \frac{x_{4n-6}x_{4n-7}}{x_{4n-3}(1+x_{4n-6}x_{4n-7})} \\
&= \frac{\frac{x_{-2}x_0^{n-1}}{x_{-4}^{n-1}} \prod_{i=0}^{n-2} \left(\frac{(1+(i+1)x_{-3}x_{-4})(1+ix_{-1}x_{-2})}{(1+(i+1)x_{-2}x_{-3})(1+ix_{-1}x_0)} \right) \frac{x_{-3}x_{-4}^{n-1}}{x_0^{n-1}} \prod_{i=0}^{n-2} \left(\frac{(1+ix_{-2}x_{-3})(1+ix_{-1}x_0)}{(1+(i+1)x_{-3}x_{-4})(1+ix_{-1}x_{-2})} \right)}{\frac{x_{-3}x_{-4}^n}{x_0^n} \prod_{i=0}^{n-1} \left(\frac{(1+ix_{-2}x_{-3})(1+ix_{-1}x_0)}{(1+(i+1)x_{-3}x_{-4})(1+ix_{-1}x_{-2})} \right)} \\
&\times \frac{1}{\left(1+\frac{x_{-2}x_0^{n-1}}{x_{-4}^{n-1}} \prod_{i=0}^{n-2} \left(\frac{(1+(i+1)x_{-3}x_{-4})(1+ix_{-1}x_{-2})}{(1+(i+1)x_{-2}x_{-3})(1+ix_{-1}x_0)} \right) \frac{x_{-3}x_{-4}^{n-1}}{x_0^{n-1}} \prod_{i=0}^{n-2} \left(\frac{(1+ix_{-2}x_{-3})(1+ix_{-1}x_0)}{(1+(i+1)x_{-3}x_{-4})(1+ix_{-1}x_{-2})} \right) \right)}
\end{aligned}$$

$$= \frac{x_{-2}\prod_{i=0}^{n-2}\left(\frac{(1+ix_{-2}x_{-3})}{(1+(i+1)x_{-2}x_{-3})}\right)}{\frac{x_{-4}^n}{x_0^n}\prod_{i=0}^{n-1}\left(\frac{(1+ix_{-2}x_{-3})\,(1+ix_{-1}x_0)}{(1+(i+1)x_{-3}x_{-4})\,(1+ix_{-1}x_{-2})}\right)\left(1+x_{-3}x_{-2}\prod_{i=0}^{n-2}\left(\frac{(1+ix_{-2}x_{-3})}{(1+(i+1)x_{-2}x_{-3})}\right)\right)}$$

$$= \prod_{i=0}^{n-1}\left(\frac{(1+(i+1)x_{-3}x_{-4})\,(1+ix_{-1}x_{-2})}{(1+ix_{-2}x_{-3})\,(1+ix_{-1}x_0)}\right)\frac{x_{-2}x_0^n}{x_{-4}^n(1+(n-1)x_{-2}x_{-3})\left(1+\frac{x_{-3}x_{-2}}{(1+(n-1)x_{-2}x_{-3})}\right)}$$

$$= \prod_{i=0}^{n-1}\left(\frac{(1+(i+1)x_{-3}x_{-4})\,(1+ix_{-1}x_{-2})}{(1+ix_{-2}x_{-3})\,(1+ix_{-1}x_0)}\right)\frac{x_{-2}x_0^n}{x_{-4}^n\,(1+(n-1)x_{-2}x_{-3}+x_{-3}x_{-2})}$$

$$= \prod_{i=0}^{n-1}\left(\frac{(1+(i+1)x_{-3}x_{-4})\,(1+ix_{-1}x_{-2})}{(1+ix_{-2}x_{-3})\,(1+ix_{-1}x_0)}\right)\frac{x_{-2}x_0^n}{x_{-4}^n\,(1+nx_{-2}x_{-3})}$$

Hence, we have

$$x_{4n-2} = \frac{x_{-2}x_0^n}{x_{-4}^n}\prod_{i=0}^{n-1}\left(\frac{(1+(i+1)x_{-3}x_{-4})\,(1+ix_{-1}x_{-2})}{(1+(i+1)x_{-2}x_{-3})\,(1+ix_{-1}x_0)}\right).$$

Similarly, we can easily obtain the other relations. Thus, the proof is completed.

Theorem 2.2. *Eq.(3) has a unique equilibrium point which is the number zero and this equilibrium point is not locally asymptotically stable.*

Proof: For the equilibrium points of Eq.(3), we can write

$$\overline{x} = \frac{\overline{x}^2}{\overline{x}\left(1+\overline{x}^2\right)}.$$

Then, we have

$$\overline{x}^2\left(1+\overline{x}^2\right) = \overline{x}^2, \qquad \Rightarrow \qquad \overline{x}^2\left(1+\overline{x}^2-1\right) = 0,$$

or,

$$\overline{x}^4 = 0.$$

Thus, the equilibrium point of Eq.(3) is $\overline{x} = 0$.
Let $f : (0,\infty)^3 \longrightarrow (0,\infty)$ be a function defined by

$$f(u,v,w) = \frac{vw}{u(1+vw)}.$$

Therefore, it follows that

$$f_u(u,v,w) = -\tfrac{vw}{u^2(1+vw)},\ f_v(u,v,w) = \tfrac{w}{u(1+vw)^2},\ f_w(u,v,w) = \tfrac{v}{u(1+vw)^2},$$

we deduce that

$$f_u(\overline{x},\overline{x},\overline{x}) = -1, \quad f_v(\overline{x},\overline{x},\overline{x}) = 1, \quad f_w(\overline{x},\overline{x},\overline{x}) = 1.$$

The proof follows by using Theorem A.

Numerical simulations

To confirming the results obtained in this section, we consider some numerical examples which represent different types of solutions to Eq. (3).

Example 1. We assume $x_{-4}=5$, $x_{-3}=3$, $x_{-2}=0.13,\ x_{-1}=2,\ x_0=7$. See Fig. 1.

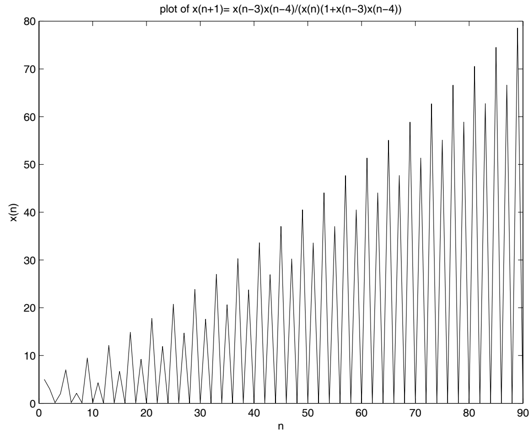

Figure 1.

Example 2. See Fig. 2, since $x_{-4}=0.1,\ x_{-3}=0.3,\ x_{-2}=3,\ x_{-1}=0.2,\ x_0=0.7$.

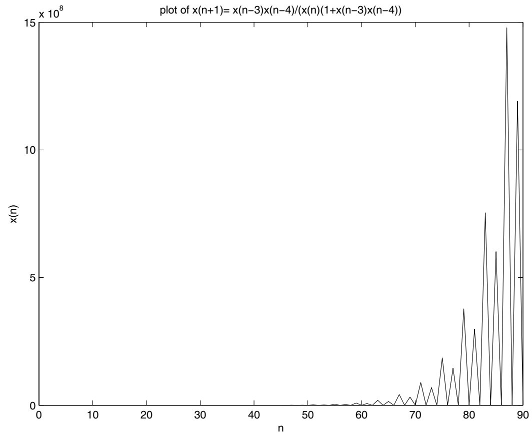

Figure 2.

3 On the Recursive Sequence $x_{n+1}=\frac{x_{n-3}x_{n-4}}{x_n(-1+x_{n-3}x_{n-4})}$

In this section, we obtain the solution of the difference equation in the form

$$x_{n+1}=\frac{x_{n-3}x_{n-4}}{x_n(-1+x_{n-3}x_{n-4})},\quad n=0,1,..., \tag{4}$$

where the initial values are arbitrary non zero real numbers with $x_{-3}x_{-4}\neq 1,\ x_{-3}x_{-2}\neq 1,\ x_{-2}x_{-1}\neq 1$ and $x_{-1}x_0\neq 1$.

Theorem 3.1. *Let $\{x_n\}_{n=-4}^{\infty}$ be a solution of Eq.(4). Then the solution of Eq.(4) is given by the following formula for $n=0,1,2,...$*

$$\begin{aligned}
x_{8n-4} &= \frac{x_0^{2n}}{x_{-4}^{2n-1}}\left(\frac{(-1+x_{-3}x_{-4})(-1+x_{-1}x_{-2})}{(-1+x_{-2}x_{-3})(-1+x_{-1}x_0)}\right)^n,\\
x_{8n-3} &= \frac{x_{-3}x_{-4}^{2n}}{x_0^{2n}}\left(\frac{(-1+x_{-2}x_{-3})(-1+x_{-1}x_0)}{(-1+x_{-3}x_{-4})(-1+x_{-1}x_{-2})}\right)^n,\\
x_{8n-2} &= \frac{x_{-2}x_0^{2n}}{x_{-4}^{2n}}\left(\frac{(-1+x_{-3}x_{-4})(-1+x_{-1}x_{-2})}{(-1+x_{-2}x_{-3})(-1+x_{-1}x_0)}\right)^n,
\end{aligned}$$

$$
\begin{aligned}
x_{8n-1} &= \frac{x_{-1}x_{-4}^{2n}}{x_0^{2n}}\left(\frac{(-1+x_{-2}x_{-3})(-1+x_{-1}x_0)}{(-1+x_{-3}x_{-4})(-1+x_{-1}x_{-2})}\right)^n, \\
x_{8n} &= \frac{x_0^{2n+1}}{x_{-4}^{2n}}\left(\frac{(-1+x_{-3}x_{-4})(-1+x_{-1}x_{-2})}{(-1+x_{-2}x_{-3})(-1+x_{-1}x_0)}\right)^n, \\
x_{8n+1} &= \frac{x_{-3}x_{-4}^{2n+1}}{x_0^{2n+1}(-1+x_{-3}x_{-4})}\left(\frac{(-1+x_{-2}x_{-3})(-1+x_{-1}x_0)}{(-1+x_{-3}x_{-4})(-1+x_{-1}x_{-2})}\right)^n, \\
x_{8n+2} &= \frac{x_{-2}x_0^{2n+1}}{x_{-4}^{2n+1}}\left(\frac{(-1+x_{-3}x_{-4})}{(-1+x_{-2}x_{-3})}\right)\left(\frac{(-1+x_{-3}x_{-4})(-1+x_{-1}x_{-2})}{(-1+x_{-2}x_{-3})(-1+x_{-1}x_0)}\right)^n, \\
x_{8n+3} &= \frac{x_{-1}x_{-4}^{2n+1}}{x_0^{2n+1}(-1+x_{-1}x_0)}\left(\frac{(-1+x_{-2}x_{-3})(-1+x_{-1}x_0)}{(-1+x_{-3}x_{-4})(-1+x_{-1}x_{-2})}\right)^{n+1}.
\end{aligned}
$$

Proof: For $n=0$ the result holds. Now suppose that $n>0$ and that our assumption holds for $n-1$. That is;

$$
\begin{aligned}
x_{8n-12} &= \frac{x_0^{2n-2}}{x_{-4}^{2n-3}}\left(\frac{(-1+x_{-3}x_{-4})(-1+x_{-1}x_{-2})}{(-1+x_{-2}x_{-3})(-1+x_{-1}x_0)}\right)^{n-1}, \\
x_{8n-11} &= \frac{x_{-3}x_{-4}^{2n-2}}{x_0^{2n-2}}\left(\frac{(-1+x_{-2}x_{-3})(-1+x_{-1}x_0)}{(-1+x_{-3}x_{-4})(-1+x_{-1}x_{-2})}\right)^{n-2}, \\
x_{8n-10} &= \frac{x_{-2}x_0^{2n-2}}{x_{-4}^{2n-2}}\left(\frac{(-1+x_{-3}x_{-4})(-1+x_{-1}x_{-2})}{(-1+x_{-2}x_{-3})(-1+x_{-1}x_0)}\right)^{n-1}, \\
x_{8n-9} &= \frac{x_{-1}x_{-4}^{2n-2}}{x_0^{2n-2}}\left(\frac{(-1+x_{-2}x_{-3})(-1+x_{-1}x_0)}{(-1+x_{-3}x_{-4})(-1+x_{-1}x_{-2})}\right)^{n-1}, \\
x_{8n-8} &= \frac{x_0^{2n-1}}{x_{-4}^{2n-2}}\left(\frac{(-1+x_{-3}x_{-4})(-1+x_{-1}x_{-2})}{(-1+x_{-2}x_{-3})(-1+x_{-1}x_0)}\right)^{n-1}, \\
x_{8n-7} &= \frac{x_{-3}x_{-4}^{2n-1}}{x_0^{2n-1}(-1+x_{-3}x_{-4})}\left(\frac{(-1+x_{-2}x_{-3})(-1+x_{-1}x_0)}{(-1+x_{-3}x_{-4})(-1+x_{-1}x_{-2})}\right)^{n-1},
\end{aligned}
$$

$$
\begin{aligned}
x_{8n-6} &= \frac{x_{-2}x_0^{2n-1}}{x_{-4}^{2n-1}}\left(\frac{(-1+x_{-3}x_{-4})}{(-1+x_{-2}x_{-3})}\right)\left(\frac{(-1+x_{-3}x_{-4})(-1+x_{-1}x_{-2})}{(-1+x_{-2}x_{-3})(-1+x_{-1}x_0)}\right)^{n-1}, \\
x_{8n-5} &= \frac{x_{-1}x_{-4}^{2n-1}}{x_0^{2n-1}(-1+x_{-1}x_0)}\left(\frac{(-1+x_{-2}x_{-3})(-1+x_{-1}x_0)}{(-1+x_{-3}x_{-4})(-1+x_{-1}x_{-2})}\right)^{n}.
\end{aligned}
$$

Now, it follows from Eq.(4) that

$$
x_{8n-4} = \frac{x_{8n-8}x_{8n-9}}{x_{8n-5}(-1+x_{8n-8}x_{8n-9})}
$$

$$
\begin{aligned}
&= \frac{\frac{x_0^{2n-1}}{x_{-4}^{2n-2}}\left(\frac{(-1+x_{-3}x_{-4})(-1+x_{-1}x_{-2})}{(-1+x_{-2}x_{-3})(-1+x_{-1}x_0)}\right)^{n-1}\frac{x_{-1}x_{-4}^{2n-2}}{x_0^{2n-2}}\left(\frac{(-1+x_{-2}x_{-3})(-1+x_{-1}x_0)}{(-1+x_{-3}x_{-4})(-1+x_{-1}x_{-2})}\right)^{n-1}}{\frac{x_{-1}x_{-4}^{2n-1}}{x_0^{2n-1}(-1+x_{-1}x_0)}\left(\frac{(-1+x_{-2}x_{-3})(-1+x_{-1}x_0)}{(-1+x_{-3}x_{-4})(-1+x_{-1}x_{-2})}\right)^{n}} \\
&\quad\times\frac{1}{\left(-1+\frac{x_0^{2n-1}}{x_{-4}^{2n-2}}\left(\frac{(-1+x_{-3}x_{-4})(-1+x_{-1}x_{-2})}{(-1+x_{-2}x_{-3})(-1+x_{-1}x_0)}\right)^{n-1}\frac{x_{-1}x_{-4}^{2n-2}}{x_0^{2n-2}}\left(\frac{(-1+x_{-2}x_{-3})(-1+x_{-1}x_0)}{(-1+x_{-3}x_{-4})(-1+x_{-1}x_{-2})}\right)^{n-1}\right)} \\
&= \frac{x_0^{2n}}{\frac{x_{-4}^{2n-1}}{(-1+x_{-1}x_0)}\left(\frac{(-1+x_{-2}x_{-3})(-1+x_{-1}x_0)}{(-1+x_{-3}x_{-4})(-1+x_{-1}x_{-2})}\right)^{n}(-1+x_{-1}x_0)} \\
&= \frac{x_0^{2n}}{x_{-4}^{2n-1}}\left(\frac{(-1+x_{-3}x_{-4})(-1+x_{-1}x_{-2})}{(-1+x_{-2}x_{-3})(-1+x_{-1}x_0)}\right)^{n},
\end{aligned}
$$

$$x_{8n-3}=\frac{x_{8n-7}x_{8n-8}}{x_{8n-4}(-1+x_{8n-7}x_{8n-8})}$$

$$\begin{aligned}
&= \frac{\frac{x_{-3}x_{-4}^{2n-1}}{x_0^{2n-1}(-1+x_{-3}x_{-4})}\left(\frac{(-1+x_{-2}x_{-3})(-1+x_{-1}x_0)}{(-1+x_{-3}x_{-4})(-1+x_{-1}x_{-2})}\right)^{n-1}\frac{x_0^{2n-1}}{x_{-4}^{2n-2}}\left(\frac{(-1+x_{-3}x_{-4})(-1+x_{-1}x_{-2})}{(-1+x_{-2}x_{-3})(-1+x_{-1}x_0)}\right)^{n-1}}{\frac{x_0^{2n}}{x_{-4}^{2n-1}}\left(\frac{(-1+x_{-3}x_{-4})(-1+x_{-1}x_{-2})}{(-1+x_{-2}x_{-3})(-1+x_{-1}x_0)}\right)^{n}}\\
&\times\frac{1}{\left(-1+\frac{x_{-3}x_{-4}^{2n-1}}{x_0^{2n-1}(-1+x_{-3}x_{-4})}\left(\frac{(-1+x_{-2}x_{-3})(-1+x_{-1}x_0)}{(-1+x_{-3}x_{-4})(-1+x_{-1}x_{-2})}\right)^{n-1}\frac{x_0^{2n-1}}{x_{-4}^{2n-2}}\left(\frac{(-1+x_{-3}x_{-4})(-1+x_{-1}x_{-2})}{(-1+x_{-2}x_{-3})(-1+x_{-1}x_0)}\right)^{n-1}\right)}\\
&= \frac{x_{-3}x_{-4}^{2n}}{x_0^{2n}(-1+x_{-3}x_{-4})\left(-1+\frac{x_{-3}x_{-4}}{(-1+x_{-3}x_{-4})}\right)}\left(\frac{(-1+x_{-2}x_{-3})(-1+x_{-1}x_0)}{(-1+x_{-3}x_{-4})(-1+x_{-1}x_{-2})}\right)^{n}.
\end{aligned}$$

Hence, we have

$$x_{8n-3}=\frac{x_{-3}x_{-4}^{2n}}{x_0^{2n}}\left(\frac{(-1+x_{-2}x_{-3})(-1+x_{-1}x_0)}{(-1+x_{-3}x_{-4})(-1+x_{-1}x_{-2})}\right)^{n}.$$

Similarly, we can easily obtain the other relations. Thus, the proof is completed.

Theorem 3.2. *Eq.(4) has three equilibrium points, which are $0,\pm\sqrt{2}$ and these equilibrium points are not locally asymptotically stable.*

Proof: For the equilibrium points of Eq.(4), we can write

$$\overline{x}=\frac{\overline{x}^2}{\overline{x}\left(-1+\overline{x}^2\right)}.$$

Then, we have

$$\overline{x}^2\left(\overline{x}^2-2\right)=0,$$

Thus, the equilibrium points of Eq.(4) are $0,\pm\sqrt{2}$.
Let $f:(0,\infty)^3\longrightarrow(0,\infty)$ be a function defined by

$$f(u,v,w)=\frac{vw}{u(-1+vw)}.$$

Therefore, it follows that

$$f_u(u,v,w)=-\frac{vw}{u^2(-1+vw)},\ f_v(u,v,w)=\frac{-w}{u(-1+vw)^2},\ f_w(u,v,w)=\frac{-v}{u(-1+vw)^2},$$

we see that,

$$f_u(\overline{x},\overline{x},\overline{x})=\pm1,\quad f_v(\overline{x},\overline{x},\overline{x})=-1,\quad f_w(\overline{x},\overline{x},\overline{x})=-1.$$

The proof follows by using Theorem A.
Lemma 1. It is easy to see that every solution of Eq.(4) is unbounded except in the following case.

Theorem 3.3. *Eq.(4) has a periodic solution of period eight iff $(-1+x_{-2}x_{-3})(-1+x_{-1}x_0)=(-1+x_{-3}x_{-4})(-1+x_{-1}x_{-2})$, $x_0=x_{-4}$ and will be taken the form*

$$\left\{x_0,x_{-3},x_{-2},x_{-1},x_0,\frac{x_{-3}}{(-1+x_{-3}x_{-4})},x_{-2}\left(\frac{(-1+x_{-3}x_{-4})}{(-1+x_{-2}x_{-3})}\right),\frac{x_{-1}}{(-1+x_{-1}x_0)},x_0,x_{-3},x_{-2},....\right\}.$$

Proof: First, suppose that there exists a prime period eight solution

$$x_0, x_{-3}, x_{-2}, x_{-1}, x_0, \frac{x_{-3}}{(-1+x_{-3}x_{-4})}, x_{-2}\left(\frac{(-1+x_{-3}x_{-4})}{(-1+x_{-2}x_{-3})}\right), \frac{x_{-1}}{(-1+x_{-1}x_0)}, x_0, x_{-3}, ...$$

of Eq.(4), we see from the form of the solution of Eq.(4) that

$$\begin{aligned}
x_{8n-4} &= \frac{x_0^{2n}}{x_{-4}^{2n-1}}\left(\frac{(-1+x_{-3}x_{-4})(-1+x_{-1}x_{-2})}{(-1+x_{-2}x_{-3})(-1+x_{-1}x_0)}\right)^n = x_0, \\
x_{8n-3} &= \frac{x_{-3}x_{-4}^{2n}}{x_0^{2n}}\left(\frac{(-1+x_{-2}x_{-3})(-1+x_{-1}x_0)}{(-1+x_{-3}x_{-4})(-1+x_{-1}x_{-2})}\right)^n = x_{-3}, \\
x_{8n-2} &= \frac{x_{-2}x_0^{2n}}{x_{-4}^{2n}}\left(\frac{(-1+x_{-3}x_{-4})(-1+x_{-1}x_{-2})}{(-1+x_{-2}x_{-3})(-1+x_{-1}x_0)}\right)^n = x_{-2}, \\
x_{8n-1} &= \frac{x_{-1}x_{-4}^{2n}}{x_0^{2n}}\left(\frac{(-1+x_{-2}x_{-3})(-1+x_{-1}x_0)}{(-1+x_{-3}x_{-4})(-1+x_{-1}x_{-2})}\right)^n = x_{-1}, \\
x_{8n} &= \frac{x_0^{2n+1}}{x_{-4}^{2n}}\left(\frac{(-1+x_{-3}x_{-4})(-1+x_{-1}x_{-2})}{(-1+x_{-2}x_{-3})(-1+x_{-1}x_0)}\right)^n = x_0, \\
x_{8n+1} &= \frac{x_{-3}x_{-4}^{2n+1}}{x_0^{2n+1}(-1+x_{-3}x_{-4})}\left(\frac{(-1+x_{-2}x_{-3})(-1+x_{-1}x_0)}{(-1+x_{-3}x_{-4})(-1+x_{-1}x_{-2})}\right)^n = \frac{x_{-3}}{(-1+x_{-3}x_{-4})}, \\
x_{8n+2} &= \frac{x_{-2}x_0^{2n+1}}{x_{-4}^{2n+1}}\left(\frac{(-1+x_{-3}x_{-4})}{(-1+x_{-2}x_{-3})}\right)\left(\frac{(-1+x_{-3}x_{-4})(-1+x_{-1}x_{-2})}{(-1+x_{-2}x_{-3})(-1+x_{-1}x_0)}\right)^n \\
&= x_{-2}\left(\frac{(-1+x_{-3}x_{-4})}{(-1+x_{-2}x_{-3})}\right), \\
x_{8n+3} &= \frac{x_{-1}x_{-4}^{2n+1}}{x_0^{2n+1}(-1+x_{-1}x_0)}\left(\frac{(-1+x_{-2}x_{-3})(-1+x_{-1}x_0)}{(-1+x_{-3}x_{-4})(-1+x_{-1}x_{-2})}\right)^{n+1} = \frac{x_{-1}}{(-1+x_{-1}x_0)}.
\end{aligned}$$

Then, we get

$$\frac{(-1+x_{-2}x_{-3})(-1+x_{-1}x_0)}{(-1+x_{-3}x_{-4})(-1+x_{-1}x_{-2})} = 1, \quad x_0 = x_{-4}.$$

Thus,

$$(-1+x_{-2}x_{-3})(-1+x_{-1}x_0) = (-1+x_{-3}x_{-4})(-1+x_{-1}x_{-2}), \quad x_0 = x_{-4}.$$

Secondly, assume that $(-1+x_{-2}x_{-3})(-1+x_{-1}x_0) = (-1+x_{-3}x_{-4})(-1+x_{-1}x_{-2}),\ x_0 = x_{-4}$. Then, we see from the form of the solution of Eq.(4) that

$$\begin{aligned}
x_{8n-4} &= x_0,\ \ x_{8n-3} = x_{-3}, \quad x_{8n-2} = x_{-2}, \quad x_{8n-1} = x_{-1},\ \ x_{8n} = x_0, \\
x_{8n+1} &= \frac{x_{-3}}{(-1+x_{-3}x_{-4})},\ \ x_{8n+2} = x_{-2}\left(\frac{(-1+x_{-3}x_{-4})}{(-1+x_{-2}x_{-3})}\right),\ x_{8n+3} = \frac{x_{-1}}{(-1+x_{-1}x_0)}.
\end{aligned}$$

Therefore, we have a periodic solution of period eight and the proof is complete.

Theorem 3.4. *Eq.(4) has a periodic solution of period two iff* $x_{-3}x_{-4} = x_{-3}x_{-2} = x_{-2}x_{-1} = x_{-1}x_0 = 2$ *and will be taken the form* $\{x_0, x_{-1}, x_0, x_{-1}, ...\}$.

Proof: The proof is consequently from the previous Theorem and will be omitted.

Numerical simulations

Here we will represent different types of solutions of Eq. (4).

Example 3. If we consider $x_{-4} = 11$, $x_{-3} = 3$, $x_{-2} = 9$, $x_{-1} = 2$, $x_0 = 7$ we get Fig. 3.

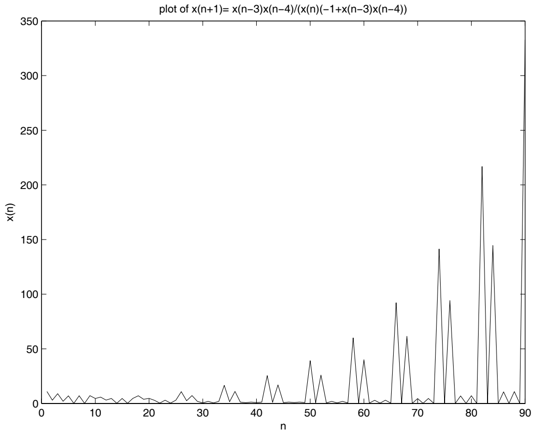

Figure 3.

Example 4. See Fig. 4, since we suppose that $x_{-4} = 15$, $x_{-3} = -2$, $x_{-2} = 15$, $x_{-1} = -2$, $x_0 = 15$.

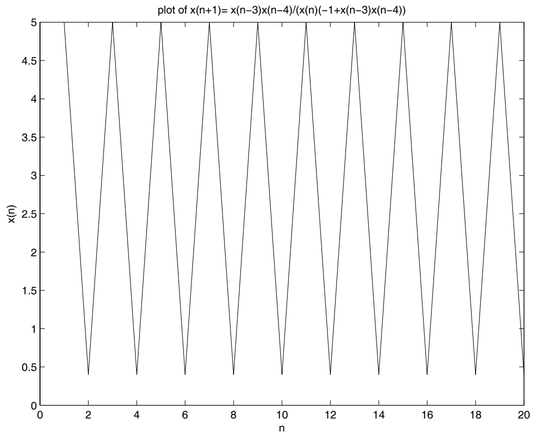

Figure 4.

Example 5. Assume that $x_{-4} = 3$, $x_{-3} = 2$, $x_{-2} = 3$, $x_{-1} = 7$, $x_0 = 3$ see Fig. 5.

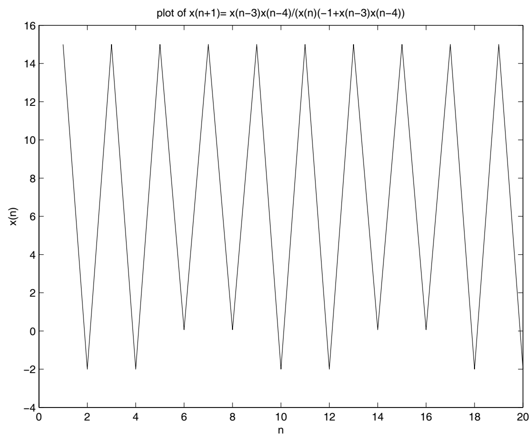

Figure 5.

The following cases can be proved similarly.

4 On the Recursive Sequence $x_{n+1}=\frac{x_{n-3}x_{n-4}}{x_n(1-x_{n-3}x_{n-4})}$

In this section, we get the solution of the third following equation

$$x_{n+1}=\frac{x_{n-3}x_{n-4}}{x_n(1-x_{n-3}x_{n-4})},\quad n=0,1,..., \tag{5}$$

where the initial values are arbitrary positive real numbers.

Theorem 4.1. *Assume that* $\{x_n\}_{n=-4}^{\infty}$ *be a solution of Eq.(5). Then for* $n=0,1,...$

$$\begin{aligned}
x_{4n-3} &= \frac{x_{-3}x_{-4}^n}{x_0^n}\prod_{i=0}^{n-1}\left(\frac{(1-ix_{-2}x_{-3})(1-ix_{-1}x_0)}{(1-(i+1)x_{-3}x_{-4})(1-ix_{-1}x_{-2})}\right),\\
x_{4n-2} &= \frac{x_{-2}x_0^n}{x_{-4}^n}\prod_{i=0}^{n-1}\left(\frac{(1-(i+1)x_{-3}x_{-4})(1-ix_{-1}x_{-2})}{(1-(i+1)x_{-2}x_{-3})(1-ix_{-1}x_0)}\right),\\
x_{4n-1} &= \frac{x_{-1}x_{-4}^n}{x_0^n}\prod_{i=0}^{n-1}\left(\frac{(1-(i+1)x_{-2}x_{-3})(1-ix_{-1}x_0)}{(1-(i+1)x_{-3}x_{-4})(1-(i+1)x_{-1}x_{-2})}\right),\\
x_{4n} &= \frac{x_0^{n+1}}{x_{-4}^n}\prod_{i=0}^{n}\left(\frac{(1-ix_{-3}x_{-4})(1-ix_{-1}x_{-2})}{(1-ix_{-2}x_{-3})(1-ix_{-1}x_0)}\right).
\end{aligned}$$

Theorem 4.2. *Eq.(5) has a unique equilibrium point which is the number zero and this equilibrium point is not locally asymptotically stable.*

Example 6. Assume that $x_{-4}=3,\ x_{-3}=2,\ x_{-2}=5,\ x_{-1}=7,\ x_0=0.13$ see Fig. 6.

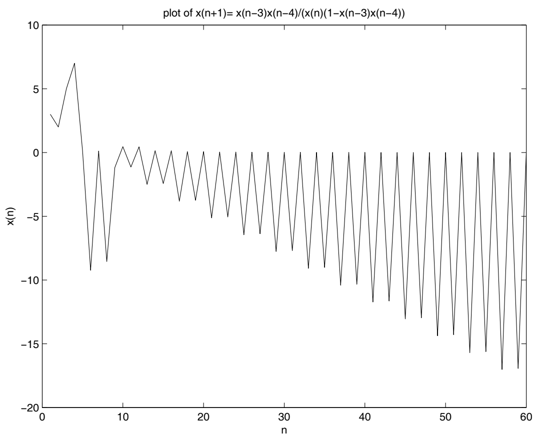

Figure 6.

5 On the Recursive Sequence $x_{n+1}=\frac{x_{n-3}x_{n-4}}{x_n(-1-x_{n-3}x_{n-4})}$

Here, we obtain a form of the solutions of the equation

$$x_{n+1}=\frac{x_{n-3}x_{n-4}}{x_n(-1-x_{n-3}x_{n-4})},\quad n=0,1,..., \tag{6}$$

where the initial values are arbitrary non zero real numbers with $x_{-3}x_{-4}\neq 1,\ x_{-3}x_{-2}\neq -1,\ x_{-2}x_{-1}\neq -1$ and $x_{-1}x_0\neq -1$.

Theorem 5.1. *Let $\{x_n\}_{n=-4}^{\infty}$ be a solution of Eq.(6). Then for $n = 0, 1, 2, ...$ the solution of Eq.(6) is given by*

$$\begin{aligned}
x_{8n-4} &= \frac{x_0^{2n}}{x_{-4}^{2n-1}} \left(\frac{(-1-x_{-3}x_{-4})(-1-x_{-1}x_{-2})}{(-1-x_{-2}x_{-3})(-1-x_{-1}x_0)} \right)^n, \\
x_{8n-3} &= \frac{x_{-3}x_{-4}^{2n}}{x_0^{2n}} \left(\frac{(-1-x_{-2}x_{-3})(-1-x_{-1}x_0)}{(-1-x_{-3}x_{-4})(-1-x_{-1}x_{-2})} \right)^n, \\
x_{8n-2} &= \frac{x_{-2}x_0^{2n}}{x_{-4}^{2n}} \left(\frac{(-1-x_{-3}x_{-4})(-1-x_{-1}x_{-2})}{(-1-x_{-2}x_{-3})(-1-x_{-1}x_0)} \right)^n, \\
x_{8n-1} &= \frac{x_{-1}x_{-4}^{2n}}{x_0^{2n}} \left(\frac{(-1-x_{-2}x_{-3})(-1-x_{-1}x_0)}{(-1-x_{-3}x_{-4})(-1-x_{-1}x_{-2})} \right)^n, \\
x_{8n} &= \frac{x_0^{2n+1}}{x_{-4}^{2n}} \left(\frac{(-1-x_{-3}x_{-4})(-1-x_{-1}x_{-2})}{(-1-x_{-2}x_{-3})(-1-x_{-1}x_0)} \right)^n,
\end{aligned}$$

$$\begin{aligned}
x_{8n+1} &= \frac{x_{-3}x_{-4}^{2n+1}}{x_0^{2n+1}(-1-x_{-3}x_{-4})} \left(\frac{(-1-x_{-2}x_{-3})(-1-x_{-1}x_0)}{(-1-x_{-3}x_{-4})(-1-x_{-1}x_{-2})} \right)^n, \\
x_{8n+2} &= \frac{x_{-2}x_0^{2n+1}}{x_{-4}^{2n+1}} \left(\frac{(-1-x_{-3}x_{-4})}{(-1-x_{-2}x_{-3})} \right) \left(\frac{(-1-x_{-3}x_{-4})(-1-x_{-1}x_{-2})}{(-1-x_{-2}x_{-3})(-1-x_{-1}x_0)} \right)^n, \\
x_{8n+3} &= \frac{x_{-1}x_{-4}^{2n+1}}{x_0^{2n+1}(-1-x_{-1}x_0)} \left(\frac{(-1-x_{-2}x_{-3})(-1-x_{-1}x_0)}{(-1-x_{-3}x_{-4})(-1-x_{-1}x_{-2})} \right)^{n+1}.
\end{aligned}$$

Theorem 5.2. *Eq.(6) has a unique equilibrium point which is the number zero and this equilibrium point is not locally asymptotically stable.*

Lemma 2. It is easy to see that every solution of Eq.(6) is unbounded except in the following case.

Theorem 5.3. *Eq.(4) has a periodic solution of period eight iff $(-1 - x_{-2}x_{-3})(-1 - x_{-1}x_0) = (-1 - x_{-3}x_{-4})(-1 - x_{-1}x_{-2})$, $x_0 = x_{-4}$ and has the form*

$$\left\{ x_0, x_{-3}, x_{-2}, x_{-1}, x_0, \frac{x_{-3}}{(-1-x_{-3}x_{-4})}, x_{-2} \left(\frac{(-1-x_{-3}x_{-4})}{(-1-x_{-2}x_{-3})} \right), \frac{x_{-1}}{(-1-x_{-1}x_0)}, x_0, x_{-3}, x_{-2}, ... \right\}.$$

Theorem 5.4. *Eq.(4) has a periodic solution of period two iff $x_{-3}x_{-4} = x_{-3}x_{-2} = x_{-2}x_{-1} = x_{-1}x_0 = -2$ and give by the form $\{x_0, x_{-1}, x_0, x_{-1}, ...\}$.*

Example 7. Fig. 7. shows the solutions when $x_{-4} = 0.21$, $x_{-3} = 2$, $x_{-2} = 0.5$, $x_{-1} = 7$, $x_0 = 0.3$.

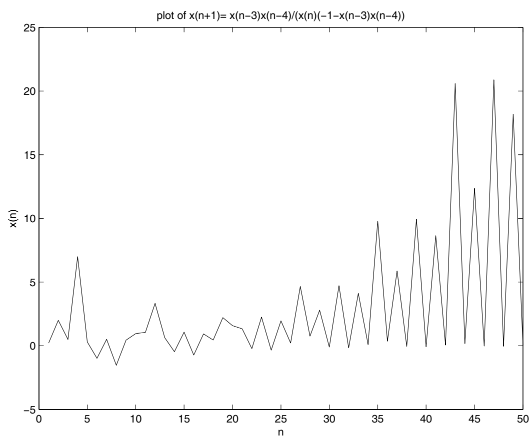

Figure 7.

Acknowledgements

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Global dynamics of two target cells HIV infection model with Beddington-DeAngelis functional response and delay-discrete or distributed

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Abstract

In this paper, we study the global analysis of virus dynamics models with discrete delay and with distributed delay. The models describe the interaction of the HIV with two classes of target cells, CD4$^+$ T cells and macrophages. The incidence rate of virus infection is given by the Beddington-DeAngelis functional response. The models have two types of discrete time delay or distributed delay describing the time needed for infection of cell and virus replication. The basic reproduction number R_0 is identified which completely determines the global dynamics of the models. By constructing suitable Lyapunov functionals, we have proven that if $R_0 \leq 1$ then the uninfected steady state is globally asymptotically stable (GAS), and if $R_0 > 1$ then the infected steady state exists and it is GAS.

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Keywords: HIV dynamics; Global stability; Delay; Beddington-DeAngelis functional response.

1 Introduction

In the last decades, mathematical modeling and model analysis have proven their importance in understanding the infection dynamics of HIV, and that provided a remarkable improvement in understanding the disease, hence determine the treatment dosages and the effectiveness of the medications [1]. A great effort has been devoted to study the basic and global properties of the HIV infection models such as positive invariance properties, boundedness of the model solutions and stability analysis which are important for understanding the associated characteristics of the HIV dynamics. Some of the existing models are given by ordinary differential equations (see e.g. [2], [3] and ([4]). Others models incorporate the delay between the time of viral entry into the target cell and the time the production of new virus particles, modeled with discrete time delay or distributed time delay using functional differential equations (see e.g. [6], [8], [20], [7], [21], [16], [23], [17], [15], [25]). In the literature, different forms of the incidence rate of infection in virus dynamics models have been used such as bilinear incidence rate βxv (e.g. [24], [7], [17], [26]), saturated incidence rate $\frac{\beta xv}{1+\gamma v}$ (e.g. [16], [23]), Beddington-DeAngelis functional response $\frac{\beta xv}{1+\alpha x+\gamma v}$, $\alpha, \gamma \geq 0$, [30], [28], [27], [29]), and unspecified function $h(x, v)$ ([15], [18]). The basic mathematical model describing the virus infection dynamics with Beddington-DeAngelis functional response has been studied in [4] is of the form:

$$\dot{x} = \lambda - dx - \frac{\beta xv}{1+\alpha x+\gamma v}, \tag{1}$$

$$\dot{y} = \frac{\beta xv}{1+\alpha x+\gamma v} - ay, \tag{2}$$

$$\dot{v} = ky - rv, \tag{3}$$

where x, y and v represent the concentrations of the uninfected CD4$^+$T cells, infected cells and free virus particles, respectively. The uninfected cells are generated from sources within the body at rate λ. The parameters d and β are the death rate constant of the uninfected cells and rate constant characterizing infections of the cells. Eq. (2) describes the concentrations dynamics of the infected cells and shows that they die with rate constant a. The virus particles are produced by the infected cells with rate constant k, and are cleared from plasma with rate constant r.

Model (1)-(3) has been extended to incorporate the discrete time delay in [30], [28] and [27]. In [29], model (1)-(3) has also been modified to take into account the cytotoxic T-lymphocyte (CTL) immune response. In model (1)-(3), it is assumed that the HIV attack one class of target cells ($CD4^{+}T$ cells). More accurate modeling being developed in 1997, when Perleson et al. [31], observed that after the rapid first phase of decay during the initial 1-2 weeks of antiretroviral treatment, plasma virus levels declined at a considerably slower rate. This second phase of viral decay was attributed to the turnover of a longer-lived virus reservoir of infected cells. These cells are called macrophages and considered as the second target cell for the HIV. Mathematical model of the HIV dynamics with two classes of target cells $CD4^{+}T$ cells and macrophages has been proposed in [5] and [22]. Recently, several papers have been presented deals with the global properties of the HIV dynamics with two classes of target cells ([9], [12], [11], [13]). Elaiw [9] studied the global properties of HIV infection model with two classes of target cells ($CD4^{+}T$ cells and macrophages). Elaiw and Azoz in [12], also studied the global properties of HIV infection models with two classes of target cells and with Beddington-DeAngelis functional response. In [12] the effect of time delay is neglected. Elaiw et al. [11] studied the global stability of HIV model with Beddington-DeAnglis functional response and one kind of discrete time delay. Elaiw [13] studied the global dynamics of a delay HIV model with two classes of target cells and saturated function response.

The aim of this paper is to study the global dynamics of two HIV infection models with Beddington-DeAngelis functional response. Model with discrete delay and model with distributed delay have been studied to take into account the time delay between the time the target cells contacted by the virus and the time the emission of infectious (matures) virus particles. The global stability of the these models is established using Lyapunov functionals. We have proven that the global dynamics of these models are determined by the basic reproduction number R_0. If $R_0 \leq 1$, then the uninfected steady state is globally asymptotically stable (GAS) and if $R_0 > 1$, then the infected steady state exists and it is GAS.

2 Model with discrete-time delay

In this section we study a viral infection model with two classes of target cells and Beddington-DeAngelis functional response. We incorporate two types of discrete-time delays into the model.

$$\dot{x_1} = \lambda_1 - d_1 x_1 - \frac{\beta_1 x_1 v}{1 + \alpha_1 x_1 + \gamma_1 v}, \tag{4}$$

$$\dot{x_2} = \lambda_2 - d_2 x_2 - \frac{\beta_2 x_2 v}{1 + \alpha_2 x_2 + \gamma_2 v}, \tag{5}$$

$$\dot{y_1} = \frac{e^{-m_1\tau_1}\beta_1 x_1(t-\tau_1)v(t-\tau_1)}{1 + \alpha_1 x_1(t-\tau_1) + \gamma_1 v(t-\tau_1)} - a_1 y_1, \tag{6}$$

$$\dot{y_2} = \frac{e^{-m_2\tau_2}\beta_2 x_2(t-\tau_2)v(t-\tau_2)}{1 + \alpha_2 x_2(t-\tau_2) + \gamma_2 v(t-\tau_2)} - a_2 y_2, \tag{7}$$

$$\dot{v} = \sum_{i=1}^{2} e^{-n_i\omega_i} k_i y_i\,(t-\omega_i) - rv, \tag{8}$$

where x_i and y_i represent the concentration of the uninfected and infected target cells, respectively, where $i = 1$ and 2 correspond to $CD4^{+}T$ cells and macrophages, v is the concentrtion of the virus particles. Here the parameter τ_i accounts for the time between the target cells of class i are contacted by the virus particles and the production of new virus particles. The factor $e^{-m_i\tau_i}$ accounts for the probability of surviving the time period from $t-\tau_i$ to t, where m_i is the death rate of infected but not yet virus producer cells. The parameter ω_i accounts for the time between the virus has penetrated into a target cell i, and the emission of infectious virus particles. The factor $e^{-n_i\omega_i}$ accounts for the probability of surviving the time period from $t-\omega_i$ to t, where n_i is positive constant. The parameters $\lambda_i, \beta_i, d_i, \alpha_i, \gamma_i, a_i,$ and k_i are positive constants with the same biological meaning given in model (1)-(3).

Initial conditions

The initial conditions for the system (4)-(8) take the form

$$\begin{aligned} &x_1(\theta) = \varphi_1(\theta),\ x_2(\theta) = \varphi_2(\theta), y_1(\theta) = \varphi_3(\theta),\ y_2(\theta) = \varphi_4(\theta), v(\theta) = \varphi_5(\theta), \\ &\varphi_i(\theta) \geq 0,\ \theta \in [-\ell, 0),\ \varphi_j(0) > 0,\ j = 1, ..., 5, \end{aligned} \tag{9}$$

where, $(\varphi_1(\theta), \varphi_2(\theta), ..., \varphi_5(\theta)) \in C$ and $C = C([-\ell, 0], \mathbb{R}^5_+)$ is the Banach space of continuous functions mapping the interval $[-\ell, 0]$ into $\mathbb{R}^5_+$, where $\ell = \max\{\tau_1, \tau_2, \omega_1, \omega_2\}$. By the fundamental theory of functional differential equations [14], the system (4)-(8) has a unique solution $(x_1(t), x_2(t), y_1(t), y_2(t), v(t))$ satisfying initial conditions (9).

We put $\beta_i' = e^{-m_i\tau_i}\beta_i$ and $p_i = e^{-n_i\omega_i}k_i$, then the system (4)-(8) can be written as:

$$\dot{x}_i = \lambda_i - d_i x_i - \frac{\beta_i x_i v}{1+\alpha_i x_i + \gamma_i v}, \qquad i=1,2 \tag{10}$$

$$\dot{y}_i = \frac{\beta_i' x_i(t-\tau_i)v(t-\tau_i)}{1+\alpha_i x_i(t-\tau_i)+\gamma_i v(t-\tau_i)} - a_i y_i, \qquad i=1,2 \tag{11}$$

$$\dot{v} = \sum_{i=1}^{2} p_i y_i\,(t-\omega_i) - rv. \tag{12}$$

2.1 Non-negativity and boundedness of solutions

In the following, we establish the non-negativity and boundedness of solutions of (10)-(12) with initial conditions (9).

Proposition 1. Let $X\,(t) = (x_1(t), x_2(t), y_1(t), y_2(t), v(t))^T$ be any solution of (10)-(12) satisfying the initial conditions (9), then $x_1(t), x_2(t), y_1(t), y_2(t)$ and $v(t)$ are all non-negative for $t \geq 0$ and ultimately bounded.

Proof. First, we prove that $x_i(t) > 0$, $i=1,2$, for all $t \geq 0$. Assume that $x_i(t)$ lose its non-negativity on some local existence interval $[0,\rho]$ for some constant ρ and let $t_1 \in [0,\rho]$ be such that $x_i(t_1) = 0$. From Eq. (10) we have $\dot{x}_i(t_1) = \lambda_i > 0$. Hence $x_i(t) > 0$ for some $t \in (t_1, t_1+\varepsilon)$, where $\varepsilon > 0$ is sufficiently small. This leads to a contradiction and hence $x_i(t) > 0$, for all $t \geq 0$. Further, from Eqs. (11) and (12) we have

$$y_i(t) = y_i(0)e^{-a_i t} + \beta_i' \int_0^t e^{-a_i(t-\eta)} \frac{x_i(\eta-\tau_i)v(\eta-\tau_i)}{1+\alpha_i x_i(\eta-\tau_i)+\gamma_i v(\eta-\tau_i)} d\eta, \quad i=1,2$$

$$v(t) = v(0)e^{-rt} + \sum_{i=1}^{2} p_i \int_0^t e^{-r(t-\eta)} y_i(\eta-\omega_i) d\eta,$$

confirming that $y_i(t) \geq 0$, $i=1,2$, and $v(t) \geq 0$ for all $t \in [0,\ell]$. By a recursive argument, we obtain $y_i(t) \geq 0$, $i=1,2$, and $v(t) \geq 0$ for all $t \geq 0$.

Next we show that the solution is ultimately bounded. From Eq. (10) we have $\dot{x}_i \leq \lambda_i - d_i x_i$. Thus $\limsup_{t\to\infty} x_i(t) \leq \frac{\lambda_i}{d_i}$ and $x_i(t)$ is ultimately bounded. Let $T_i(t) = \frac{\beta_i'}{\beta_i} x_i(t-\tau_i) + y_i(t)$, then

$$\begin{aligned}\dot{T}_i(t) &= \frac{\beta_i'}{\beta_i}\left(\lambda_i - d_i x_i\,(t-\tau_i) - \frac{\beta_i x_i\,(t-\tau_i)\,v\,(t-\tau_i)}{1+\alpha_i x_i\,(t-\tau_i)+\gamma_i v\,(t-\tau_i)}\right) + \frac{\beta_i' x_i\,(t-\tau_i)\,v\,(t-\tau_i)}{1+\alpha_i x_i\,(t-\tau_i)+\gamma_i v\,(t-\tau_i)} - a_i y_i \\ &= \frac{\beta_i'}{\beta_i}\lambda_i - \frac{\beta_i'}{\beta_i} d_i x_i\,(t-\tau_i) - a_i y_i \leq \frac{\beta_i'\lambda_i}{\beta_i} - \sigma_i T_i(t),\end{aligned}$$

where $\sigma_i = \min\{d_i, a_i\}$. It follows that $\limsup_{t\to\infty} T_i(t) \leq L_i$, where $L_i = \frac{\lambda_i \beta_i'}{\sigma_i \beta_i}$. This in turn implies, by the non-negativity of $x_i(t)$ and $y_i(t)$, that $\limsup_{t\to\infty} y_i(t) \leq L_i$ and $y_i(t)$ is ultimately bounded. On the other hand, from Eq. (12) we have $\dot{v}(t) \leq \sum_{i=1}^{2} p_i L_i - rv$, then $\limsup_{t\to\infty} v(t) \leq L^*$, where $L^* = \sum_{i=1}^{2} \frac{p_i L_i}{r}$ and $v(t)$ is ultimately bounded. □

2.2 Steady States

It is clear that, system (10)-(12) has an uninfected steady state $E_0(x_1^0, x_2^0, y_1^0, y_2^0, v^0)$, where $x_i^0 = \frac{\lambda_i}{d_i}$, $y_i^0 = 0$, $i=1,2$ and $v^0 = 0$. The system can also has an infected steady state $E_*(x_1^*, x_2^*, y_1^*, y_2^*, v^*)$ which is the infected steady state. The coordinates of E_*, if they exist, satisfies the equalities:

$$\lambda_i = d_i x_i^* + \frac{\beta_i x_i^* v^*}{1+\alpha_i x_i^* + \gamma_i v^*}, \qquad i=1,2 \tag{13}$$

$$a_i y_i^* = \frac{\beta_i' x_i^* v^*}{1+\alpha_i x_i^* + \gamma_i v^*}, \qquad i=1,2 \tag{14}$$

$$rv^* = \sum_{i=1}^{2} p_i y_i^*. \tag{15}$$

We define the intracellular delay-dependent basic reproduction number for system (10)-(12) as

$$R_0 = \sum_{i=1}^{2} R_i = \sum_{i=1}^{2} \frac{p_i \beta_i' x_i^0}{a_i r (1 + \alpha_i x_i^0)}, \tag{16}$$

where R_i is the basic reproduction number for the dynamics of the virus and the target cell of class i.

Lemma 1. If $R_0 > 1$, then there exists a positive steady state E_*.

Proof. To compute the steady state of the system (10)-(12), we let the right-hand sides of Eqs. (10)-(12) equal zero,

$$\lambda_i - d_i x_i - \frac{\beta_i x_i v}{1 + \alpha_i x_i + \gamma_i v} = 0, \qquad i = 1, 2 \tag{17}$$

$$\frac{\beta_i' x_i v}{1 + \alpha_i x_i + \gamma_i v} - a_i y_i = 0, \qquad i = 1, 2 \tag{18}$$

$$\sum_{i=1}^{2} p_i y_i - r v = 0. \tag{19}$$

Solving Eq. (17) with respect to x_i, we get x_i as a function of v as:

$$x_{i\pm} = \frac{1}{2\alpha_i} \left(\alpha_i x_i^0 - (1 + \xi_i v) \pm \sqrt{\left((1 + \xi_i v) - \alpha_i x_i^0\right)^2 + 4\alpha_i x_i^0 (1 + \gamma_i v)} \right),$$

where, $\xi_i = \gamma_i + \frac{\beta_i}{d_i}$. It is clear that if $v > 0$ then $x_{i+} > 0$ and $x_{i-} < 0$. Let us choose

$$x_i = \frac{1}{2\alpha_i} \left(\alpha_i x_i^0 - (1 + \xi_i v) + \sqrt{\left((1 + \xi_i v) - \alpha_i x_i^0\right)^2 + 4\alpha_i x_i^0 (1 + \gamma_i v)} \right). \tag{20}$$

From Eqs. (17)-(19) we have

$$\sum_{i=1}^{2} \frac{p_i \beta_i'}{a_i \beta_i} (\lambda_i - d_i x_i) - r v = 0. \tag{21}$$

Since x_i is a function of v, then we can define a function $A_1(v)$ as:

$$A_1(v) = \sum_{i=1}^{2} \frac{p_i \beta_i'}{a_i \beta_i} (\lambda_i - d_i x_i) - r v = 0.$$

When $v = 0$, then $x_i = x_i^0$, and $A_1(0) = 0$, and when $v = \bar{v} = \sum_{i=1}^{2} \frac{p_i \beta_i' \lambda_i}{a_i \beta_i r} > 0$, then substituting it in Eq. (20) we get $\bar{x}_i > 0$ and

$$A_1(\bar{v}) = -\sum_{i=1}^{2} \frac{p_i \beta_i' d_i}{a_i \beta_i} \bar{x}_i < 0.$$

Since $A_1(v)$ is continuous for all $v \geq 0$, we have

$$A_1'(0) = \sum_{i=1}^{2} \frac{p_i \beta_i' x_i^0}{a_i (1 + \alpha_i x_i^0)} - r = r(R_0 - 1).$$

Therefore, if $R_0 > 1$, then $A_1'(0) > 0$. It follows that there exists $v^* \in (0, \bar{v})$ such that $A_1'(v^*) = 0$. From Eq. (20), we obtain $x_i^* > 0, i = 1, 2$. Also, from Eq. (18) we get $y_i^* > 0, i = 1, 2$.

2.3 Global stability analysis

In this section, we study the global stability of the uninfected and infected steady states of system (10)-(12). The strategy of the proofs is to use suitable Lyapunov functionals which are similar in nature to those used in [32].

Preliminary:

We shall use the following notation: $z = z(t)$, for any $z \in \{x_i, y_i, v, \ i = 1, 2\}$. We also define a function $H : (0, \infty) \to [0, \infty)$ as

$$H(z) = z - 1 - \log z.$$

It is clear that $H(z) \geq 0$ for any $z > 0$ and H has the global minimum $H(1) = 0$. The function $H(z)$ can also be used in driving an extension of the arithmetic-geometric mean inequality which is important in proving the global stability of the steady states.

To extend the arithmetic-geometric mean inequality we put

$$-H(z_i) = 1 - z_i + \log z_i \leq 0, \quad \text{for } z_1, ..., z_n > 0, \tag{22}$$

summing $-H(z_i)$ from $i = 1$ to n

$$n - \sum_{i=1}^{n} z_i + \log \prod_{i=1}^{n} z_i \leq 0. \tag{23}$$

For $a_1, ..., a_n, b_1, ..., b_n > 0$, it holds that

$$n - \sum_{i=1}^{n} \frac{b_i}{a_i} + \log \prod_{i=1}^{n} \frac{b_i}{a_i} \leq 0. \tag{24}$$

If $z_1, z_2, ..., z_n > 0$ satisfy $z_1 z_2 ... z_n = 1$, then it holds that

$$n - \sum_{i=1}^{n} z_i \leq 0. \tag{25}$$

When $a_1 a_2 ... a_n = b_1 b_2 ... b_n$, and by substituting $z_i = \frac{b_i}{a_i}$ in (25), we get

$$n - \sum_{i=1}^{n} \frac{b_i}{a_i} \leq 0. \tag{26}$$

If we put $z_i = \frac{a_i}{\sqrt[n]{a_1 a_2 ... a_n}}$ in (25), we obtain

$$\frac{a_1 + a_2 + ... + a_n}{n} \geq \sqrt[n]{a_1 a_2 ... a_n}$$

which is the arithmetic-geometric mean inequality. Thus the inequalities (23) and (24) are considered as extensions of the arithmetic-geometric inequality.

Now let us assume that $a_1 a_2 ... a_{m-1} = b_1 b_2 ... b_{m-1}$, $m < n$, and replace $b_m, ..., b_n$ by $b'_m, ... b'_n$, then we have

$$n - \sum_{i=1}^{m-1} \frac{b_i}{a_i} - \sum_{i=m}^{n} \frac{b'_i}{a_i} + \log \frac{b_1 ... b_{m-1} b'_m ... b'_n}{a_1 ... a_{m-1} a_m ... a_n} \leq 0,$$

$$n - \sum_{i=1}^{m-1} \frac{b_i}{a_i} - \sum_{i=m}^{n} \frac{b'_i}{a_i} + \log \prod_{i=m}^{n} \frac{b'_i}{a_i} \leq 0. \tag{27}$$

This holds true for any positive a_i, b_j, b'_k, $(i = 1, ..., n; j = 1, ..., m-1; k = m, ..., n)$. The inequality (27) is crucial in proving the global stability of the infected steady states.

First we prove the global stability of the uninfected steady state E_0 by using a suitable Lyapunov functional.

Theorem 1. Consider the system (10)-(12), if $R_0 \leq 1$ then E_0 is GAS.

Proof. Define a Lyapunov functional W_0 as follows:

$$W_0 = \sum_{i=1}^{2} \frac{p_i}{a_i} \left[\frac{\beta'_i x_i^0}{\beta_i (1 + \alpha_i x_i^0)} \left(\frac{x_i}{x_i^0} - 1 - \log \frac{x_i}{x_i^0} \right) + y_i + \beta'_i \int_0^{\tau_i} \frac{x_i(t-\theta) v(t-\theta)}{1 + \alpha_i x_i(t-\theta) + \gamma_i v(t-\theta)} d\theta \right.$$
$$\left. + a_i \int_0^{\omega_i} y_i(t-\theta) d\theta \right] + v. \tag{28}$$

We note that W_0 is defined and continuous for all $x_i, y_i, v > 0$. Also, the global minimum $W_0 = 0$ occurs at the

uninfected steady state E_0. The time derivative of W_0 along the solution of (10)-(12) is given by

$$\begin{aligned}
\frac{dW_0}{dt} &= \sum_{i=1}^{2}\frac{p_i}{a_i}\left[\frac{\beta_i'}{\beta_i(1+\alpha_i x_i^0)}\left(1-\frac{x_i^0}{x_i}\right)\left(\lambda_i-d_ix_i-\frac{\beta_i x_i v}{1+\alpha_i x_i+\gamma_i v}\right)+\frac{\beta_i' x_i(t-\tau_i)v(t-\tau_i)}{1+\alpha_i x_i(t-\tau_i)+\gamma_i v(t-\tau_i)}-a_iy_i\right.\\
&\quad\left.+\frac{\beta_i' x_i v}{1+\alpha_i x_i+\gamma_i v}-\frac{\beta_i' x_i(t-\tau_i)v(t-\tau_i)}{1+\alpha_i x_i(t-\tau_i)+\gamma_i v(t-\tau_i)}+a_iy_i-a_iy_i(t-\omega_i)\right]+\sum_{i=1}^{2}p_iy_i(t-\omega_i)-rv\\
&= \sum_{i=1}^{2}\frac{p_i}{a_i}\left[\frac{-\beta_i'\lambda_i}{\beta_i x_i x_i^0(1+a_ix_i^0)}\left(x_i-x_i^0\right)^2-\frac{\beta_i' x_i v}{(1+\alpha_i x_i^0)(1+\alpha_i x_i+\gamma_i v)}\right.\\
&\quad\left.+\frac{\beta_i' x_i^0 v}{(1+\alpha_i x_i^0)(1+\alpha_i x_i+\gamma_i v)}+\frac{\beta_i' x_i v}{1+\alpha_i x_i+\gamma_i v}\right]-rv\\
&= \sum_{i=1}^{2}\frac{p_i}{a_i}\left[\frac{-\beta_i'\lambda_i}{\beta_i x_i x_i^0(1+a_ix_i^0)}\left(x_i-x_i^0\right)^2+\frac{\beta_i' x_i^0 v+\alpha_i x_i^0\beta_i' x_i v}{(1+\alpha_i x_i^0)(1+\alpha_i x_i+\gamma_i v)}\right]-rv\\
&= \sum_{i=1}^{2}\frac{p_i}{a_i}\left[\frac{-\beta_i'\lambda_i}{\beta_i x_i x_i^0(1+a_ix_i^0)}\left(x_i-x_i^0\right)^2+\frac{\beta_i' x_i^0 v(1+\alpha_i x_i+\gamma_i v-\gamma_i v)}{(1+\alpha_i x_i^0)(1+\alpha_i x_i+\gamma_i v)}\right]-rv\\
&= \sum_{i=1}^{2}\frac{p_i}{a_i}\left[\frac{-\beta_i'\lambda_i}{\beta_i x_i x_i^0(1+a_ix_i^0)}\left(x_i-x_i^0\right)^2+\frac{\beta_i' x_i^0 v(1+\alpha_i x_i+\gamma_i v)}{(1+\alpha_i x_i^0)(1+\alpha_i x_i+\gamma_i v)}-\frac{\gamma_i\beta_i' x_i^0 v^2}{(1+\alpha_i x_i^0)(1+\alpha_i x_i+\gamma_i v)}\right]-rv\\
&= -\sum_{i=1}^{2}\frac{p_i}{a_i}\frac{\beta_i'\lambda_i}{\beta_i x_i x_i^0(1+a_ix_i^0)}\left(x_i-x_i^0\right)^2-\sum_{i=1}^{2}\frac{p_i}{a_i}\frac{\gamma_i\beta_i' x_i^0 v^2}{(1+\alpha_i x_i^0)(1+\alpha_i x_i+\gamma_i v)}+\sum_{i=1}^{2}\frac{p_i}{a_i}\frac{\beta_i' x_i^0 v}{(1+\alpha_i x_i^0)}-rv\\
&= -\sum_{i=1}^{2}\frac{p_i}{a_i}\frac{\beta_i'\lambda_i}{\beta_i x_i x_i^0(1+a_ix_i^0)}\left(x_i-x_i^0\right)^2-\sum_{i=1}^{2}\frac{p_i}{a_i}\frac{\gamma_i\beta_i' x_i^0 v^2}{(1+\alpha_i x_i^0)(1+\alpha_i x_i+\gamma_i v)}+(R_0-1)rv. \qquad (29)
\end{aligned}$$

It can be seen that, if $R_0 \leq 1$ then $\frac{dW_0}{dt} \leq 0$ for all $x_i, v > 0$, $i = 1, 2$. By Theorem 5.3.1 in [14], the solutions of system (10)-(12) limit to M, the largest invariant subset of $\left\{\frac{dW_0}{dt} = 0\right\}$. Clearly, it follows from (29) that $\frac{dW_0}{dt} = 0$ if and only if $x_i = x_i^0$, $i = 1, 2$ and $v = 0$. Noting that M is invariant, for each element of M we have $v = 0$, then $\dot{v} = 0$. From Eq. (12) we drive that

$$0 = \dot{v} = \sum_{i=1}^{2} p_i y_i(t-\omega_i).$$

Since $y_i(t-\theta) \geq 0$ for all $\theta \in [0, \ell]$, then $\sum_{i=1}^{2} p_i y_i(t-\omega_i) = 0$ if and only if $y_i(t-\omega_i) = 0$, $i = 1, 2$. Hence $\frac{dW_0}{dt} = 0$ if and only if $x_i = x^0$, $y_i = 0$, $i = 1, 2$ and $v = 0$. From LaSalle's invariance principle, E_0 is GAS.

Theorem 2. If $R_0 > 1$, then E_* is GAS.

Proof. For proving the global stability of the infected steady state E_*, we first use the following system

$$\dot{x}_i = \lambda_i - d_i x_i - \frac{\beta_i x_i v}{1+\alpha_i x_i+\gamma_i v}, \qquad i = 1, 2 \qquad (30)$$

$$\dot{y}_i = \frac{\beta_i' x_i v}{1+\alpha_i x_i+\gamma_i v} - a_i y_i, \qquad i = 1, 2 \qquad (31)$$

$$\dot{v} = \sum_{i=1}^{2} p_i y_i - rv. \qquad (32)$$

let $X = (x_1, x_2, y_1, y_2, v)^T$, and denote $F(X)$ as the vector field given by (30)-(32), and define Lyapunov functional W_* as follows:

$$W_* = \sum_{i=1}^{2}\frac{p_i}{a_i}\left[\frac{\beta_i'}{\beta_i}\left(x_i - x_i^* - \int_{x_i^*}^{x_i}\frac{x_i^*(1+\alpha_i\mu+\gamma_i v^*)}{\mu(1+\alpha_i x_i^*+\gamma_i v^*)}d\mu\right)+y_i^*\left(\frac{y_i}{y_i^*}-1-\log\frac{y_i}{y_i^*}\right)\right]+v^*\left(\frac{v}{v^*}-1-\log\frac{v}{v^*}\right). \qquad (33)$$

By calculating the time derivative along (30)-(32), we get

$$\nabla W_* \cdot F(X) = \sum_{i=1}^{2} \frac{p_i}{a_i} \left[\frac{\beta_i'}{\beta_i} \left(1 - \frac{x_i^*(1+\alpha_i x_i + \gamma_i v^*)}{x_i(1+\alpha_i x_i^* + \gamma_i v^*)}\right) \left(\lambda_i - d_i x_i - \frac{\beta_i x_i v}{1+\alpha_i x_i + \gamma_i v}\right) \right.$$
$$\left. + \left(1 - \frac{y_i^*}{y_i}\right)\left(\frac{\beta_i' x_i v}{1+\alpha_i x_i + \gamma_i v} - a_i y_i\right)\right] + \left(1 - \frac{v^*}{v}\right)\left(\sum_{i=1}^{2} p_i y_i - rv\right).$$

Using the infection steady state conditions (13)-(15), we get

$$\nabla W_* \cdot F(X) = \sum_{i=1}^{2} \frac{p_i}{a_i} \left[\frac{\beta_i'}{\beta_i} \left(1 - \frac{x_i^*(1+\alpha_i x_i + \gamma_i v^*)}{x_i(1+\alpha_i x_i^* + \gamma_i v^*)}\right) \left(d_i x_i^* + \frac{\beta_i x_i^* v^*}{1+\alpha_i x_i^* + \gamma_i v^*} - d_i x_i - \frac{\beta_i x_i v}{1+\alpha_i x_i + \gamma_i v}\right) \right.$$
$$\left. + \frac{\beta_i' x_i v}{1+\alpha_i x_i + \gamma_i v} - a_i y_i - \frac{y_i^*}{y_i}\frac{\beta_i' x_i v}{1+\alpha_i x_i + \gamma_i v} + a_i y_i^*\right] + \sum_{i=1}^{2} p_i y_i - rv - \frac{v^*}{v}\sum_{i=1}^{2} p_i y_i + rv^*$$
$$= \sum_{i=1}^{2} \frac{p_i}{a_i}\left[\frac{-d_i\beta_i'(x_i - x_i^*)}{\beta_i}\left(1 - \frac{x_i^*(1+\alpha_i x_i + \gamma_i v^*)}{x_i(1+\alpha_i x_i^* + \gamma_i v^*)}\right) + a_i y_i^*\left(\frac{v(1+\alpha_i x_i + \gamma_i v^*)}{v_i^*(1+\alpha_i x_i + \gamma_i v)} - \frac{v}{v^*}\right)\right.$$
$$\left. + a_i y_i^*\left(3 - \frac{x_i^*(1+\alpha_i x_i + \gamma_i v^*)}{x_i(1+\alpha_i x_i^* + \gamma_i v^*)} - \frac{y_i^*}{y_i}\frac{x_i v(1+\alpha_i x_i^* + \gamma_i v^*)}{x_i^* v^*(1+\alpha_i x_i + \gamma_i v)} - \frac{y_i v^*}{y_i^* v}\right)\right]$$
$$= \sum_{i=1}^{2} \frac{p_i}{a_i}\left[\frac{-d_i\beta_i'(x_i - x_i^*)}{\beta_i}\left(\frac{x_i(1+\alpha_i x_i^* + \gamma_i v^*) - x_i^*(1+\alpha_i x_i + \gamma_i v^*)}{x_i(1+\alpha_i x_i^* + \gamma_i v^*)}\right)\right.$$
$$+ a_i y_i^*\left(\frac{v(1+\alpha_i x_i + \gamma_i v^*)}{v^*(1+\alpha_i x_i + \gamma_i v)} - \frac{v}{v^*} + \frac{1+\alpha_i x_i + \gamma_i v}{1+\alpha_i x_i + \gamma_i v^*} - \frac{1+\alpha_i x_i + \gamma_i v}{1+\alpha_i x_i + \gamma_i v^*}\right)$$
$$\left. + a_i y_i^*\left(3 - \frac{x_i^*(1+\alpha_i x_i + \gamma_i v^*)}{x_i(1+\alpha_i x_i^* + \gamma_i v^*)} - \frac{y_i^*}{y_i}\frac{x_i v(1+\alpha_i x_i^* + \gamma_i v^*)}{x_i^* v^*(1+\alpha_i x_i + \gamma_i v)} - \frac{y_i v^*}{y_i^* v}\right)\right]$$

$$= \sum_{i=1}^{2} \frac{p_i}{a_i}\left[\frac{-d_i\beta_i'(x_i - x_i^*)}{\beta_i}\left(\frac{x_i + \alpha_i x_i x_i^* + \gamma_i x_i v^* - x_i^* - \alpha_i x_i x_i^* - \gamma_i v^* x_i^*}{x_i(1+\alpha_i x_i^* + \gamma_i v^*)}\right)\right.$$
$$- a_i y_i^*\left(1 + \frac{v}{v^*} - \frac{v(1+\alpha_i x_i + \gamma_i v^*)}{v^*(1+\alpha_i x_i + \gamma_i v)} - \frac{1+\alpha_i x_i + \gamma_i v}{1+\alpha_i x_i + \gamma_i v^*}\right)$$
$$\left. + a_i y_i^*\left(4 - \frac{x_i^*(1+\alpha_i x_i + \gamma_i v^*)}{x_i(1+\alpha_i x_i^* + \gamma_i v^*)} - \frac{y_i^*}{y_i}\frac{x_i v(1+\alpha_i x_i^* + \gamma_i v^*)}{x_i^* v^*(1+\alpha_i x_i + \gamma_i v)} - \frac{y_i v^*}{y_i^* v} - \frac{1+\alpha_i x_i + \gamma_i v}{1+\alpha_i x_i + \gamma_i v^*}\right)\right]$$

$$= \sum_{i=1}^{2} \frac{p_i}{a_i}\left[\frac{-d_i\beta_i'(x_i - x_i^*)^2(1+\gamma_i v^*)}{\beta_i x_i(1+\alpha_i x_i^* + \gamma_i v^*)} - a_i y_i^*\left(\frac{\gamma_i(1+\alpha_i x_i)(v - v^*)^2}{v^*(1+\alpha_i x_i + \gamma_i v)(1+\alpha_i x_i + \gamma_i v^*)}\right)\right.$$
$$\left. + a_i y_i^*\left(4 - \frac{x_i^*(1+\alpha_i x_i + \gamma_i v^*)}{x_i(1+\alpha_i x_i^* + \gamma_i v^*)} - \frac{y_i^*}{y_i}\frac{x_i v(1+\alpha_i x_i^* + \gamma_i v^*)}{x_i^* v^*(1+\alpha_i x_i + \gamma_i v)} - \frac{y_i v^*}{y_i^* v} - \frac{1+\alpha_i x_i + \gamma_i v}{1+\alpha_i x_i + \gamma_i v^*}\right)\right] \quad (34)$$

Since the arithmetic mean is greater than or equal to the geometric mean then

$$\left(4 - \frac{x_i^*(1+\alpha_i x_i + \gamma_i v^*)}{x_i(1+\alpha_i x_i^* + \gamma_i v^*)} - \frac{y_i^*}{y_i}\frac{x_i v(1+\alpha_i x_i^* + \gamma_i v^*)}{x_i^* v^*(1+\alpha_i x_i + \gamma_i v)} - \frac{y_i v^*}{y_i^* v} - \frac{1+\alpha_i x_i + \gamma_i v}{1+\alpha_i x_i + \gamma_i v^*}\right) \le 0,$$

and $\nabla W_* \cdot F(X) \le 0$.

Now, we compute the time derivative of W_* along the solution of the system (10)-(12)

$$\frac{dW_*}{dt} = \sum_{i=1}^{2} \frac{p_i}{a_i}\left[\frac{\beta_i'}{\beta_i}\left(1 - \frac{x_i^*(1+\alpha_i x_i + \gamma_i v^*)}{x_i(1+\alpha_i x_i^* + \gamma_i v^*)}\right)\left(\lambda_i - d_i x_i - \frac{\beta_i x_i v}{1+\alpha_i x_i + \gamma_i v}\right)\right.$$
$$\left. + \left(1 - \frac{y_i^*}{y_i}\right)\left(\frac{\beta_i' x_i(t-\tau_i)v(t-\tau_i)}{1+\alpha_i x_i(t-\tau_i) + \gamma_i v(t-\tau_i)} - a_i y_i\right)\right] + \left(1 - \frac{v^*}{v}\right)\left(\sum_{i=1}^{2} p_i y_i(t-\omega_i) - rv\right)$$

$$
\begin{aligned}
&= \sum_{i=1}^{2} \frac{p_i}{a_i} \left[\frac{\beta_i'}{\beta_i} \left(1 - \frac{x_i^*(1+\alpha_i x_i + \gamma_i v^*)}{x_i(1+\alpha_i x_i^* + \gamma_i v^*)} \right) \left(\lambda_i - d_i x_i - \frac{\beta_i x_i v}{1+\alpha_i x_i + \gamma_i v} \right) \right.\\
&+ \left. \left(1 - \frac{y_i^*}{y_i} \right) \left(\frac{\beta_i' x_i(t-\tau_i) v(t-\tau_i)}{1+\alpha_i x_i(t-\tau_i) + \gamma_i v(t-\tau_i)} - a_i y_i + \frac{\beta_i' x_i v}{1+\alpha_i x_i + \gamma_i v} - \frac{\beta_i' x_i v}{1+\alpha_i x_i + \gamma_i v} \right) \right]\\
&+ \left(1 - \frac{v^*}{v} \right) \left(\sum_{i=1}^{2} p_i y_i(t-\omega_i) - rv + \sum_{i=1}^{2} p_i y_i - \sum_{i=1}^{2} p_i y_i \right)
\end{aligned}
$$

$$
\begin{aligned}
&= \left\{ \sum_{i=1}^{2} \frac{p_i}{a_i} \left[\frac{\beta_i'}{\beta_i} \left(1 - \frac{x_i^*(1+\alpha_i x_i + \gamma_i v^*)}{x_i(1+\alpha_i x_i^* + \gamma_i v^*)} \right) \left(\lambda_i - d_i x_i - \frac{\beta_i x_i v}{1+\alpha_i x_i + \gamma_i v} \right) \right. \right.\\
&+ \left. \left. \left(1 - \frac{y_i^*}{y_i} \right) \left(\frac{\beta_i' x_i v}{1+\alpha_i x_i + \gamma_i v} - a_i y_i \right) \right] + \left(1 - \frac{v^*}{v} \right) \left(\sum_{i=1}^{2} p_i y_i - rv \right) \right\}\\
&+ \sum_{i=1}^{2} \frac{p_i}{a_i} \left[\left(1 - \frac{y_i^*}{y_i} \right) \left(\frac{\beta_i' x_i(t-\tau_i) v(t-\tau_i)}{1+\alpha_i x_i(t-\tau_i) + \gamma_i v(t-\tau_i)} - \frac{\beta_i' x_i v}{1+\alpha_i x_i + \gamma_i v} \right) \right]\\
&+ \left(1 - \frac{v^*}{v} \right) \left(\sum_{i=1}^{2} p_i y_i(t-\omega_i) - \sum_{i=1}^{2} p_i y_i \right). \qquad (35)
\end{aligned}
$$

Substituting from (34) into (35), and using the infected steady state conditions (13)-(15) we get

$$
\begin{aligned}
\frac{dW_*}{dt} &= \nabla W_* \cdot F(X) + \sum_{i=1}^{2} \frac{p_i}{a_i} a_i y_i^* \left[\left(1 - \frac{y_i^*}{y_i} \right) \left(\frac{x_i(t-\tau_i) v(t-\tau_i)(1+\alpha_i x_i^* + \gamma_i v^*)}{x_i^* v^*(1+\alpha_i x_i(t-\tau_i) + \gamma_i v(t-\tau_i))} - \frac{x_i v(1+\alpha_i x_i^* + \gamma_i v^*)}{x_i^* v^*(1+\alpha_i x_i + \gamma_i v)} \right) \right.\\
&+ \left. \left(1 - \frac{v^*}{v} \right) \left(\frac{y_i(t-\omega_i)}{y_i^*} - \frac{y_i}{y_i^*} \right) \right]\\
&= \sum_{i=1}^{2} \frac{p_i}{a_i} \left[\frac{-d_i \beta_i'(x_i - x_i^*)^2 (1+\gamma_i v^*)}{\beta_i x_i (1+\alpha_i x_i^* + \gamma_i v^*)} - a_i y_i^* \frac{\gamma_i (1+\alpha_i x_i)(v-v^*)^2}{v^*(1+\alpha_i x_i + \gamma_i v)(1+\alpha_i x_i + \gamma_i v^*)} \right.\\
&+ a_i y_i^* \left(4 - \frac{x_i^*(1+\alpha_i x_i + \gamma_i v^*)}{x_i(1+\alpha_i x_i^* + \gamma_i v^*)} - \frac{y_i^*}{y_i} \frac{x_i v(1+\alpha_i x_i^* + \gamma_i v^*)}{x_i^* v^*(1+\alpha_i x_i + \gamma_i v)} - \frac{y_i v^*}{y_i^* v} - \frac{1+\alpha_i x_i + \gamma_i v}{1+\alpha_i x_i + \gamma_i v^*} \right)\\
&+ a_i y_i^* \left(\frac{x_i(t-\tau_i) v(t-\tau_i)(1+\alpha_i x_i^* + \gamma_i v^*)}{x_i^* v^*(1+\alpha_i x_i(t-\tau_i) + \gamma_i v(t-\tau_i))} - \frac{x_i v(1+\alpha_i x_i^* + \gamma_i v^*)}{x_i^* v^*(1+\alpha_i x_i + \gamma_i v)} \right.\\
&- \frac{y_i^* x_i(t-\tau_i) v(t-\tau_i)(1+\alpha_i x_i^* + \gamma_i v^*)}{y_i x_i^* v^*(1+\alpha_i x_i(t-\tau_i) + \gamma_i v(t-\tau_i))} + \frac{y_i^* x_i v(1+\alpha_i x_i^* + \gamma_i v^*)}{y_i x_i^* v^*(1+\alpha_i x_i + \gamma_i v)}\\
&+ \left. \left. \frac{y_i(t-\omega_i)}{y_i^*} - \frac{y_i}{y_i^*} - \frac{v^* y_i(t-\omega_i)}{v y_i^*} + \frac{v^* y_i}{v y_i^*} \right) \right]
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{2} \frac{p_i}{a_i} \left[\frac{-d_i \beta_i'(x_i - x_i^*)^2 (1+\gamma_i v^*)}{\beta_i x_i (1+\alpha_i x_i^* + \gamma_i v^*)} - a_i y_i^* \left(\frac{\gamma_i (1+\alpha_i x_i)(v-v^*)^2}{v^*(1+\alpha_i x_i + \gamma_i v)(1+\alpha_i x_i + \gamma_i v^*)} \right) \right.\\
&+ a_i y_i^* \left(4 - \frac{x_i^*(1+\alpha_i x_i + \gamma_i v^*)}{x_i(1+\alpha_i x_i^* + \gamma_i v^*)} - \frac{y_i^* x_i(t-\tau_i) v(t-\tau_i)(1+\alpha_i x_i^* + \gamma_i v^*)}{y_i x_i^* v^*(1+\alpha_i x_i(t-\tau_i) + \gamma_i v(t-\tau_i))} - \frac{v^* y_i(t-\omega_i)}{v y_i^*} \right.\\
&- \left. \left. \frac{1+\alpha_i x_i + \gamma_i v}{1+\alpha_i x_i + \gamma_i v^*} + \log \frac{y_i(t-\omega_i) x_i(t-\tau_i) v(t-\tau_i)(1+\alpha_i x_i + \gamma_i v)}{y_i x_i v (1+\alpha_i x_i(t-\tau_i) + \gamma_i v(t-\tau_i))} \right) \right]\\
&+ \sum_{i=1}^{2} \frac{p_i}{a_i} a_i y_i^* \left[\frac{x_i(t-\tau_i) v(t-\tau_i)(1+\alpha_i x_i^* + \gamma_i v^*)}{x_i^* v^*(1+\alpha_i x_i(t-\tau_i) + \gamma_i v(t-\tau_i))} - \frac{x_i v(1+\alpha_i x_i^* + \gamma_i v^*)}{x_i^* v^*(1+\alpha_i x_i + \gamma_i v)} \right.\\
&- \left. \log \frac{x_i(t-\tau_i) v(t-\tau_i)(1+\alpha_i x_i + \gamma_i v)}{x_i v(1+\alpha_i x_i(t-\tau_i) + \gamma_i v(t-\tau_i))} \right] + \sum_{i=1}^{2} p_i y_i^* \left(\frac{y_i(t-\omega_i)}{y_i^*} - \frac{y_i}{y_i^*} - \log \frac{y_i(t-\omega_i)}{y_i} \right). \qquad (36)
\end{aligned}
$$

Using the arithmetic-geometric inequality (27) we obtain

$$\sum_{i=1}^{2}\frac{p_i}{a_i}\left[\frac{-d_i\beta_i'(x_i-x_i^*)^2(1+\gamma_i v^*)}{\beta_i x_i(1+\alpha_i x_i^*+\gamma_i v^*)}-a_i y_i^*\left(\frac{\gamma_i(1+\alpha_i x_i)(v-v^*)^2}{v^*(1+\alpha_i x_i+\gamma_i v)(1+\alpha_i x_i+\gamma_i v^*)}\right)\right.$$
$$+a_i y_i^*\left(4-\frac{x_i^*(1+\alpha_i x_i+\gamma_i v^*)}{x_i(1+\alpha_i x_i^*+\gamma_i v^*)}-\frac{y_i^* x_i(t-\tau_i)v(t-\tau_i)(1+\alpha_i x_i^*+\gamma_i v^*)}{y_i x_i^* v^*(1+\alpha_i x_i(t-\tau_i)+\gamma_i v(t-\tau_i))}-\frac{v^* y_i(t-\omega_i)}{v y_i^*}\right.$$
$$\left.\left.-\frac{1+\alpha_i x_i+\gamma_i v}{1+\alpha_i x_i+\gamma_i v^*}+\log\frac{y_i(t-\omega_i)x_i(t-\tau_i)v(t-\tau_i)(1+\alpha_i x_i+\gamma_i v)}{y_i x_i v(1+\alpha_i x_i(t-\tau_i)+\gamma_i v(t-\tau_i))}\right)\right]\le 0.$$

Define the following functionals

$$\overline{W_i}=\int_0^{\tau_i}H\left(\frac{x_i(t-\theta)v(t-\theta)((1+\alpha_i x_i^*+\gamma_i v^*)}{x_i^* v^*(1+\alpha_i x_i(t-\theta)+\gamma_i v(t-\theta))}\right)d\theta,\quad i=1,2$$

$$\widetilde{W_i}=\int_0^{\omega_i}H\left(\frac{y_i(t-\theta)}{y_i^*}\right)d\theta,\quad i=1,2$$

then we have,

$$\frac{d\overline{W_i}}{dt}=\frac{x_i v(1+\alpha_i x_i^*+\gamma_i v^*)}{x_i^* v^*(1+\alpha_i x_i+\gamma_i v)}-\frac{x_i(t-\tau_i)v(t-\tau_i)(1+\alpha_i x_i^*+\gamma_i v^*)}{x_i^* v^*(1+\alpha_i x_i(t-\tau_i)+\gamma_i v(t-\tau_i))}$$
$$+\log\frac{x_i(t-\tau_i)v(t-\tau_i)(1+\alpha_i x_i+\gamma_i v)}{x_i v(1+\alpha_i x_i(t-\tau_i)+\gamma_i v(t-\tau_i))}.$$

$$\frac{d\widetilde{W_i}}{dt}=\frac{y_i}{y_i^*}-\frac{y_i(t-\omega_i)}{y_i^*}+\log\frac{y_i(t-\omega_i)}{y_i}.$$

Therefore, we construct Lyapunov functional as follows:

$$W_1=W_*+\sum_{i=1}^{2}p_i y_i^*\overline{W_i}+\sum_{i=1}^{2}p_i y_i^*\widetilde{W_i}.$$

Then the time derivative of W_1 along the trajectory of (10)-(12) is given as

$$\frac{dW_1}{dt}=\sum_{i=1}^{2}\frac{p_i}{a_i}\left[\frac{-d_i\beta_i'(x_i-x_i^*)^2(1+\gamma_i v^*)}{\beta_i x_i(1+\alpha_i x_i^*+\gamma_i v^*)}-a_i y_i^*\left(\frac{\gamma_i(1+\alpha_i x_i)(v-v^*)^2}{v^*(1+\alpha_i x_i+\gamma_i v)(1+\alpha_i x_i+\gamma_i v^*)}\right)\right.$$
$$+a_i y_i^*\left(4-\frac{x_i^*(1+\alpha_i x_i+\gamma_i v^*)}{x_i(1+\alpha_i x_i^*+\gamma_i v^*)}-\frac{y_i^* x_i(t-\tau_i)v(t-\tau_i)(1+\alpha_i x_i^*+\gamma_i v^*)}{y_i x_i^* v^*(1+\alpha_i x_i(t-\tau_i)+\gamma_i v(t-\tau_i))}-\frac{v^* y_i(t-\omega_i)}{v y_i^*}\right.$$
$$\left.\left.-\frac{1+\alpha_i x_i+\gamma_i v}{1+\alpha_i x_i+\gamma_i v^*}+\log\frac{y_i(t-\omega_i)x_i(t-\tau_i)v(t-\tau_i)(1+\alpha_i x_i+\gamma_i v)}{y_i x_i v(1+\alpha_i x_i(t-\tau_i)+\gamma_i v(t-\tau_i))}\right)\right]\le 0.$$

It is clear that, if $R_0>1$, then $x_i^*, y_i^*, v^*>0$, then $\frac{dW_1}{dt}\le 0$ for all $x_i, y_i, v>0, i=1,2$, and $\frac{dW_1}{dt}=0$ if and only if $x_i=x_i^*, y_i=y_i^*, i=1,2$ and $v=v^*$ which is the infected steady state E_*, then E_* is GAS.

3 Model with distributed-time delays

In this section we study an HIV infection model with two classes of target cells and Beddington-DeAngelis functional response. Two types of distributed delays are incorporated into the model.

$$\dot{x}_i=\lambda_i-d_i x_i-\frac{\beta_i x_i v}{1+\alpha_i x_i+\gamma_i v},\qquad i=1,2 \tag{37}$$

$$\dot{y}_i=\int_0^{h_i}f_i(\tau)\frac{\beta_i' x_i(t-\tau)v(t-\tau)}{1+\alpha_i x_i(t-\tau)+\gamma_i v(t-\tau)}d\tau-a_i y_i,\qquad i=1,2, \tag{38}$$

$$\dot{v}=\sum_{i=1}^{2}p_i\int_0^{l_i}g_i(\omega)y_i\,(t-\omega)\,d\omega-rv, \tag{39}$$

where $\beta_i' < \beta_i$ and $p_i < k_i, i = 1, 2$. All the variables and other parameters of the model have the same meanings as given in (4)-(8). To account for the time lag between viral contacting a target cell and the production of new virus particles, two distributed intracellular delays are introduced. It is assumed that the target cells of class i are contacted by the virus particles at time $t - \tau$ become infected cells at time t, where τ is a random variable with a probability distribution $f_i(\tau)$ over the interval $[0, h_i]$ and h_i is limit superior of this delay. On the other hand, it is assumed that, a cell infected at time $t - \omega$ starts to yield new infectious virus at time t where ω is distributed according to a probability distribution $g_i(\omega)$ over the interval $[0, l_i]$ and l_i is limit superior of this delay. The probability distribution functions $f_i(\tau) : [0, h_i] \to \mathbb{R}_+$ and $g_i(\omega) : [0, l_i] \to \mathbb{R}_+$ are integral functions with

$$\int_0^{h_i} f_i(\tau)d\tau = \int_0^{l_i} g_i(\omega)d\omega = 1, \quad i = 1, 2.$$

We have

$$\int_0^{h_i}\int_0^{l_i} f_i(\tau)g_i(\omega)d\omega d\tau = 1.$$

We put

$$D_{f_i,h_i}[x_i v] = \int_0^{h_i} f_i(\tau)\frac{x_i(t-\tau)v(t-\tau)}{1+\alpha_i x_i(t-\tau)+\gamma_i v(t-\tau)}d\tau,$$

$$D_{g_i,l_i}[y_i] = \int_0^{l_i} g_i(\omega)y_i(t-\omega)d\omega.$$

Therefore, we have

$$D_{f_i,h_i}[x_i v] = \int_0^{l_i} g_i(\omega)D_{f_i,h_i}[x_i v]d\omega = \int_0^{h_i}\int_0^{l_i} f_i(\tau)g_i(\omega)\frac{x_i(t-\tau)v(t-\tau)}{1+\alpha_i x_i(t-\tau)+\gamma_i v(t-\tau)}d\omega d\tau,$$

$$D_{g_i,l_i}[y_i] = \int_0^{l_i} f_i(\tau)D_{g_i,l_i}[y_i]d\tau = \int_0^{h_i}\int_0^{l_i} f_i(\tau)g_i(\omega)y_i(t-\omega)d\omega d\tau.$$

Then system (37)-(39) can be rewritten as follows

$$\dot{x_i} = \lambda_i - d_i x_i - \frac{\beta_i x_i v}{1+\alpha_i x_i+\gamma_i v}, \qquad i = 1, 2 \tag{40}$$

$$\dot{y_i} = \beta_i' D_{f_i,h_i}[x_i v] - a_i y_i, \qquad i = 1, 2 \tag{41}$$

$$\dot{v} = \sum_{i=1}^{2} p_i D_{g_i,l_i}[y_i] - rv. \tag{42}$$

Initial conditions

The initial conditions for system (40)-(42) take the form

$$\begin{aligned} &x_1(\theta) = \varphi_1(\theta),\ x_2(\theta) = \varphi_2(\theta), y_1(\theta) = \varphi_3(\theta),\ y_2(\theta) = \varphi_4(\theta), v(\theta) = \varphi_5(\theta),\\ &\varphi_i(\theta) \geq 0,\ \theta \in [-\ell, 0),\ \varphi_j(0) > 0,\ j = 1, ..., 5. \end{aligned} \tag{43}$$

where, $(\varphi_1(\theta), \varphi_2(\theta), ..., \varphi_5(\theta)) \in C$ and $C = C([-\ell, 0], \mathbb{R}^5_+)$ is the Banach space of continuous functions mapping the interval $[-\ell, 0]$ into $\mathbb{R}^5_+$, and $\ell = \max\{h_1, h_2, l_1, l_2\}$. By the fundamental theory of functional differential equations [14], system (40)-(42) has a unique solution $(x_1(t), x_2(t), y_1(t), y_2(t), v(t))$ satisfying initial conditions (43).

3.1 Non-negativity and boundedness of solutions

In the following, we establish the non-negativity and boundedness of solutions of (40)-(42) with initial conditions (43).

Proposition 2. Let $X(t) = (x_1(t), x_2(t), y_1(t), y_2(t), v(t))^T$ be any solution of (40)-(42) satisfying the initial conditions (43), then $X(t)$ is non-negative for $t \geq 0$ and ultimately bounded.

Proof. Similar to the proof of Proposition 1, we have $x_i(t) > 0$, $i = 1, 2$ for all $t \geq 0$. Moreover, from Eqs. (41) and (42) we have

$$y_i(t) = y_i(0)e^{-a_i t} + \beta_i' \int_0^t e^{-a_i(t-\eta)} \int_0^{h_i} f_i(\tau) \frac{x_i(\eta-\tau)v(\eta-\tau)}{1+\alpha_i x_i(\eta-\tau)+\gamma_i v(\eta-\tau)} d\tau d\eta, \quad i = 1, 2$$

$$v(t) = v(0)e^{-rt} + \sum_{i=1}^{2} p_i \int_0^t e^{-r(t-\eta)} \int_0^{l_i} g_i(\omega) y_i(\eta-\tau) d\omega d\eta,$$

confirming that $y_i(t) \geq 0$, $i = 1, 2$, and $v(t) \geq 0$ for all $t \in [0, \ell]$. By a recursive argument, we obtain $y_i(t) \geq 0$, $i = 1, 2$, and $v(t) \geq 0$ for all $t \geq 0$.

Now we show the boundedness of the solutions of (40)-(42). Eqs. (40) imply that $\limsup_{t\to\infty} x_i(t) \leq x_i^0$, where $x_i^0 = \lambda_i/d_i$, and thus $x_i(t)$ is ultimately bounded. It follows that $\int_0^{h_i} f_i(\tau) x_i(t-\tau) d\tau \leq x_i^0$.

Let $T_i(t) = \frac{\beta_i'}{\beta_i} \int_0^{h_i} f_i(\tau) x_i(t-\tau) d\tau + y_i(t)$, $i = 1, 2$, then

$$\begin{aligned}\frac{dT_i(t)}{dt} &= \frac{\beta_i'}{\beta_i} \int_0^{h_i} f_i(\tau) \left(\lambda_i - d_i x_i(t-\tau) - \frac{\beta_i x_i(t-\tau)v(t-\tau)}{1+\alpha_i x_i(t-\tau)+\gamma_i v(t-\tau)} \right) d\tau \\ &+ \int_0^{h_i} f_i(\tau) \frac{\beta_i' x_i(t-\tau)v(t-\tau)}{1+\alpha_i x_i(t-\tau)+\gamma_i v(t-\tau)} d\tau - a_i y_i(t) \leq \frac{\beta_i'}{\beta_i}\lambda_i - \sigma_i T_i(t),\end{aligned}$$

where $\sigma_i = \min\{d_i, a_i\}$. Hence $\limsup_{t\to\infty} T_i(t) \leq L_i$, where $L_i = \beta_i'\lambda_i/\beta_i\sigma_i$. Since $\int_0^{h_i} f_i(\tau) x_i(t-\tau) d\tau > 0$, we get $\limsup_{t\to\infty} y_i(t) \leq L_i$. On the other hand,

$$\dot{v}(t) \leq \sum_{i=1}^{2} p_i L_i \int_0^{l_i} g_i(\omega) d\omega - rv = \sum_{i=1}^{2} p_i L_i - rv,$$

then $\limsup_{t\to\infty} v(t) \leq L^*$, where $L^* = \sum_{i=1}^{2} \frac{p_i L_i}{r}$. Therefore, $X(t)$ is ultimately bounded.$\Box$

3.2 Steady States

It is clear that, system (40)-(42) has an uninfected steady state $E_0(x_1^0, x_2^0, y_1^0, y_2^0, v^0)$, where $x_i^0 = \frac{\lambda_i}{d_i}$, $y_i^0 = 0$, $i = 1, 2$ and $v^0 = 0$. The system can also has another steady state which is the infected steady state $E_*(x_1^*, x_2^*, y_1^*, y_2^*, v^*)$, with coordinates if exist, they satisfy Eqs. (13)-(15). The basic reproduction number for system (40)-(42) is given by Eq. (16).

Lemma 2. If $R_0 > 1$, then there exists a positive steady state E_*.

The proof is the same as given in Lemma 1.

3.3 Global stability analysis

In this section, we study the global stability of the uninfected and infected steady states of system (40)-(42). The strategy of the proofs is to use suitable Lyapunov functionals which are similar in nature to those used in [19] and [32].

Define

$$\delta_{f_i h_i}(\tau) = \int_\tau^{h_i} f_i(\sigma) d\sigma, \quad \delta_{g_i, l_i}(\omega) = \int_\omega^{l_i} g_i(\sigma) d\sigma, \quad i = 1, 2, \tag{44}$$

implies that

$$\delta_{f_i h_i}(0) = 1, \delta_{f_i h_i}(h_i) = 0, \qquad \frac{d\delta_{f_i h_i}(\tau)}{d\tau} = -f_i(\tau), \tag{45}$$

$$\delta_{g_i, l_i}(0) = 1, \delta_{g_i, l_i}(l_i) = 0, \qquad \frac{d\delta_{g_i l_i}(\omega)}{d\tau} = -g_i(\omega). \tag{46}$$

Also, for a continuous function x , we have that

$$\frac{d}{dt}\int_0^h \delta_{f,h}(\tau)R(x(t-\tau))d\tau = \int_0^h f(\tau)\left[R(x(t)) - R\left(x\left(t-\tau\right)\right)\right]d\tau.$$

First we prove the global stability of the infected steady state E_0 employing the method of Lyapunov functional.

Theorem 3. Consider the system (40)-(42), if $R_0 \le 1$ then E_0 is GAS.

Proof. Define a Lyapunov functional W_0 as follows:

$$W_0 = \sum_{i=1}^{2}\frac{p_i}{a_i}\left[\frac{\beta_i' x_i^0}{\beta_i(1+\alpha_i x_i^0)}\left(\frac{x_i}{x_i^0} - 1 - \log\frac{x_i}{x_i^0}\right) + y_i + \beta_i'\int_0^{h_i}\delta_{f_i,h_i}(\tau)\frac{x_i(t-\tau)v(t-\tau)}{1+\alpha_i x_i(t-\tau)+\gamma_i v(t-\tau)}d\tau\right.$$
$$\left. + a_i\int_0^{l_i}\delta_{g_i,l}(\omega)y_i(t-\omega)d\omega\right] + v, \qquad i=1,2. \tag{47}$$

Now, We note that W_0 is defined and continuous for all $(x_1, x_2, y_1, y_2, v) > 0$. Also, the global minimum $W_0 = 0$ occurs at the uninfected steady state E_0. The time derivative of W_0 along the solution of (40)-(42) is given by

$$\begin{aligned}
\frac{dW_0}{dt} &= \sum_{i=1}^{2}\frac{p_i}{a_i}\left[\frac{\beta_i'}{\beta_i(1+\alpha_i x_i^0)}\left(1-\frac{x_i^0}{x_i}\right)\left(\lambda_i - d_i x_i - \frac{\beta_i x_i v}{1+\alpha_i x_i+\gamma_i v}\right)\right.\\
&+\beta_i' D_{f_i h_i}[x_i v] - a_i y_i + \int_0^{h_i} f_i(\tau)\left(\frac{\beta_i' x_i v}{1+\alpha_i x_i+\gamma_i v} - \frac{\beta_i' x_i(t-\tau)v(t-\tau)}{1+\alpha_i x_i(t-\tau)+\gamma_i v(t-\tau)}\right)d\tau\\
&\left.+a_i\int_0^{l_i} g_i(\omega)(y_i - y_i(t-\omega))d\omega\right] + \sum_{i=1}^{2} p_i D_{g_i l_i}[y_i] - rv\\
&= \sum_{i=1}^{2}\frac{p_i}{a_i}\left[\frac{-\beta_i'\lambda_i}{\beta_i x_i x_i^0(1+a_i x_i^0)}\left(x_i - x_i^0\right)^2 - \frac{\beta_i' x_i v}{\left(1+\alpha_i x_i^0\right)\left(1+\alpha_i x_i+\gamma_i v\right)}\right.\\
&\left.+\frac{\beta_i' x_i^0 v}{\left(1+\alpha_i x_i^0\right)\left(1+\alpha_i x_i+\gamma_i v\right)} + \frac{\beta_i' x_i v}{1+\alpha_i x_i+\gamma_i v}\right] - rv\\
&= \sum_{i=1}^{2}\frac{p_i}{a_i}\left[\frac{-\beta_i'\lambda_i}{\beta_i x_i x_i^0(1+a_i x_i^0)}\left(x_i - x_i^0\right)^2 + \frac{\beta_i' x_i^0 v + \alpha_i x_i^0\beta_i' x_i v}{\left(1+\alpha_i x_i^0\right)\left(1+\alpha_i x_i+\gamma_i v\right)}\right] - rv\\
&= \sum_{i=1}^{2}\frac{p_i}{a_i}\left[\frac{-\beta_i'\lambda_i}{\beta_i x_i x_i^0(1+a_i x_i^0)}\left(x_i - x_i^0\right)^2 + \frac{\beta_i' x_i^0 v(1+\alpha_i x_i+\gamma_i v-\gamma_i v)}{\left(1+\alpha_i x_i^0\right)\left(1+\alpha_i x_i+\gamma_i v\right)}\right] - rv\\
&= \sum_{i=1}^{2}\frac{p_i}{a_i}\left[\frac{-\beta_i'\lambda_i}{\beta_i x_i x_i^0(1+a_i x_i^0)}\left(x_i - x_i^0\right)^2 + \frac{\beta_i' x_i^0 v(1+\alpha_i x_i+\gamma_i v)}{\left(1+\alpha_i x_i^0\right)\left(1+\alpha_i x_i+\gamma_i v\right)} - \frac{\gamma_i\beta_i' x_i^0 v^2}{\left(1+\alpha_i x_i^0\right)\left(1+\alpha_i x_i+\gamma_i v\right)}\right] - rv\\
&= -\sum_{i=1}^{2}\frac{p_i}{a_i}\frac{\beta_i'\lambda_i}{\beta_i x_i x_i^0(1+a_i x_i^0)}\left(x_i - x_i^0\right)^2 - \sum_{i=1}^{2}\frac{p_i}{a_i}\frac{\gamma_i\beta_i' x_i^0 v^2}{\left(1+\alpha_i x_i^0\right)\left(1+\alpha_i x_i+\gamma_i v\right)} + \sum_{i=1}^{2}\frac{p_i}{a_i}\frac{\beta_i' x_i^0 v}{\left(1+\alpha_i x_i^0\right)} - rv\\
&= -\sum_{i=1}^{2}\frac{\beta_i'\lambda_i p_i}{a_i\beta_i x_i x_i^0(1+a_i x_i^0)}\left(x_i - x_i^0\right)^2 - \sum_{i=1}^{2}\frac{p_i\gamma_i\beta_i' x_i^0 v^2}{a_i\left(1+\alpha_i x_i^0\right)\left(1+\alpha_i x_i+\gamma_i v\right)} + (R_0 - 1)rv.
\end{aligned} \tag{48}$$

It can be seen that, if $R_0 \le 1$ then $\frac{dW_0}{dt} \le 0$ for all $x_i, v > 0$, $i = 1, 2$. By Theorem 5.3.1 in [14], the solutions of system (40)-(42) limit to M, the largest invariant subset of $\left\{\frac{dW_0}{dt} = 0\right\}$. Clearly, it follows from (48) that $\frac{dW_0}{dt} = 0$ if and only if $x_i = x_i^0$, $i = 1, 2$ and $v = 0$. Noting that M is invariant, for each element of M we have $v = 0$, then $\dot{v} = 0$. From Eq. (42) we drive that

$$0 = \dot{v} = \sum_{i=1}^{2} p_i\int_0^{l_i} g_i(\omega)y_i(t-\omega)d\omega.$$

Since $y_i(t-\theta) \geq 0$ for all $\theta \in [0,\ell]$, then $\sum_{i=1}^{2} p_i \int_{\omega}^{l_i} g_i(\omega) y_i(t-\omega) = 0$ if and only if $y_i(t-\omega) = 0$, $i = 1,2$. Hence $\frac{dW_0}{dt} = 0$ if and only if $x_i = x_i^0$, $y_i = 0$, $i = 1,2$ and $v = 0$. From LaSalle's invariance principle, E_0 is GAS.

Theorem 4. If $R_0 > 1$, then E_* is GAS.

Proof. The strategy of proving the global stability of the infection steady state E_* is similar to the proof of Theorem 2. Then, we first consider the system (30)-(32), and define Lyapunov functional W_* as follows:

$$W_* = \sum_{i=1}^{2} \frac{p_i}{a_i} \left[\frac{\beta_i'}{\beta_i} \left(x_i - x_i^* - \int_{x_i^*}^{x_i} \frac{x_i^*(1+\alpha_i\mu+\gamma_i v^*)}{\mu(1+\alpha_i x_i^*+\gamma_i v^*)} d\mu \right) + y_i^* \left(\frac{y_i}{y_i^*} - 1 - \log\frac{y_i}{y_i^*} \right) \right] + v^* \left(\frac{v}{v^*} - 1 - \log\frac{v}{v^*} \right). \tag{49}$$

By calculating the time derivative along (30)-(32) we get

$$\begin{aligned} \nabla W_* \cdot F(X) = \sum_{i=1}^{2} \frac{p_i}{a_i} \Bigg[\frac{\beta_i'}{\beta_i} \left(1 - \frac{x_i^*(1+\alpha_i x_i+\gamma_i v^*)}{x_i(1+\alpha_i x_i^*+\gamma_i v^*)} \right) \left(\lambda_i - d_i x_i - \frac{\beta_i x_i v}{1+\alpha_i x_i+\gamma_i v} \right) \\ + \left(1 - \frac{y_i^*}{y_i} \right) \left(\frac{\beta_i' x_i v}{1+\alpha_i x_i+\gamma_i v} - a_i y_i \right) \Bigg] + \left(1 - \frac{v^*}{v} \right) \left(\sum_{i=1}^{2} p_i y_i - rv \right). \end{aligned}$$

Similar calculation as given in Theorem 2, we have

$$\begin{aligned} \nabla W_* \cdot F(X) = \sum_{i=1}^{2} \frac{p_i}{a_i} \Bigg[\frac{-d_i\beta_i'(x_i-x_i^*)^2(1+\gamma_i v^*)}{\beta_i x_i(1+\alpha_i x_i^*+\gamma_i v^*)} - a_i y_i^* \frac{\gamma_i(1+\alpha_i x_i)(v-v^*)^2}{v^*(1+\alpha_i x_i+\gamma_i v)(1+\alpha_i x_i+\gamma_i v^*)} \\ + a_i y_i^* \left(4 - \frac{x_i^*(1+\alpha_i x_i+\gamma_i v^*)}{x_i(1+\alpha_i x_i^*+\gamma_i v^*)} - \frac{y_i^*}{y_i} \frac{x_i v(1+\alpha_i x_i^*+\gamma_i v^*)}{x_i^* v^*(1+\alpha_i x_i+\gamma_i v)} - \frac{y_i v^*}{y_i^* v} - \frac{1+\alpha_i x_i+\gamma_i v}{1+\alpha_i x_i+\gamma_i v^*} \right) \Bigg] \leq 0. \end{aligned} \tag{50}$$

Now, we compute the time derivative of W_* along the solution of the system (40)-(42)

$$\begin{aligned} \frac{dW_*}{dt} &= \sum_{i=1}^{2} \frac{p_i}{a_i} \Bigg[\frac{\beta_i'}{\beta_i} \left(1 - \frac{x_i^*(1+\alpha_i x_i+\gamma_i v^*)}{x_i(1+\alpha_i x_i^*+\gamma_i v^*)} \right) \left(\lambda_i - d_i x_i - \frac{\beta_i x_i v}{1+\alpha_i x_i+\gamma_i v} \right) \\ &+ \left(1 - \frac{y_i^*}{y_i} \right) \left(\beta_i' D_{f_i,h_i}[x_i v] - a_i y_i + \frac{\beta_i' x_i v}{1+\alpha_i x_i+\gamma_i v} - \frac{\beta_i' x_i v}{1+\alpha_i x_i+\gamma_i v} \right) \Bigg] \\ &+ \left(1 - \frac{v^*}{v} \right) \left(\sum_{i=1}^{2} p_i D_{g_i,l_i}[y_i] - rv + \sum_{i=1}^{2} p_i y_i - \sum_{i=1}^{2} p_i y_i \right) \\ &= \Bigg\{ \sum_{i=1}^{2} \frac{p_i}{a_i} \Bigg[\frac{\beta_i'}{\beta_i} \left(1 - \frac{x_i^*(1+\alpha_i x_i+\gamma_i v^*)}{x_i(1+\alpha_i x_i^*+\gamma_i v^*)} \right) \left(\lambda_i - d_i x_i - \frac{\beta_i x_i v}{1+\alpha_i x_i+\gamma_i v} \right) \\ &+ \left(1 - \frac{y_i^*}{y_i} \right) \left(\frac{\beta_i' x_i v}{1+\alpha_i x_i+\gamma_i v} - a_i y_i \right) \Bigg] + \left(1 - \frac{v^*}{v} \right) \left(\sum_{i=1}^{2} p_i y_i - rv \right) \Bigg\} \\ &+ \sum_{i=1}^{2} \frac{p_i}{a_i} \left[\left(1 - \frac{y_i^*}{y_i} \right) \left(\beta_i' D_{f_i h_i}[x_i v] - \frac{\beta_i' x_i v}{1+\alpha_i x_i+\gamma_i v} \right) \right] + \left(1 - \frac{v^*}{v} \right) \left(\sum_{i=1}^{2} p_i D_{g_i,l_i}[y_i] - \sum_{i=1}^{2} p_i y_i \right) \end{aligned}$$

$$
\begin{aligned}
&= \nabla W_* \cdot F(X) + \sum_{i=1}^{2} \frac{p_i}{a_i} a_i y_i^* \left[\left(1 - \frac{y_i^*}{y_i}\right) \left(\frac{D_{f_i h_i}[x_i v](1+\alpha_i x_i^* + \gamma_i v^*)}{x_i^* v^*} - \frac{x_i v (1+\alpha_i x_i^* + \gamma_i v^*)}{x_i^* v^* (1+\alpha_i x_i + \gamma_i v)} \right) \right. \\
&\left. + \left(1 - \frac{v^*}{v}\right) \left(\frac{D_{g_i,l_i}[y_i]}{y_i^*} - \frac{y_i}{y_i^*} \right) \right] \\
&= \sum_{i=1}^{2} \frac{p_i}{a_i} \left[\frac{-d_i \beta_i'(x_i - x_i^*)^2 (1+\gamma_i v^*)}{\beta_i x_i (1+\alpha_i x_i^* + \gamma_i v^*)} - a_i y_i^* \left(\frac{\gamma_i (1+\alpha_i x_i)(v - v^*)^2}{v^*(1+\alpha_i x_i + \gamma_i v)(1+\alpha_i x_i + \gamma_i v^*)} \right) \right. \\
&+ a_i y_i^* \left(4 - \frac{x_i^*(1+\alpha_i x_i + \gamma_i v^*)}{x_i (1+\alpha_i x_i^* + \gamma_i v^*)} - \frac{y_i^*}{y_i} \frac{x_i v (1+\alpha_i x_i^* + \gamma_i v^*)}{x_i^* v^* (1+\alpha_i x_i + \gamma_i v)} - \frac{y_i v^*}{y_i^* v} - \frac{1+\alpha_i x_i + \gamma_i v}{1+\alpha_i x_i + \gamma_i v^*} \right) \\
&+ a_i y_i^* \left(\frac{D_{f_i h_i}[x_i v](1+\alpha_i x_i^* + \gamma_i v^*)}{x_i^* v^*} - \frac{x_i v (1+\alpha_i x_i^* + \gamma_i v^*)}{x_i^* v^* (1+\alpha_i x_i + \gamma_i v)} \right. \\
&- \frac{y_i^* D_{f_i h_i}[x_i v](1+\alpha_i x_i^* + \gamma_i v^*)}{y_i x_i^* v^*} + \frac{y_i^* x_i v (1+\alpha_i x_i^* + \gamma_i v^*)}{y_i x_i^* v^* (1+\alpha_i x_i + \gamma_i v)} \\
&\left. \left. + \frac{D_{g_i,l_i}[y_i]}{y_i^*} - \frac{y_i}{y_i^*} - \frac{v^* D_{g_i,l_i}[y_i]}{v y_i^*} + \frac{v^* y_i}{v y_i^*} \right) \right] \\
&= \sum_{i=1}^{2} \frac{p_i}{a_i} \left[\frac{-d_i \beta_i'(x_i - x_i^*)^2 (1+\gamma_i v^*)}{\beta_i x_i (1+\alpha_i x_i^* + \gamma_i v^*)} - a_i y_i^* \left(\frac{\gamma_i (1+\alpha_i x_i)(v - v^*)^2}{v^*(1+\alpha_i x_i + \gamma_i v)(1+\alpha_i x_i + \gamma_i v^*)} \right) \right. \\
&+ a_i y_i^* \int_0^{h_i} \int_0^{l_i} f_i(\tau) g_i(\omega) \left(4 - \frac{x_i^*(1+\alpha_i x_i + \gamma_i v^*)}{x_i (1+\alpha_i x_i^* + \gamma_i v^*)} - \frac{y_i^* x_i(t-\tau) v(t-\tau)(1+\alpha_i x_i^* + \gamma_i v^*)}{y_i x_i^* v^* (1+\alpha_i x_i(t-\tau) + \gamma_i v(t-\tau))} - \frac{v^* y_i(t-\omega)}{v y_i^*} \right. \\
&\left. \left. - \frac{1+\alpha_i x_i + \gamma_i v}{1+\alpha_i x_i + \gamma_i v^*} + \log \frac{y_i(t-\omega) x_i(t-\tau) v(t-\tau)(1+\alpha_i x_i + \gamma_i v)}{y_i x_i v (1+\alpha_i x_i(t-\tau) + \gamma_i v(t-\tau))} \right) d\omega d\tau \right] \\
&+ \sum_{i=1}^{2} \frac{p_i}{a_i} a_i y_i^* \left[\int_0^{h_i} f_i(\tau) \left(\frac{x_i(t-\tau) v(t-\tau)(1+\alpha_i x_i^* + \gamma_i v^*)}{x_i^* v^* (1+\alpha_i x_i(t-\tau) + \gamma_i v(t-\tau))} - \frac{x_i v (1+\alpha_i x_i^* + \gamma_i v^*)}{x_i^* v^* (1+\alpha_i x_i + \gamma_i v)} \right. \right. \\
&\left. \left. - \log \frac{x_i(t-\tau) v(t-\tau)(1+\alpha_i x_i + \gamma_i v)}{x_i v (1+\alpha_i x_i(t-\tau) + \gamma_i v(t-\tau))} \right) d\tau \right] \\
&+ \sum_{i=1}^{2} p_i y_i^* \int_0^{l_i} g_i(\omega) \left(\frac{y_i(t-\omega)}{y_i^*} - \frac{y_i}{y_i^*} - \log \frac{y_i(t-\omega)}{y_i} \right) d\omega. \qquad (51)
\end{aligned}
$$

We define a functional

$$
\overline{W_i} = \int_0^{h_i} \delta_{f_i,h_i}(\tau) H \left(\frac{x_i(t-\tau) v(t-\tau)((1+\alpha_i x_i^* + \gamma_i v^*)}{x_i^* v^* (1+\alpha_i x_i(t-\tau) + \gamma_i v(t-\tau))} \right) d\tau, \qquad i = 1, 2 \qquad (52)
$$

$$
\widetilde{W_i} = \int_0^{l_i} \delta_{g_i,l_i}(\omega) H \left(\frac{y_i(t-\omega)}{y_i^*} \right) d\omega, \qquad i = 1, 2 \qquad (53)
$$

then we have,

$$
\begin{aligned}
\frac{d\overline{W}_i}{dt} &= \int_0^{h_i} f_i(\tau) \left[\frac{x_i v (1+\alpha_i x_i^* + \gamma_i v^*)}{x_i^* v^* (1+\alpha_i x_i + \gamma_i v)} - \frac{x_i(t-\tau) v(t-\tau)(1+\alpha_i x_i^* + \gamma_i v^*)}{x_i^* v^* (1+\alpha_i x_i(t-\tau) + \gamma_i v(t-\tau))} + \right. \\
&\left. + \log \frac{x_i(t-\tau) v(t-\tau)(1+\alpha_i x_i + \gamma_i v)}{x_i v (1+\alpha_i x_i(t-\tau) + \gamma_i v(t-\tau))} \right] d\tau. \qquad (54)
\end{aligned}
$$

$$
\frac{d\widetilde{W_i}}{dt} = \int_0^{l_i} g_i(\omega) \left[\frac{y_i}{y_i^*} - \frac{y_i(t-\omega)}{y_i^*} + \log \frac{y_i(t-\omega)}{y_i} \right] d\omega. \qquad (55)
$$

Therefore, we construct Lyapunov functional as follows:

$$
W_1 = W_* + \sum_{i=1}^{2} p_i y_i^* \overline{W}_i + \sum_{i=1}^{2} p_i y_i^* \widetilde{W}_i.
$$

Then, from (51), (54), and (55) and using the arithmetic-geometric inequality (27) we obtain

$$\frac{dW_1}{dt}=\sum_{i=1}^{2}\frac{p_i}{a_i}\left[\frac{-d_i\beta_i'(x_i-x_i^*)^2(1+\gamma_i v^*)}{\beta_i x_i(1+\alpha_i x_i^*+\gamma_i v^*)}-a_i y_i^*\left(\frac{\gamma_i(1+\alpha_i x_i)(v-v^*)^2}{v^*(1+\alpha_i x_i+\gamma_i v)(1+\alpha_i x_i+\gamma_i v^*)}\right)\right.$$
$$+a_i y_i^*\int_0^{h_i}\int_0^{l_i} f_i(\tau)g_i(\omega)\left(4-\frac{x_i^*(1+\alpha_i x_i+\gamma_i v^*)}{x_i(1+\alpha_i x_i^*+\gamma_i v^*)}-\frac{y_i^* x_i(t-\tau)v(t-\tau)(1+\alpha_i x_i^*+\gamma_i v^*)}{y_i x_i^* v^*(1+\alpha_i x_i(t-\tau)+\gamma_i v(t-\tau))}-\frac{v^* y_i(t-\omega)}{v y_i^*}\right.$$
$$\left.\left.-\frac{1+\alpha_i x_i+\gamma_i v}{1+\alpha_i x_i+\gamma_i v^*}+\log\frac{y_i(t-\omega)x_i(t-\tau)v(t-\tau)(1+\alpha_i x_i+\gamma_i v)}{y_i x_i v(1+\alpha_i x_i(t-\tau)+\gamma_i v(t-\tau))}\right)d\omega d\tau\right]\leq 0.$$

Hence, $\frac{dW_1}{dt}\leq 0$ where the equality occurs at the infected steady state E_*. Then, E_* is GAS.

4 Conclusion

In this paper, we have proposed HIV infection models describing the interaction of the HIV with two classes of target cells, CD4$^+$ T cells and macrophages taking into account the Beddington-DeAngelis functional response. Two types of discrte or distributed time delays describing time needed for infection of target cell and virus replication have been incorporated into the models. The global stability of the uninfected and infected steady states of the model has been established by using suitable Lyapunov functionals and LaSalle invariant principle. We have proven that, if the basic reproduction number R_0 is less than or equal unity, then the uninfected steady state is GAS and if $R_0 > 1$, then the infected steady state exists and it is GAS.

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Legendre spectral collocation method for solving fractional SIRC model and influenza A

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Abstract

In this paper, Legendre spectral method is presented to study the approximate solution of fractional SIRC model. The fractional derivative is described in the Caputo sense. The properties of the Legendre polynomials are used to reduce the proposed method to the solution of nonlinear system of algebraic equations using Newton iteration method. Moreover, the convergence analysis and an upper bound of the error for the derived formula are given. We compared our numerical solutions with those numerical solutions using fourth-order Rung-Kutta method. The obtained results of the SIRC model show the simplicity and the efficiency of the proposed method.

Keywords: Fractional SIRC model; Caputo fractional derivative; Legendre spectral method; Fourth-order Rung-Kutta method; Convergence analysis.

1. Introduction

It is well known that the fractional differential equations (FDEs) have been the focus of many studies due to their frequent appearance in various applications such as in fluid mechanics, viscoelasticity, biology, physics and engineering applications, for more details see for example ([21], [23]). Consequently, considerable attention has been given to the efficient numerical solutions of FDEs of physical interest, because it is difficult to find exact solutions. Different numerical methods have been proposed in the literature for solving FDEs ([4], [25]-[29]). Recently, several numerical and approximate methods to solve the FDEs have been given such as variational iteration method [13], homotopy perturbation method [14] and collocation method ([12], [15]-[18], [20]).

Mathematical models have become important tools in analyzing the spread and control of infectious diseases. Under-standing the transmission characteristics of infectious diseases in communities, regions, and countries can lead to better approaches to decrease the transmission of these diseases [10].

Influenza is transmitted by a virus that can be of three different types, namely A, B, and C [22]. Among these, the virus A is epidemiologically the most important one for human beings, because it can recombine its genes with those of strains circulating in animal populations such as birds, swine, horses, and so forth ([2], [30]). Over the last two decades, a number of epidemic models for predicting the spread of influenza through human population have been proposed based on either the classical susceptible-infected-removed (SIR) model developed by Kermack and McKendrick [11].

In this paper, we use the collocation spectral method to study the behavior of the approximate solution of the following fractional model of SIRC

$$\begin{aligned} D^{\alpha}S(t) &= \mu(1-S) - \beta SI + \gamma C,\\ D^{\alpha}I(t) &= \beta SI + \sigma\beta CI - (\mu+\theta)I,\\ D^{\alpha}R(t) &= (1-\sigma)\beta CI + \theta I - (\mu+\delta)R,\\ D^{\alpha}C(t) &= \delta R - \beta CI - (\mu+\gamma)C, \end{aligned} \tag{1}$$

with the following initial conditions

$$S(0)=s_0, \quad I(0)=i_0, \quad R(0)=r_0, \quad C(0)=c_0. \tag{2}$$

Where D^{α} is the Caputo fractional derivative, with respect to time t. In which $S(t)$, $I(t)$, $R(t)$ and $C(t)$ represent the proportions of susceptible, infectious, recovered and cross-immune. The model assumes a population of constant size, N, so that $N = S + I + R + C$, where provides an interpretation of the model parameters, μ is the mortality rate, θ is the rate of progression from infective to recovered per year, δ is the rate of progression from recovered to cross-immune per year, γ is the rate of progression from recovered to susceptible per year, σ is the recruitment rate of cross-immune into the infective $(0 \leq \sigma \leq 1)$ and β is the contact rate per year. Because model (1) monitors the dynamics of human populations, all the parameters are assumed to be nonnegative. Furthermore, it can be shown that all state variables of the model are nonnegative for all time $t \geq 0$ ([7]-[9], [24]). Further details on the biological motivation and the associated assumptions are given in ([5], [6]).

Note that, when $\alpha = 1$, we get the standard form of this system as follows

$$\begin{aligned} \frac{dS}{dt} &= \mu(1-S) - \beta SI + \gamma C,\\ \frac{dI}{dt} &= \beta SI + \sigma\beta CI - (\mu+\theta)I,\\ \frac{dR}{dt} &= (1-\sigma)\beta CI + \theta I - (\mu+\delta)R,\\ \frac{dC}{dt} &= \delta R - \beta CI - (\mu+\gamma)C, \end{aligned} \tag{3}$$

where the parameter β is the contact rate for the influenza disease also called the rate of transmission for susceptible to infected, γ^{-1} is the cross-immune period, θ^{-1} is the infectious period, δ^{-1} is the total immune period and σ is the fraction of the exposed cross-immune individuals who are recruited in a unit time into the infective subpopulation ([5], [9]).

Khader et al. [19] introduced a new approximate formula of the fractional derivative using Legendre series expansion and used it to solve numerically the fractional diffusion equation. Also, Khader and Hendy in [15] used this formula to solve numerically the fractional delay differential equations.

In this article, we extended this work to study the numerical solution of the fractional SIRC model. An approximate formula of the fractional derivative is presented. Also, special attention is given to study the convergence analysis and estimate an upper bound of the error for the introduced formula.

We present some necessary definitions and mathematical preliminaries of the fractional calculus theory that will be required in the present paper.

Definition 1.

The Caputo fractional derivative operator D^α of order α is defined in the following form

$$D^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^{(m)}(t)}{(x-t)^{\alpha-m+1}} dt, \quad \alpha > 0, \quad x > 0,$$

where $m-1 < \alpha \le m,\ m \in \mathbb{N}$, $\Gamma(.)$ is the Gamma function.
Similar to integer-order differentiation, Caputo fractional derivative operator is a linear operation

$$D^\alpha\left(\lambda\, f(x) + \mu\, g(x)\right) = \lambda\, D^\alpha\, f(x) + \mu\, D^\alpha\, g(x),$$

where λ and μ are constants. For the Caputo's derivative we have [23]

$$D^\alpha\, C = 0, \qquad \text{C is a constant}, \tag{4}$$

$$D^\alpha\, x^n = \begin{cases} 0, & \text{for} \quad n \in \mathbb{N}_0 \text{ and } n < \lceil\alpha\rceil; \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} x^{n-\alpha}, & \text{for} \quad n \in \mathbb{N}_0 \text{ and } n \ge \lceil\alpha\rceil. \end{cases} \tag{5}$$

We use the ceiling function $\lceil\alpha\rceil$ to denote the smallest integer greater than or equal to α, and $\mathbb{N}_0 = \{0, 1, 2, ...\}$. Recall that for $\alpha \in \mathbb{N}$, the Caputo differential operator coincides with the usual differential operator of integer order.
For more details on fractional derivatives definitions and its properties see ([21], [23]).

Our paper is organized as follows: In section 2, we derive an approximate formula for the fractional derivatives using Legendre series expansion. In section 3, we study the error analysis

of the introduced approximate formula. In section 4, we present the procedure solution using Legendre collocation method. In section 5, we present the numerical implementation of the proposed method. Finally, in section 6 the paper ends with a brief conclusion and some remarks.

2. An approximate formula of the Caputo fractional derivative

The well known Legendre polynomials are defined on the interval $[-1, 1]$ and can be determined with the aid of the following recurrence formula [3]

$$L_{k+1}(z) = \frac{2k+1}{k+1}\, z\, L_k(z) - \frac{k}{k+1} L_{k-1}(z), \qquad k = 1, 2, ...,$$

where $L_0(z) = 1$ and $L_1(z) = z$. In order to use these polynomials on the interval $[0, 1]$ we define the so called shifted Legendre polynomials by introducing the change of variable $z = 2t - 1$. Let the shifted Legendre polynomials $L_k(2t-1)$ be denoted by $L^*_k(t)$. Then $L^*_k(t)$ can be obtained as follows

$$L^*_{k+1}(t) = \frac{(2k+1)(2t-1)}{(k+1)}\, L^*_k(t) - \frac{k}{k+1}\, L^*_{k-1}(t), \qquad k = 1, 2, ...,$$

where $L^*_0(t) = 1$ and $L^*_1(t) = 2t-1$. The analytic form of the shifted Legendre polynomials $L^*_k(t)$ of degree k is given by

$$L^*_k(t) = \sum_{i=0}^{k} (-1)^{k+i} \frac{(k+i)!}{(k-i)!(i!)^2}\, t^i. \tag{6}$$

Note that $L^*_k(0) = (-1)^k$ and $L^*_k(1) = 1$. The orthogonality condition is

$$\int_0^1 L^*_i(t) L^*_j(t)\, dt = \begin{cases} \frac{1}{2i+1}, & \text{for} \quad i = j; \\ 0, & \text{for} \quad i \neq j. \end{cases}$$

The function $u(t)$, which is square integrable in $[0, 1]$, may be expressed in terms of shifted Legendre polynomials as

$$u(t) = \sum_{i=0}^{\infty} c_i L^*_i(t), \tag{7}$$

where the coefficients c_i are given by

$$c_i = (2i+1) \int_0^1 u(t) L^*_i(t)\, dt, \qquad i = 0, 1, \ldots . \tag{8}$$

In practice, only the first $(m+1)$-terms of shifted Legendre polynomials are considered. Then we have

$$u_m(t) = \sum_{i=0}^{m} c_i L^*_i(t). \tag{9}$$

The main approximate formula of the fractional derivative is given in the following theorem.

Theorem 1.

Let $u(t)$ be approximated by shifted Legendre polynomials as (9) and also suppose $\alpha > 0$, then

$$D^{\alpha}(u_m(t)) = \sum_{i=\lceil\alpha\rceil}^{m} \sum_{k=\lceil\alpha\rceil}^{i} c_i\, w_{i,k}^{(\alpha)} t^{k-\alpha}, \tag{10}$$

where $w_{i,k}^{(\alpha)}$ is given by

$$w_{i,k}^{(\alpha)} = \frac{(-1)^{(i+k)}(i+k)!}{(i-k)!(k)!\Gamma(k-\alpha+1)}. \tag{11}$$

Proof. Since the Caputo's fractional differentiation is a linear operation we have

$$D^{\alpha}(u_m(t)) = \sum_{i=0}^{m} c_i D^{\alpha}(L_i^*(t)). \tag{12}$$

Employing Eqs.(4)-(5) in Eq.(6) we have

$$D^{\alpha} L_i^*(t) = 0, \qquad i = 0, 1, ..., \lceil\alpha\rceil - 1, \quad \alpha > 0. \tag{13}$$

Therefore, for $i = \lceil\alpha\rceil, \lceil\alpha\rceil + 1, ..., m$ and by using Eqs.(4)-(5) in formula (6) we get

$$D^{\alpha} L_i^*(t) = \sum_{k=0}^{i} \frac{(-1)^{i+k}(i+k)!}{(i-k)!(k!)^2} D^{\alpha}(t^k) = \sum_{k=\lceil\alpha\rceil}^{i} \frac{(-1)^{i+k}(i+k)!}{(i-k)!(k!)\Gamma(k-\alpha+1)} t^{k-\alpha}. \tag{14}$$

A combination of Eqs.(12), (13) and (14) leads to the desired result (10). □

Test example:

Consider the function $u(t) = t^2$ and $m = 2$, $\alpha = 0.5$, the shifted Legendre series of t^2 is

$$t^2 = \frac{1}{3} L_0^*(t) + \frac{1}{2} L_1^*(t) + \frac{1}{6} L_2^*(t).$$

Hence,

$$D^{\frac{1}{2}} t^2 = \sum_{i=1}^{2} \sum_{k=1}^{i} c_i\, w_{i,k}^{(\frac{1}{2})} t^{k-\frac{1}{2}}, \text{ where, } \quad w_{1,1}^{(\frac{1}{2})} = \frac{2}{\Gamma(\frac{3}{2})}, \;\; w_{2,1}^{(\frac{1}{2})} = \frac{-6}{\Gamma(\frac{3}{2})}, \;\; w_{2,2}^{(\frac{1}{2})} = \frac{12}{\Gamma(\frac{5}{2})},$$

therefore

$$D^{\frac{1}{2}} t^2 = t^{-\frac{1}{2}} [c_1\, w_{1,1}^{(\frac{1}{2})} t + c_2\, w_{2,1}^{(\frac{1}{2})} t + c_2\, w_{2,2}^{(\frac{1}{2})} t^2] = \frac{2}{\Gamma(\frac{5}{2})} t^{\frac{3}{2}},$$

which agree with the exact derivative (5).

3. Error analysis

In this section, special attention is given to study the convergence analysis and evaluate an upper bound of the error for the proposed approximate formula.

Theorem 2. *(Legendre truncation theorem) [3]*

The error in approximating $u(t)$ by the sum of its first m terms is bounded by the sum of the absolute values of all the neglected coefficients. If

$$u_m(t) = \sum_{k=0}^{m} c_k \, L_k(t), \tag{15}$$

then

$$E_T(m) \equiv |u(t) - u_m(t)| \leq \sum_{k=m+1}^{\infty} |c_k|, \tag{16}$$

for all $u(t)$, all m, and all $t \in [-1, 1]$.

Lemma 1.

For any continuous function $u(t)$ defined on $[0, 1]$ with bounded second derivative (i.e., $|u''(t)| \leq \delta$ for some constant δ), then the coefficients of the shifted Legendre expansion (9) is bounded in the following form

$$|c_i| \leq \frac{\sqrt{6}\delta}{\sqrt{2i-3}(2i-1)}. \tag{17}$$

Proof. Suppose that $dv = (2i+1)L_i^*(t)dt$ and from the recurrence relation between the derivatives of shifted Legendre polynomials $L'^*_{n+1}(t) - L'^*_{n-1}(t) = (2n+1)L_n^*(t), \quad n \geq 1$,

we get $\quad dv = (L'^*_{i+1}(t) - L'^*_{i-1}(t))dt = d(L^*_{i+1}(t) - L^*_{i-1}(t))$.

Now, we use the integration by parts two times, in the integral (8) to obtain

$$\begin{aligned}
c_i &= u(t)[L^*_{i+1}(t) - L^*_{i-1}(t)]_0^1 - \int_0^1 u'(t)(L^*_{i+1}(t) - L^*_{i-1}(t))dt \\
&= -\int_0^1 u'(t)(L^*_{i+1}(t) - L^*_{i-1}(t))dt \\
&= -u'(t)\left[\frac{L^*_{i+2}(t) - L^*_i(t)}{2i+3} - \frac{L^*_i(t) - L^*_{i-2}(t)}{2i-1}\right]_0^1 \\
&+ \int_0^1 u''(t)\left[\frac{L^*_{i+2}(t) - L^*_i(t)}{2i+3} - \frac{L^*_i(t) - L^*_{i-2}(t)}{2i-1}\right]dt \\
&= \int_0^1 u''(t)\left[\frac{L^*_{i+2}(t) - L^*_i(t)}{2i+3} - \frac{L^*_i(t) - L^*_{i-2}(t)}{2i-1}\right]dt.
\end{aligned}$$

Now, by using the properties of the shifted Legendre polynomials and the orthogonal condition we can see that

$$
\begin{aligned}
|c_i|^2 &= \left| \int_0^1 u''(t) \left[\frac{L^*_{i+2}(t) - L^*_i(t)}{2i+3} - \frac{L^*_i(t) - L^*_{i-2}(t)}{2i-1} \right] dt \right|^2 \\
&\leq \int_0^1 |u''(t)|^2 dt \int_0^1 \left| \frac{(2i-1)L^*_{i+2}(t) - 2(2i+1)L^*_i(t) + (2i+3)L^*_{i-2}(t)}{(2i+3)(2i-1)} \right|^2 dt \\
&\leq \delta^2 \int_0^1 \left| \frac{(2i-1)^2 L^{*2}_{i+2}(t) + (4i+2)^2 L^{*2}_i(t) + (2i+3)^2 L^{*2}_{i-2}(t)}{(2i+3)^2(2i-1)^2} \right| dt \\
&= \frac{\delta^2}{(2i+3)^2(2i-1)^2} \left(\frac{(2i-1)^2}{2i+5} + \frac{(4i+2)^2}{2i+1} + \frac{(2i+3)^2}{2i-3} \right) \\
&\leq \frac{\delta^2}{(2i+3)^2(2i-1)^2} \left(\frac{(2i+3)^2}{2i-3} + \frac{4(2i+3)^2}{2i-3} + \frac{(2i+3)^2}{2i-3} \right) \\
&\leq \frac{6\delta^2}{(2i-3)(2i-1)^2},
\end{aligned}
$$

this implies and proves the required relation (17). □

Theorem 3.

For a function $u(t)$. Under the two assumptions:

1. $u(t)$ is continuous function on $[0, 1]$;
2. $u(t)$ has bounded second derivative (i.e., $|u''(t)| \leq \delta$ for some constant δ).

Then its shifted Legendre approximation $u_m(t)$ defined in (9) converges uniformly. Moreover, we have the following accuracy estimation

$$
||u(t) - u_m(t)||_{L^2[0,1]} \leq \sqrt{6}\delta \left(\sum_{i=m+1}^{\infty} \frac{1}{(2i-3)^4} \right)^{0.5}. \tag{18}
$$

Proof. Using the properties of the shifted Legendre polynomials, the orthogonal condition and the proved formula (17) we can get

$$
\begin{aligned}
||u(t) - u_m(t)||^2_{L^2[0,1]} &= \int_0^1 \left[\sum_{i=0}^{\infty} c_i L^*_i(t) - \sum_{i=0}^{m} c_i L^*_i(t) \right]^2 dt \\
&= \int_0^1 \left[\sum_{i=m+1}^{\infty} c_i L^*_i(t) \right]^2 dt \leq \sum_{i=m+1}^{\infty} |c_i|^2 \int_0^1 L^{*2}_i(t) dt \\
&= \sum_{i=m+1}^{\infty} |c_i|^2 \frac{1}{2i+1} \leq \sum_{i=m+1}^{\infty} \frac{6\delta^2}{(2i-3)^3} \frac{1}{2i+1} \\
&\leq 6\delta^2 \sum_{i=m+1}^{\infty} \frac{1}{(2i-3)^4},
\end{aligned}
$$

this implies and proves the required relation (18). □

Theorem 4.

The Caputo fractional derivative of order α for the shifted Legendre polynomials can be expressed in terms of the shifted Legendre polynomials themselves in the following form

$$D^{\alpha}(L_i^*(t)) = \sum_{k=\lceil\alpha\rceil}^{i} \sum_{j=0}^{k-\lceil\alpha\rceil} \Theta_{i,j,k}\, L_j^*(t), \tag{19}$$

where

$$\Theta_{i,j,k} = \frac{(-1)^{i+k}(i+k)!(2j+1)}{(i-k)!(k)!\Gamma(k-\alpha+1)} \times \sum_{r=0}^{j} \frac{(-1)^{j+r}(j+r)!}{(j-r)!\,(r!)^2(k-\alpha+r+1)}, \quad j=0,1,\dots . \tag{20}$$

Proof. Using the properties of the shifted Legendre polynomials [3], then $t^{k-\alpha}$ in (14) can be expanded in the following form

$$t^{k-\alpha} = \sum_{j=0}^{k-\lceil\alpha\rceil} c_{kj} L_j^*(t), \tag{21}$$

where c_{kj} can be obtained using (8) such that $u(t) = t^{k-\alpha}$, then we can claim the following

$$c_{kj} = (2j+1)\int_0^1 t^{k-\alpha} L_j^*(t)dt, \qquad j=0,1,\dots .$$

But at $j=0$ we have, $\quad c_{k0} = \int_0^1 t^{k-\alpha}dt = \dfrac{1}{k-\alpha+1},$

also, for any j, using the formula (6), we can claim

$$c_{kj} = (2j+1)\sum_{r=0}^{j}(-1)^{j+r}\frac{(j+r)!}{(j-r)!(r!)^2(k-\alpha+r+1)}, \qquad j=1,2,...,$$

employing Eqs.(14) and (21) gives

$$D^{\alpha}(L_i^*(t)) = \sum_{k=\lceil\alpha\rceil}^{i} \sum_{j=0}^{k-\lceil\alpha\rceil} \Theta_{i,j,k} L_j^*(t), \quad i=\lceil\alpha\rceil, \lceil\alpha\rceil+1,\dots ,$$

where

$$\Theta_{i,j,k} = \frac{(-1)^{i+k}(i+k)!(2j+1)}{(i-k)!(k)!\Gamma(k-\alpha+1)} \times \sum_{r=0}^{j} \frac{(-1)^{j+r}(j+r)!}{(j-r)!\,(r!)^2(k-\alpha+r+1)}, \qquad j=0,1,\dots ,$$

and this completes the proof of the theorem. □

Theorem 5.

The error $|E_T(m)| = |D^\alpha u(t) - D^\alpha u_m(t)|$ in approximating $D^\alpha u(t)$ by $D^\alpha u_m(t)$ is bounded by

$$|E_T(m)| \leq \Big| \sum_{i=m+1}^{\infty} c_i \Big(\sum_{k=\lceil\alpha\rceil}^{i} \sum_{j=0}^{k-\lceil\alpha\rceil} \Theta_{i,j,k} \Big) \Big|. \tag{22}$$

Proof. A combination of Eqs.(7), (9) and (19) leads to

$$|E_T(m)| = \Big| D^\alpha u(t) - D^\alpha u_m(t) \Big| = \Big| \sum_{i=m+1}^{\infty} c_i \Big(\sum_{k=\lceil\alpha\rceil}^{i} \sum_{j=0}^{k-\lceil\alpha\rceil} \Theta_{i,j,k} L_j^*(t) \Big) \Big|,$$

but $|L_j^*(t)| \leq 1$, so, we can obtain

$$|E_T(m)| \leq \Big| \sum_{i=m+1}^{\infty} c_i \Big(\sum_{k=\lceil\alpha\rceil}^{i} \sum_{j=0}^{k-\lceil\alpha\rceil} \Theta_{i,j,k} \Big) \Big|,$$

and subtracting the truncated series from the infinite series, bounding each term in the difference, and summing the bounds completes the proof of the theorem. □

4. Procedure solution of the fractional SIRC model and influenza A

Consider the fractional SIRC model and influenza A of the type given in Eq.(1). In order to use Legendre collocation method, we first approximate $S(t)$, $I(t)$, $R(t)$ and $C(t)$ as

$$S_m(t) = \sum_{i=0}^{m} a_i\, L_i^*(t), \quad I_m(t) = \sum_{i=0}^{m} b_i\, L*_i(t), \quad R_m(t) = \sum_{i=0}^{m} c_i\, L_i^*(t), \quad C_m(t) = \sum_{i=0}^{m} d_i\, L_i^*(t). \tag{23}$$

From Eqs.(1) and Theorem 1 we have

$$\begin{aligned}
&\sum_{i=\lceil\alpha\rceil}^{m} \sum_{k=\lceil\alpha\rceil}^{i} a_i\, w_{i,k}^{(\alpha)}\, t^{k-\alpha} = \mu(1 - \sum_{i=0}^{m} a_i\, L_i^*(t)) - \beta(\sum_{i=0}^{m} a_i\, L_i^*(t))(\sum_{i=0}^{m} b_i\, L_i^*(t)) + \gamma(\sum_{i=0}^{m} d_i\, L_i^*(t)),\\
&\sum_{i=\lceil\alpha\rceil}^{m} \sum_{k=\lceil\alpha\rceil}^{i} b_i\, w_{i,k}^{(\alpha)}\, t^{k-\alpha} = \beta(\sum_{i=0}^{m} a_i\, L_i^*(t))(\sum_{i=0}^{m} b_i\, L_i^*(t)) + \sigma\beta(\sum_{i=0}^{m} d_i\, L_i^*(t))(\sum_{i=0}^{m} b_i\, L_i^*(t)) - (\mu+\theta)(\sum_{i=0}^{m} b_i\, L_i^*(t)),\\
&\sum_{i=\lceil\alpha\rceil}^{m} \sum_{k=\lceil\alpha\rceil}^{i} c_i\, w_{i,k}^{(\alpha)}\, t^{k-\alpha} = (1-\sigma)\beta(\sum_{i=0}^{m} d_i\, L_i^*(t))(\sum_{i=0}^{m} b_i\, L_i^*(t)) + \theta(\sum_{i=0}^{m} b_i\, L_i^*(t)) - (\mu+\delta)\sum_{i=0}^{m} c_i\, L_i^*(t),\\
&\sum_{i=\lceil\alpha\rceil}^{m} \sum_{k=\lceil\alpha\rceil}^{i} d_i\, w_{i,k}^{(\alpha)}\, t^{k-\alpha} = \delta(\sum_{i=0}^{m} c_i\, L_i^*(t)) - \beta(\sum_{i=0}^{m} d_i\, L_i^*(t))(\sum_{i=0}^{m} b_i\, L_i^*(t)) - (\mu+\gamma)(\sum_{i=0}^{m} d_i\, L_i^*(t)).
\end{aligned} \tag{24}$$

We now collocate Eqs.(24) at $(m+1-\lceil\alpha\rceil)$ points t_p $(p=0,1,...,m-\lceil\alpha\rceil)$ as

$$\begin{aligned}
\sum_{i=\lceil\alpha\rceil}^{m}\sum_{k=\lceil\alpha\rceil}^{i} a_i\, w_{i,k}^{(\alpha)}\, t_p^{k-\alpha} &= \mu(1-\sum_{i=0}^{m} a_i\, L_i^*(t_p)) - \beta(\sum_{i=0}^{m} a_i\, L_i^*(t_p))(\sum_{i=0}^{m} b_i\, L_i^*(t_p)) + \gamma(\sum_{i=0}^{m} d_i\, L_i^*(t_p)),\\
\sum_{i=\lceil\alpha\rceil}^{m}\sum_{k=\lceil\alpha\rceil}^{i} b_i\, w_{i,k}^{(\alpha)}\, t_p^{k-\alpha} &= \beta(\sum_{i=0}^{m} a_i\, L_i^*(t_p))(\sum_{i=0}^{m} b_i\, L_i^*(t_p)) + \sigma\beta(\sum_{i=0}^{m} d_i\, L_i^*(t_p))(\sum_{i=0}^{m} b_i\, L_i^*(t_p))\\
&\quad - (\mu+\theta)(\sum_{i=0}^{m} b_i\, L_i^*(t_p)),\\
\sum_{i=\lceil\alpha\rceil}^{m}\sum_{k=\lceil\alpha\rceil}^{i} c_i\, w_{i,k}^{(\alpha)}\, t_p^{k-\alpha} &= (1-\sigma)\beta(\sum_{i=0}^{m} d_i\, L_i^*(t_p))(\sum_{i=0}^{m} b_i\, L_i^*(t_p)) + \theta(\sum_{i=0}^{m} b_i\, L_i^*(t_p)) - (\mu+\delta)\sum_{i=0}^{m} c_i\, L_i^*(t_p),\\
\sum_{i=\lceil\alpha\rceil}^{m}\sum_{k=\lceil\alpha\rceil}^{i} d_i\, w_{i,k}^{(\alpha)}\, t_p^{k-\alpha} &= \delta(\sum_{i=0}^{m} c_i\, L_i^*(t_p)) - \beta(\sum_{i=0}^{m} d_i\, L_i^*(t_p))(\sum_{i=0}^{m} b_i\, L_i^*(t_p)) - (\mu+\gamma)(\sum_{i=0}^{m} d_i\, L_i^*(t_p)).
\end{aligned} \tag{25}$$

For suitable collocation points we use roots of shifted Legendre polynomial $L_{m+1-\lceil\alpha\rceil}^*(t)$.
Also, by substituting Eq.(23) in the initial conditions (2) we can obtain $4\lceil\alpha\rceil$ equations as follows

$$\sum_{i=0}^{m}(-1)^i\, a_i = s_0, \quad \sum_{i=0}^{m}(-1)^i b_i = i_0, \quad \sum_{i=0}^{m}(-1)^i c_i = r_0, \quad \sum_{i=0}^{m}(-1)^i d_i = c_0. \tag{26}$$

Equations (25), together with the equations of the initial conditions (26), give $(4m+4)$ of nonlinear algebraic equations which can be solved using the Newton iteration method, for the unknowns a_i, b_i, c_i and d_i for $i=0,1,...,m$.

5. Numerical implementation

In this section, we implement the suggested technique to solve the system SIRC (1) with the constants $\mu=0.02$, $\beta=100$, $\delta=1$, $\gamma=0.5$, $\sigma=0.05$, $\theta=73$, and the initial conditions

$$S(0)=0.8, \qquad I(0)=0.1, \qquad R(0)=0.05, \qquad C(0)=0.05. \tag{27}$$

The approximate solutions of the proposed system are given in figures 1-6 at different values of α. In figures 1-2 we compared the obtained approximate solution at $\alpha=1$ using the proposed method with the numerical solution using fourth order Runge-Kutta method, respectively. Also, in the figures 3-6, we presented the behavior of the approximate solution $S_m(t)$, $I_m(t)$, $R_m(t)$ and $C_m(t)$, respectively, with different values of $\alpha(\alpha = 0.4,\ 0.6,\ 0.8)$.

From figures 1 and 2, we can confirm that the approximate solution is in excellent agreement with the solution using fourth order Runge-Kutta method. Also, from figures 3-6, we can conclude that the behavior of the approximate solution depends on the order of the fractional derivative.

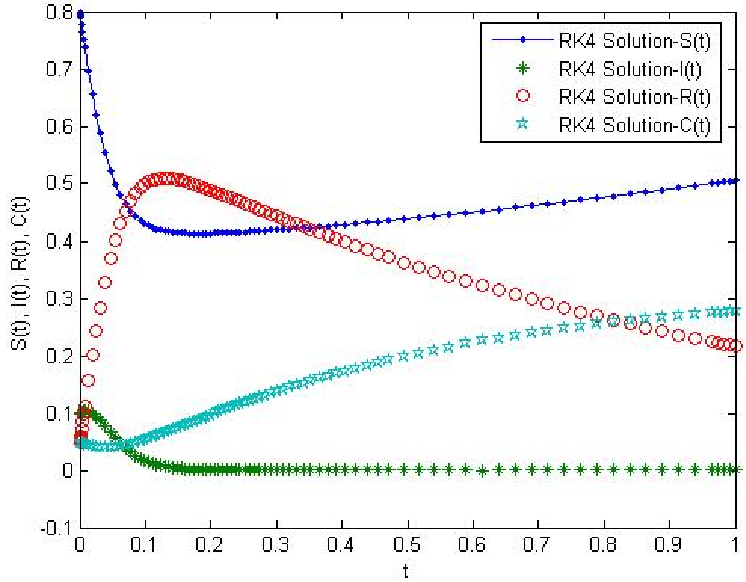

Figure 1. The behavior of the approximate solution using the proposed method at $\alpha = 1$.

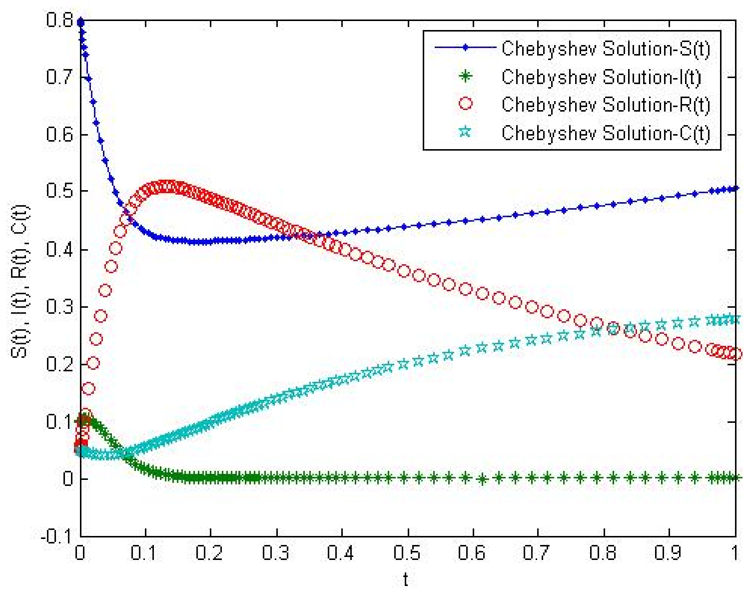

Figure 2. The numerical solution using the fourth order Runge-Kutta method at $\alpha = 1$.

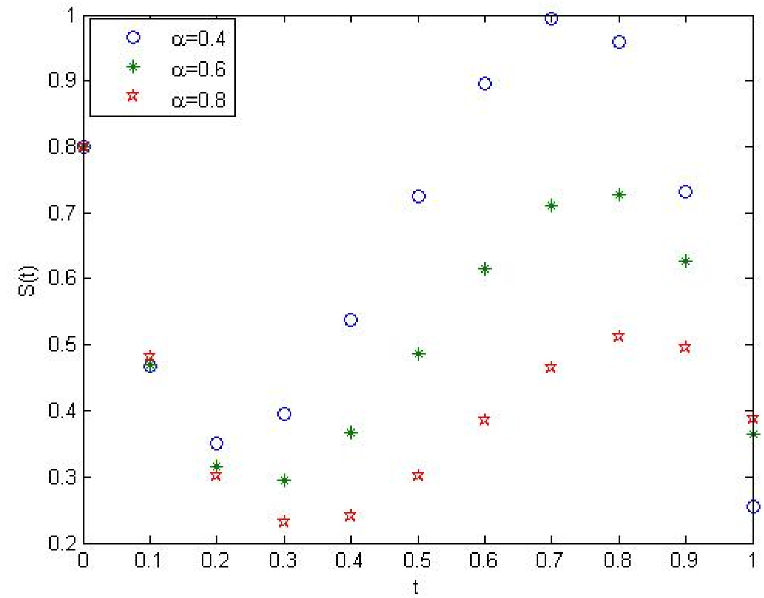

Figure 3. The behavior of the approximate solution $S(t)$.

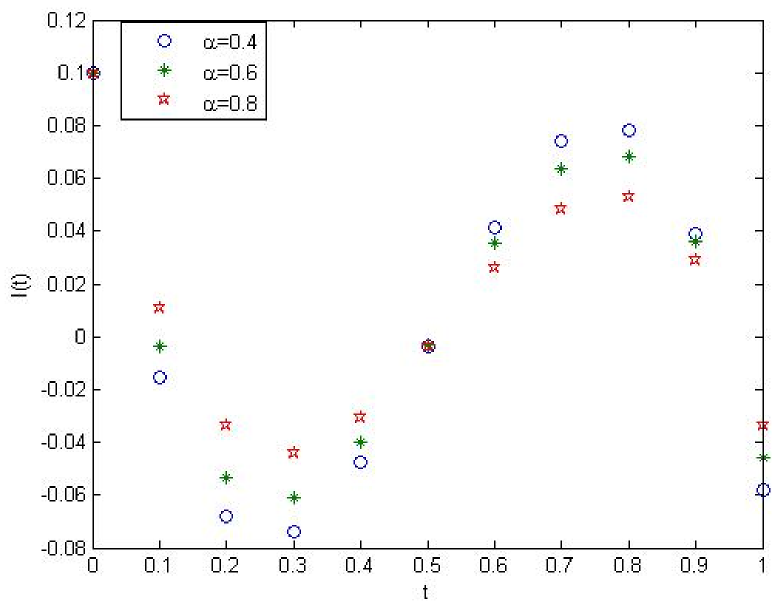

Figure 4. The behavior of the approximate solution $I(t)$.

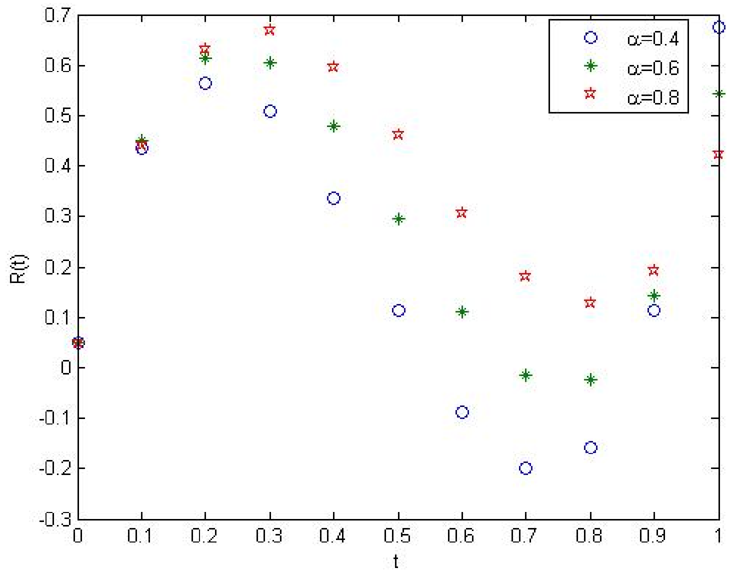

Figure 5. The behavior of the approximate solution $R(t)$.

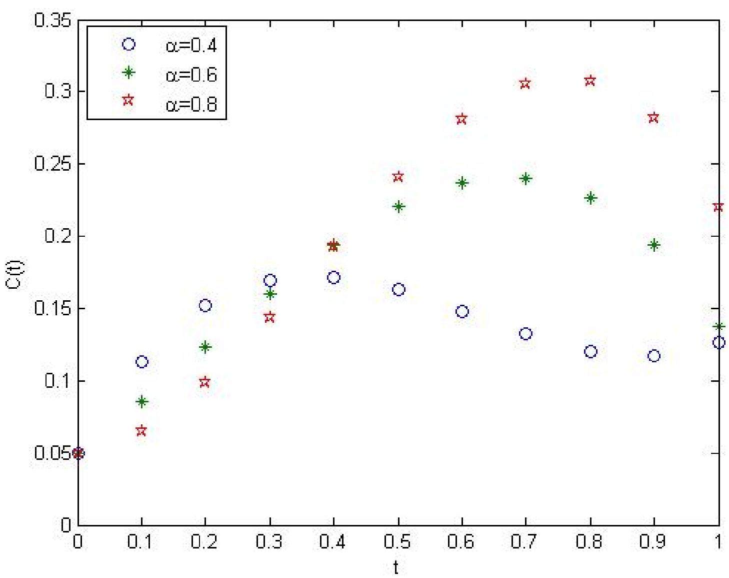

Figure 6. The behavior of the approximate solution $C(t)$.

6. Conclusion and remarks

In this article, we used Legendre colloction method for solving the fractional SIRC model and influenza A. The properties of the Legendre polynomials are used to reduce the proposed model to a non-linear system of algebraic equations which solved by Newton iteration method. The convergence analysis and derivation of an upper bound for the error of the approximate formula are given. From the obtained numerical results, we can conclude that this method gives results with an excellent agreement with those numerical solutions using fourth-order Rung-Kutta method. All numerical results are obtained using Matlab 8.

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Global properties of HIV infection models with nonlinear incidence rate and delay-discrete or distributed

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Abstract

In this paper, we study the global properties of two mathematical models which describe the interaction of the human immunodeficiency virus (HIV) with two classes of target cells, CD4$^+$ T cells and macrophages. The incidence rate of virus infection is given by the Crowley-Martin functional response. The first model has two types of discrete delays while the second one incorporates two types of distributed delays to describe the time needed for infection of cell and virus replication. The basic reproduction number R_0 is identified which completely determines the global dynamics of the models. By constructing suitable Lyapunov functionals, we have proven that if $R_0 \leq 1$ then the uninfected steady state is globally asymptotically stable (GAS), and if $R_0 > 1$ then the infected steady state exists and it is GAS.

Keywords: HIV dynamics; Global stability; Delay; Crowley-Martin functional response.

1 Introduction

Over the last decade, much collaborated effort involving biologists and mathematicians has been devoted towards designing mathematical models of the dynamics of the human immunodeficiency virus (HIV) [1]. The interaction of the virus and target cells has been formulated as ordinary differential equations in several works (see e.g. [2], [3], [21] and [23]). The basic mathematical model describing the HIV dynamics is given by [2]:

$$\dot{x} = \lambda - dx - \beta xv, \tag{1}$$
$$\dot{y} = \beta xv - ay, \tag{2}$$
$$\dot{v} = ky - rv, \tag{3}$$

where x, y and v represent the populations of the uninfected CD4$^+$ T cells, infected cells and free virus particles, respectively. The uninfected cells are generated from sources within the body at rate λ. The parameter d is the death rate constant of the uninfected cells. Eq. (2) describes the population dynamics of the infected cells and shows that they die with rate constant a. The virus particles are produced by the infected cells with rate constant k, and are cleared from plasma with rate constant r. Bilinear incidence rate associated with the mass action principle is given by βxv where β is the infection rate constant.

Many researchers suggested that this bilinear incidence rate is insufficient to describe the infection process in detail. Therefore, different forms of the incidence rate of infection have been proposed such as, saturated incidence rate $\frac{\beta xv}{1+\gamma v}$ [23], Holling type II functional response $\frac{\beta xv}{1+\alpha x}$ [27], Beddington-DeAngelis infection rate $\frac{\beta xv}{1+\alpha x+\gamma v}$ [21], Crowley-Martin functional response $\frac{\beta xv}{(1+\alpha x)(1+\gamma v)}$ [24], [25], [26], where $\alpha, \gamma \geq 0$.

In model (1)-(3), it is assumed that the infection could occur and the viruses are produced from infected target cells instantaneously, once the uninfected cells are contacted by the virus particles. However, this assumption is unrealistic. Therefore, more realistic HIV dynamics models incorporate the delay between the time of viral entry into the target cell and the time the production of new virus particles, modeled with discrete time delay or distributed time delay using functional differential equations (see e.g. [5], [7], [6], [20], [22], [17], [19], [15], [14], [27]).

It is observed that, most of the proposed HIV dynamics models assume that the HIV has one class of target cells, CD4$^+$ T cells (see e.g. [2], and the book Nowak and May [1]). In [4], [18], [8], [9], [11], [12] and [10], some HIV models with two classes of target cells, CD4$^+$ T cells and macrophages have been proposed. The global stability of these models has been investigated in ([8], [12], [10] and [11]). Elaiw [8] studied the global

properties of HIV infection model with two classes of target cells (CD4$^+$T cells and macrophages). Elaiw and Azoz in [12], also studied the global properties of HIV infection models with two classes of target cells and with Beddington-DeAngelis functional response. In [12] and [8], the effect of time delay is neglected. Elaiw et al. [10] studied the global stability of HIV model with Beddington-DeAnglis functional response and one kind of discrete time delay. Elaiw [11] studied the global dynamics of a delay HIV model with two classes of target cells and saturated function response.

The purpose of this paper is to study the global dynamics of two HIV infection models with Crowley-Martin functional response. We take into account that the HIV attack two classes of target cells CD4$^+$T cells and macrophages. Model with discrete delay and model with distributed delay have been studied to take into account the time delay between the time the target cells contacted by the virus and the time the emission of infectious (matures) virus particles. The global stability of the these models is established using Lyapunov functionals. We have proven that the global dynamics of these models are determined by the basic reproduction number R_0. If $R_0 \leq 1$, then the uninfected steady state is globally asymptotically stable (GAS) and if $R_0 > 1$, then the infected steady state exists and it is GAS.

2 Model with discrete-time delays

In this section we study a viral infection model with two classes of target cells and Crowley-Martin functional response. We incorporate two types of discrete-time delays into the model.

$$\dot{x_i} = \lambda_i - d_i x_i - \frac{\beta_i x_i v}{(1+\alpha_i x_i)(1+\gamma_i v)}, \qquad i = 1,2 \tag{4}$$

$$\dot{y_i} = \frac{\beta_i' x_i(t-\tau_i) v(t-\tau_i)}{(1+\alpha_i x_i(t-\tau_i))(1+\gamma_i v(t-\tau_i))} - a_i y_i, \qquad i = 1,2 \tag{5}$$

$$\dot{v} = \sum_{i=1}^{2} p_i y_i (t-\omega_i) - rv, \tag{6}$$

where x_i and y_i represent the populations of the uninfected and infected target cells, respectively, where $i = 1$ and 2 correspond to CD4$^+$T cells and macrophages. We put $\beta_i' = e^{-m_i\tau_i}\beta_i$ and $p_i = e^{-n_i\omega_i}k_i$. Here the parameter τ_i accounts for the time between the target cells of class i are contacted by the virus particles and the production of new virus particles. The factor $e^{-m_i\tau_i}$ accounts for the probability of surviving the time period from $t-\tau_i$ to t, where m_i is the death rate of infected but not yet virus producer cells. The parameter ω_i accounts for the time between the virus has penetrated into a target cell i, and the emission of infectious virus particles. The factor $e^{-n_i\omega_i}$ accounts for the probability of surviving the time period from $t-\omega_i$ to t, where n_i is positive constant. The parameters $\lambda_i, \beta_i, d_i, \alpha_i, \gamma_i, a_i,$ and k_i are positive constants with the same biological meaning given above.

Initial conditions

The initial conditions for system (4)-(6) take the form

$$\begin{aligned} &x_1(\theta) = \varphi_1(\theta),\ x_2(\theta) = \varphi_2(\theta), y_1(\theta) = \varphi_3(\theta),\ y_2(\theta) = \varphi_4(\theta), v(\theta) = \varphi_5(\theta), \\ &\varphi_i(\theta) \geq 0,\ \theta \in [-\ell, 0),\ \varphi_j(0) > 0,\ j = 1, ..., 5. \end{aligned} \tag{7}$$

where, $(\varphi_1(\theta), \varphi_2(\theta), ..., \varphi_5(\theta)) \in C$ and $C = C([-\ell, 0], \mathbb{R}^5_+)$ is the Banach space of continuous functions mapping the interval $[-\ell, 0]$ into $\mathbb{R}^5_+$,where $\ell = \max\{\tau_1, \tau_2, \omega_1, \omega_2\}$. By the fundamental theory of functional differential equations [13], system (4)-(6) has a unique solution $(x_1(t), x_2(t), y_1(t), y_2(t), v(t))$ satisfying initial conditions (7).

2.1 Non-negativity and boundedness of solutions

In the following, we establish the non-negativity and boundedness of solutions of (4)-(6) with initial conditions (7).

Proposition 1. Let $X(t) = (x_1(t), x_2(t), y_1(t), y_2(t), v(t))^T$ be any solution of (4)-(6) satisfying the initial conditions (7), then $X(t)$ is non-negative for $t \geq 0$ and ultimately bounded.

Proof. First, we prove that $x_i(t) > 0$, $i = 1, 2$, for all $t \geq 0$. Assume that $x_i(t)$ lose its non-negativity on some local existence interval $[0, \rho]$ for some constant ρ and let $t_1 \in [0, \rho]$ be such that $x_i(t_1) = 0$. From Eq. (4) we have $\dot{x}_i(t_1) = \lambda_i > 0$. Hence $x_i(t) > 0$ for some $t \in (t_1, t_1 + \varepsilon)$, where $\varepsilon > 0$ is sufficiently small. This leads

to a contradiction and hence $x_i(t) > 0$, for all $t \geq 0$. Further more, from Eqs. (5) and (6) we have

$$y_i(t) = y_i(0)e^{-a_i t} + \beta_i' \int_0^t e^{-a_i(t-\eta)} \frac{x_i(\eta-\tau_i)v(\eta-\tau_i)}{(1+\alpha_i x_i(\eta-\tau_i))(1+\gamma_i v(\eta-\tau_i))} d\eta, \quad i=1,2$$

$$v(t) = v(0)e^{-rt} + \sum_{i=1}^{2} p_i \int_0^t e^{-r(t-\eta)} y_i(\eta-\omega_i) d\eta,$$

confirming that $y_i(t) \geq 0$, $i=1,2$, and $v(t) \geq 0$ for all $t \in [0,\ell]$. By a recursive argument, we obtain $y_i(t) \geq 0$, $i=1,2$, and $v(t) \geq 0$ for all $t \geq 0$.

Next we show that the solution is ultimately bounded. From (4) we have $\dot{x}_i \leq \lambda_i - d_i x_i$. Thus $\limsup_{t\to\infty} x_i(t) \leq \frac{\lambda_i}{d_i}$ and $x_i(t)$ is ultimately bounded. Let $T_i(t) = \frac{\beta_i'}{\beta_i} x_i(t-\tau_i) + y_i(t)$, then

$$\dot{T}_i(t) \leq \frac{\lambda_i \beta_i'}{\beta_i} - \sigma_i T_i(t),$$

where $\sigma_i = \min\{d_i, a_i\}$. It follows that $\limsup_{t\to\infty} T_i(t) \leq L_i$, where $L_i = \frac{\lambda_i \beta_i'}{\sigma_i \beta_i}$. This in turn implies, by the non-negativity of $x_i(t)$ and $y_i(t)$, that $\limsup_{t\to\infty} y_i(t) \leq L_i$ and $y_i(t)$ is ultimately bounded. On the other hand, from Eq. (6) we have

$$\dot{v}(t) \leq \sum_{i=1}^{2} p_i L_i - rv,$$

then $\limsup_{t\to\infty} v(t) \leq L^*$, where $L^* = \sum_{i=1}^{2} \frac{p_i L_i}{r}$ and $v(t)$ is ultimately bounded. □

2.2 Steady States

It is clear that, system (4)-(6) has an uninfected steady state $E_0(x_1^0, x_2^0, y_1^0, y_2^0, v^0)$, where $x_i^0 = \frac{\lambda_i}{d_i}$, $y_i^0 = 0$, $i=1,2$ and $v^0 = 0$, and there may another steady state $E_*(x_1^*, x_2^*, y_1^*, y_2^*, v^*)$ which is the infected steady state with coordinates that if exist, satisfy the equalities:

$$\lambda_i = d_i x_i^* + \frac{\beta_i x_i^* v^*}{(1+\alpha_i x_i^*)(1+\gamma_i v^*)}, \qquad i=1,2 \tag{8}$$

$$a_i y_i^* = \frac{\beta_i' x_i^* v^*}{(1+\alpha_i x_i^*)(1+\gamma_i v^*)}, \qquad i=1,2 \tag{9}$$

$$rv^* = \sum_{i=1}^{2} p_i y_i^*. \tag{10}$$

We define the basic reproduction number for system (4)-(6) as

$$R_0 = \sum_{i=1}^{2} R_i = \sum_{i=1}^{2} \frac{p_i \beta_i' x_i^0}{a_i r(1+\alpha_i x_i^0)}, \tag{11}$$

where R_i is the basic reproduction number for the dynamics of the virus and the target cell of class i.

Lemma 1. If $R_0 > 1$, then there exists a positive steady state E_*.

Proof. To compute the steady state of the system (4)-(6), we let the right-hand sides of Eqs. (4)-(6) equal zero,

$$\lambda_i - d_i x_i - \frac{\beta_i x_i v}{(1+\alpha_i x_i)(1+\gamma_i v)} = 0, \qquad i=1,2 \tag{12}$$

$$\frac{\beta_i' x_i v}{(1+\alpha_i x_i)(1+\gamma_i v)} - a_i y_i = 0, \qquad i=1,2 \tag{13}$$

$$\sum_{i=1}^{2} p_i y_i - rv = 0. \tag{14}$$

Solving Eq. (12) with respect to x_i, we get x_i as a function of v as:

$$x_{i\pm} = \frac{1}{2\alpha_i(1+\gamma_i v)}\left(\alpha_i x_i^0(1+\gamma_i v) - (1+\phi_i v) \pm \sqrt{((1+\phi_i v) - \alpha_i x_i^0(1+\gamma_i v))^2 + 4\alpha_i x_i^0(1+\gamma_i v)^2}\right),$$

where $\phi_i = \gamma_i + \frac{\beta_i}{d_i}$. It is clear that if $v > 0$ then $x_{i+} > 0$ and $x_{i-} < 0$. Let us choose

$$x_i = \frac{1}{2\alpha_i(1+\gamma_i v)}\left(\alpha_i x_i^0(1+\gamma_i v) - (1+\phi_i v) + \sqrt{((1+\phi_i v) - \alpha_i x_i^0(1+\gamma_i v))^2 + 4\alpha_i x_i^0(1+\gamma_i v)^2}\right). \quad (15)$$

From Eqs. (12)-(14), we have

$$\sum_{i=1}^{2}\frac{p_i\beta_i'}{a_i\beta_i}(\lambda_i - d_i x_i) - rv = 0. \quad (16)$$

Since x_i is a function of v, then we can define a function $B_1(v)$ as:

$$B_1(v) = \sum_{i=1}^{2}\frac{p_i\beta_i'}{a_i\beta_i}(\lambda_i - d_i x_i) - rv = 0.$$

When $v = 0$, then $x_i = x_i^0$, and $B_1(0) = 0$, and when $v = \bar{v} = \sum_{i=1}^{2}\frac{p_i\beta_i'\lambda_i}{a_i\beta_i r} > 0$, then substituting it in Eq. (15) we get the corresponding $\bar{x}_i > 0$ and

$$B_1(\bar{v}) = -\sum_{i=1}^{2}\frac{p_i\beta_i' d_i}{a_i\beta_i}\bar{x}_i < 0.$$

Since $B_1(v)$ is continuous for all $v \geq 0$, we have

$$B_1'(0) = \sum_{i=1}^{2}\frac{p_i\beta_i' x_i^0}{a_i(1+\alpha_i x_i^0)} - r = r(R_0 - 1).$$

Therefore, if $R_0 > 1$, then $B_1'(0) > 0$. It follows that there exists $v^* \in (0, \bar{v})$ such that $B_1'(v^*) = 0$. From Eq. (15), we obtain $x_i^* > 0, i = 1, 2$. Also, from Eq. (13) we get $y_i^* > 0, i = 1, 2$.

2.3 Global stability analysis

In this section, we study the global stability of the uninfected and infected steady states of system (4)-(6). The strategy of the proofs is to use suitable Lyapunov functionals which are similar in nature to those used in [28].

Preliminary:

We shall use the following notation: $z = z(t)$, for any $z \in \{x_i, y_i, v,\ i = 1, 2\}$. We also define a function $H : (0, \infty) \to [0, \infty)$ as

$$H(z) = z - 1 - \log z.$$

It is clear that $H(z) \geq 0$ for any $z > 0$ and H has the global minimum $H(1) = 0$. The function $H(z)$ can also be used in driving an extension of the arithmetic-geometric mean inequality which is important in proving the global stability of the steady states.

To extend the arithmetic-geometric mean inequality we put

$$-H(z_i) = 1 - z_i + \log z_i \leq 0, \quad \text{for } z_1, ..., z_n > 0, \quad (17)$$

summing $-H(z_i)$ from $i = 1$ to n

$$n - \sum_{i=1}^{n} z_i + \log\prod_{i=1}^{n} z_i \leq 0. \quad (18)$$

For $a_1, ..., a_n, b_1, ..., b_n > 0$, it holds that

$$n - \sum_{i=1}^{n}\frac{b_i}{a_i} + \log\prod_{i=1}^{n}\frac{b_i}{a_i} \leq 0. \quad (19)$$

If $z_1, z_2, ..., z_n > 0$ satisfy $z_1 z_2 ... z_n = 1$, then it holds that

$$n - \sum_{i=1}^{n} z_i \leq 0. \quad (20)$$

When $a_1a_2...a_n = b_1b_2...b_n$, and by substituting $z_i = \frac{b_i}{a_i}$ in (20), we get

$$n - \sum_{i=1}^{n} \frac{b_i}{a_i} \leq 0. \tag{21}$$

If we put $z_i = \frac{a_i}{\sqrt[n]{a_1a_2...a_n}}$ in (20), we obtain

$$\frac{a_1 + a_2 + ... + a_n}{n} \geq \sqrt[n]{a_1a_2...a_n} \tag{22}$$

which is the arithmetic-geometric mean inequality. Thus the inequalities (18) and (19) are considered as extensions of the arithmetic-geometric inequality.

Now let us assume that $a_1a_2...a_{m-1} = b_1b_2...b_{m-1}$, $m < n$, and replace $b_m, ..., b_n$ by $b'_m, ...b'_n$, then we have

$$n - \sum_{i=1}^{m-1} \frac{b_i}{a_i} - \sum_{i=m}^{n} \frac{b'_i}{a_i} + \log \frac{b_1...b_{m-1}b'_m...b'_n}{a_1...a_{m-1}a_m...a_n} \leq 0,$$

$$n - \sum_{i=1}^{m-1} \frac{b_i}{a_i} - \sum_{i=m}^{n} \frac{b'_i}{a_i} + \log \prod_{i=m}^{n} \frac{b'_i}{a_i} \leq 0. \tag{23}$$

This holds true for any positive a_i, b_j, b'_k, $(i = 1, ..., n; j = 1, ..., m-1; k = m, ..., n)$. The inequality (23) is crucial in proving the global stability of the infected steady states.

The next theorem establish the global stability of the uninfected steady state.

Theorem 1. Consider the system (4)-(6), if $R_0 \leq 1$ then E_0 is GAS.

Proof. Define a Lyapunov functional W_0 as follows:

$$W_0 = \sum_{i=1}^{2} \frac{p_i}{a_i} \left[\frac{\beta'_i x_i^0}{\beta_i(1+\alpha_i x_i^0)} \left(\frac{x_i}{x_i^0} - 1 - \log \frac{x_i}{x_i^0} \right) + y_i + \beta'_i \int_0^{\tau_i} \frac{x_i(t-\theta)v(t-\theta)}{(1+\alpha_i x_i(t-\theta))(1+\gamma_i v(t-\theta))} d\theta \right.$$
$$\left. + a_i \int_0^{\omega_i} y_i(t-\theta)d\theta \right] + v. \tag{24}$$

We note that W_0 is defined and continuous for all $(x_1(t), x_2(t), y_1(t), y_2(t), v(t)) > 0$. Also, the global minimum $W_0 = 0$ occurs at the uninfected steady state E_0. The time derivative of W_0 along the solution of (4)-(6) is given by

$$\begin{aligned}
\frac{dW_0}{dt} &= \sum_{i=1}^{2} \frac{p_i}{a_i} \left[\frac{\beta'_i}{\beta_i(1+\alpha_i x_i^0)} \left(1 - \frac{x_i^0}{x_i}\right) \left(\lambda_i - d_i x_i - \frac{\beta_i x_i v}{(1+\alpha_i x_i)(1+\gamma_i v)}\right) \right. \\
&\quad + \frac{\beta'_i x_i(t-\tau_i)v(t-\tau_i)}{(1+\alpha_i x_i(t-\tau_i))(1+\gamma_i v(t-\tau_i))} - a_i y_i + \frac{\beta'_i x_i v}{(1+\alpha_i x_i)(1+\gamma_i v)} \\
&\quad \left. - \frac{\beta'_i x_i(t-\tau_i)v(t-\tau_i)}{(1+\alpha_i x_i(t-\tau_i))(1+\gamma_i v(t-\tau_i))} + a_i y_i - a_i y_i(t-\omega_i) \right] + \sum_{i=1}^{2} p_i y_i(t-\omega_i) - rv \\
&= \sum_{i=1}^{2} \frac{p_i}{a_i} \left[\frac{-\beta'_i \lambda_i}{\beta_i x_i x_i^0 (1+a_i x_i^0)} \left(x_i - x_i^0\right)^2 - \frac{\beta'_i x_i v}{(1+\alpha_i x_i^0)(1+\alpha_i x_i)(1+\gamma_i v)} \right. \\
&\quad \left. + \frac{\beta'_i x_i^0 v}{(1+\alpha_i x_i^0)(1+\alpha_i x_i)(1+\gamma_i v)} + \frac{\beta'_i x_i v}{(1+\alpha_i x_i)(1+\gamma_i v)} \right] - rv
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{2} \frac{p_i}{a_i}\left[\frac{-\beta_i'\lambda_i}{\beta_i x_i x_i^0(1+a_i x_i^0)}\left(x_i - x_i^0\right)^2 + \frac{\beta_i' x_i^0 v + \alpha_i x_i^0 \beta_i' x_i v}{(1+\alpha_i x_i^0)\,(1+\alpha_i x_i)(1+\gamma_i v)}\right] - rv \\
&= \sum_{i=1}^{2} \frac{p_i}{a_i}\left[\frac{-\beta_i'\lambda_i}{\beta_i x_i x_i^0(1+a_i x_i^0)}\left(x_i - x_i^0\right)^2 + \frac{\beta_i' x_i^0 v}{(1+\alpha_i x_i^0)\,(1+\gamma_i v)}\right] - rv \\
&= -\sum_{i=1}^{2} \frac{p_i}{a_i}\frac{\beta_i'\lambda_i}{\beta_i x_i x_i^0(1+a_i x_i^0)}\left(x_i - x_i^0\right)^2 + \sum_{i=1}^{2} rR_i \frac{v+\gamma_i v^2 - \gamma_i v^2}{1+\gamma_i v} - rv \\
&= -\sum_{i=1}^{2} \frac{p_i}{a_i}\frac{\beta_i'\lambda_i}{\beta_i x_i x_i^0(1+a_i x_i^0)}\left(x_i - x_i^0\right)^2 - \sum_{i=1}^{2} \frac{rR_i\gamma_i v^2}{1+\gamma_i v} + \left(\sum_{i=1}^{2} R_i - 1\right) rv \\
&= -\sum_{i=1}^{2} \frac{p_i}{a_i}\frac{\beta_i'\lambda_i}{\beta_i x_i x_i^0(1+a_i x_i^0)}\left(x_i - x_i^0\right)^2 - \sum_{i=1}^{2} \frac{rR_i\gamma_i v^2}{1+\gamma_i v} + (R_0 - 1)rv. \qquad (25)
\end{aligned}
$$

It can be seen that, if $R_0 \le 1$ then $\frac{dW_0}{dt} \le 0$ for all $x_i, v > 0$, $i = 1, 2$. By Theorem 5.3.1 in [13], the solutions of system (4)-(6) limit to M, the largest invariant subset of $\left\{\frac{dW_0}{dt} = 0\right\}$. Clearly, it follows from (25) that $\frac{dW_0}{dt} = 0$ if and only if $x_i = x^0$, $i = 1, 2$ and $v = 0$. Noting that M is invariant, for each element of M we have $v = 0$, then $\dot{v} = 0$. From Eq. (6) we drive that

$$
0 = \dot{v} = \sum_{i=1}^{2} p_i y_i(t - \omega_i).
$$

Since $y_i\,(t-\theta) \ge 0$ for all $\theta \in [0, \ell]$, then $\sum_{i=1}^{2} p_i y_i(t-\omega_i) = 0$ if and only if $y_i(t-\omega_i) = 0$, $i = 1, 2$. Hence $\frac{dW_0}{dt} = 0$ if and only if $x_i = x^0$, $y_i = 0$, $i = 1, 2$ and $v = 0$. From LaSalle's invariance principle, E_0 is GAS.

Theorem 2. If $R_0 > 1$, then E_* is GAS.

Proof. For proving the global stability of the infection steady state E_*, we use the same strategy as in [28]. First, we consider the following system

$$
\dot{x}_i = \lambda_i - d_i x_i - \frac{\beta_i x_i v}{(1+\alpha_i x_i)(1+\gamma_i v)}, \qquad i = 1, 2 \qquad (26)
$$

$$
\dot{y}_i = \frac{\beta_i' x_i v}{(1+\alpha_i x_i)(1+\gamma_i v)} - a_i y_i, \qquad i = 1, 2 \qquad (27)
$$

$$
\dot{v} = \sum_{i=1}^{2} p_i y_i - rv. \qquad (28)
$$

Let $X = (x_1, x_2, y_1, y_2, v)^T$, and denote the vector field given by (26)-(28) as $G(X)$, and define Lyapunov functional W_* as follows:

$$
W_* = \sum_{i=1}^{2} \frac{p_i}{a_i}\left[\frac{\beta_i'}{\beta_i}\left(x_i - x_i^* - \int_{x_i^*}^{x_i} \frac{x_i^*(1+\alpha_i\mu)}{\mu(1+\alpha_i x_i^*)}d\mu\right) + y_i^*\left(\frac{y_i}{y_i^*} - 1 - \log\frac{y_i}{y_i^*}\right)\right] + v^*\left(\frac{v}{v^*} - 1 - \log\frac{v}{v^*}\right). \qquad (29)
$$

By calculating the time derivative along (26)-(28) we get

$$
\begin{aligned}
\nabla W_* \cdot G(X) = \sum_{i=1}^{2} \frac{p_i}{a_i}&\left[\frac{\beta_i'}{\beta_i}\left(1 - \frac{x_i^*(1+\alpha_i x_i)}{x_i(1+\alpha_i x_i^*)}\right)\left(\lambda_i - d_i x_i - \frac{\beta_i x_i v}{(1+\alpha_i x_i)(1+\gamma_i v)}\right)\right. \\
&\left. + \left(1 - \frac{y_i^*}{y_i}\right)\left(\frac{\beta_i' x_i v}{(1+\alpha_i x_i)(1+\gamma_i v)} - a_i y_i\right)\right] + \left(1 - \frac{v^*}{v}\right)\left(\sum_{i=1}^{2} p_i y_i - rv\right).
\end{aligned}
$$

Using the conditions (8)-(10) for the infected steady state E_* and the arithmetic-geometric mean inequality

(22), we get

$$
\begin{aligned}
\nabla W_* \cdot G(X) &= \sum_{i=1}^{2} \frac{p_i}{a_i} \left[\frac{\beta_i'}{\beta_i} \left(1 - \frac{x_i^*(1+\alpha_i x_i)}{x_i(1+\alpha_i x_i^*)}\right) \left(d_i x_i^* + \frac{\beta_i x_i^* v^*}{(1+\alpha_i x_i^*)(1+\gamma_i v^*)} - d_i x_i - \frac{\beta_i x_i v}{(1+\alpha_i x_i)(1+\gamma_i v)}\right) \right. \\
&\left. + \frac{\beta_i' x_i v}{(1+\alpha_i x_i)(1+\gamma_i v)} - a_i y_i - \frac{y_i^*}{y_i} \frac{\beta_i' x_i v}{(1+\alpha_i x_i)(1+\gamma_i v)} + a_i y_i^* \right] + \sum_{i=1}^{2} p_i y_i - rv - \frac{v^*}{v} \sum_{i=1}^{2} p_i y_i + rv^* \\
&= \sum_{i=1}^{2} \frac{p_i}{a_i} \left[\frac{-d_i \beta_i'(x_i - x_i^*)}{\beta_i} \left(1 - \frac{x_i^*(1+\alpha_i x_i)}{x_i(1+\alpha_i x_i^*)}\right) + a_i y_i^* \left(\frac{v(1+\gamma_i v^*)}{v^*(1+\gamma_i v)} - \frac{v}{v^*}\right) \right. \\
&\left. + a_i y_i^* \left(3 - \frac{x_i^*(1+\alpha_i x_i)}{x_i(1+\alpha_i x_i^*)} - \frac{y_i^*}{y_i} \frac{x_i v(1+\alpha_i x_i^*)(1+\gamma_i v^*)}{x_i^* v^*(1+\alpha_i x_i)(1+\gamma_i v)} - \frac{y_i v^*}{y_i^* v}\right) \right] \\
&= \sum_{i=1}^{2} \frac{p_i}{a_i} \left[\frac{-d_i \beta_i'(x_i - x_i^*)^2}{\beta_i x_i(1+\alpha_i x_i^*)} + a_i y_i^* \left(-1 + \frac{v(1+\gamma_i v^*)}{v^*(1+\gamma_i v)} - \frac{v}{v^*} + \frac{1+\gamma_i v}{1+\gamma_i v^*} - \frac{1+\gamma_i v}{1+\gamma_i v^*}\right) \right. \\
&\left. + a_i y_i^* \left(4 - \frac{x_i^*(1+\alpha_i x_i)}{x_i(1+\alpha_i x_i^*)} - \frac{y_i^*}{y_i} \frac{x_i v(1+\alpha_i x_i^*)(1+\gamma_i v^*)}{x_i^* v^*(1+\alpha_i x_i)(1+\gamma_i v)} - \frac{y_i v^*}{y_i^* v}\right) \right] \\
&= \sum_{i=1}^{2} \frac{p_i}{a_i} \left[\frac{-d_i \beta_i'(x_i - x_i^*)^2}{\beta_i x_i(1+\alpha_i x_i^*)} + a_i y_i^* \left(-1 + \frac{v(1+\gamma_i v^*)}{v^*(1+\gamma_i v)} - \frac{v}{v^*} + \frac{1+\gamma_i v}{1+\gamma_i v^*}\right) \right. \\
&\left. + a_i y_i^* \left(4 - \frac{x_i^*(1+\alpha_i x_i)}{x_i(1+\alpha_i x_i^*)} - \frac{y_i^*}{y_i} \frac{x_i v(1+\alpha_i x_i^*)(1+\gamma_i v^*)}{x_i^* v^*(1+\alpha_i x_i)(1+\gamma_i v)} - \frac{y_i v^*}{y_i^* v} - \frac{1+\gamma_i v}{1+\gamma_i v^*}\right) \right] \\
&= \sum_{i=1}^{2} \frac{p_i}{a_i} \left[\frac{-d_i \beta_i'(x_i - x_i^*)^2}{\beta_i x_i(1+\alpha_i x_i^*)} - a_i y_i^* \left(\frac{\gamma_i (v - v^*)^2}{v^*(1+\gamma_i v)(1+\gamma_i v^*)}\right) \right. \\
&\left. + a_i y_i^* \left(4 - \frac{x_i^*(1+\alpha_i x_i)}{x_i(1+\alpha_i x_i^*)} - \frac{y_i^*}{y_i} \frac{x_i v(1+\alpha_i x_i^*)(1+\gamma_i v^*)}{x_i^* v^*(1+\alpha_i x_i)(1+\gamma_i v)} - \frac{y_i v^*}{y_i^* v} - \frac{1+\gamma_i v}{1+\gamma_i v^*}\right) \right] \leq 0.
\end{aligned} \tag{30}
$$

Now, we compute the time derivative of W_* along the solution of the system (4)-(6)

$$
\begin{aligned}
\frac{dW_*}{dt} &= \sum_{i=1}^{2} \frac{p_i}{a_i} \left[\frac{\beta_i'}{\beta_i} \left(1 - \frac{x_i^*(1+\alpha_i x_i)}{x_i(1+\alpha_i x_i^*)}\right) \left(\lambda_i - d_i x_i - \frac{\beta_i x_i v}{(1+\alpha_i x_i)(1+\gamma_i v)}\right) \right. \\
&\left. + \left(1 - \frac{y_i^*}{y_i}\right) \left(\frac{\beta_i' x_i(t-\tau_i) v(t-\tau_i)}{(1+\alpha_i x_i(t-\tau_i))(1+\gamma_i v(t-\tau_i))} - a_i y_i + \frac{\beta_i' x_i v}{(1+\alpha_i x_i)(1+\gamma_i v)} - \frac{\beta_i' x_i v}{(1+\alpha_i x_i)(1+\gamma_i v)} \right) \right] \\
&+ \left(1 - \frac{v^*}{v}\right) \left(\sum_{i=1}^{2} p_i y_i(t-\omega_i) - rv + \sum_{i=1}^{2} p_i y_i - \sum_{i=1}^{2} p_i y_i \right) \\
&= \left\{ \sum_{i=1}^{2} \frac{p_i}{a_i} \left[\frac{\beta_i'}{\beta_i} \left(1 - \frac{x_i^*(1+\alpha_i x_i)}{x_i(1+\alpha_i x_i^*)}\right) \left(\lambda_i - d_i x_i - \frac{\beta_i x_i v}{(1+\alpha_i x_i)(1+\gamma_i v)}\right) \right. \right. \\
&\left. \left. + \left(1 - \frac{y_i^*}{y_i}\right) \left(\frac{\beta_i' x_i v}{(1+\alpha_i x_i)(1+\gamma_i v)} - a_i y_i \right) \right] + \left(1 - \frac{v^*}{v}\right) \left(\sum_{i=1}^{2} p_i y_i - rv \right) \right\} \\
&+ \sum_{i=1}^{2} \frac{p_i}{a_i} \left[\left(1 - \frac{y_i^*}{y_i}\right) \left(\frac{\beta_i' x_i(t-\tau_i) v(t-\tau_i)}{(1+\alpha_i x_i(t-\tau_i))(1+\gamma_i v(t-\tau_i))} - \frac{\beta_i' x_i v}{(1+\alpha_i x_i)(1+\gamma_i v)} \right) \right] \\
&+ \left(1 - \frac{v^*}{v}\right) \left(\sum_{i=1}^{2} p_i y_i(t-\omega_i) - \sum_{i=1}^{2} p_i y_i \right).
\end{aligned} \tag{31}
$$

From (30) into (31) and using the infected steady conditions, we obtain

$$\begin{aligned}
\frac{dW_*}{dt} &= \nabla W_* \cdot G(X) + \sum_{i=1}^{2}\frac{p_i}{a_i}a_i y_i^* \left[\left(1-\frac{y_i^*}{y_i}\right)\left(\frac{x_i(t-\tau_i)v(t-\tau_i)(1+\alpha_i x_i^*)(1+\gamma_i v^*)}{x_i^* v^*(1+\alpha_i x_i(t-\tau_i))(1+\gamma_i v(t-\tau_i))} - \frac{x_i v(1+\alpha_i x_i^*)(1+\gamma_i v^*)}{x_i^* v^*(1+\alpha_i x_i)(1+\gamma_i v)}\right)\right.\\
&\left.+\left(1-\frac{v^*}{v}\right)\left(\frac{y_i(t-\omega_i)}{y_i^*}-\frac{y_i}{y_i^*}\right)\right]\\
&= \sum_{i=1}^{2}\frac{p_i}{a_i}\left[\frac{-d_i\beta_i'(x_i-x_i^*)^2}{\beta_i x_i(1+\alpha_i x_i^*)} - a_i y_i^*\frac{\gamma_i(v-v^*)^2}{v^*(1+\gamma_i v)(1+\gamma_i v^*)}\right.\\
&+a_i y_i^*\left(4-\frac{x_i^*(1+\alpha_i x_i)}{x_i(1+\alpha_i x_i^*)} - \frac{y_i^*}{y_i}\frac{x_i v(1+\alpha_i x_i^*)(1+\gamma_i v^*)}{x_i^* v^*(1+\alpha_i x_i)(1+\gamma_i v)} - \frac{y_i v^*}{y_i^* v} - \frac{1+\gamma_i v}{1+\gamma_i v^*}\right)\\
&+a_i y_i^*\left(\frac{x_i(t-\tau_i)v(t-\tau_i)(1+\alpha_i x_i^*)(1+\gamma_i v^*)}{x_i^* v^*(1+\alpha_i x_i(t-\tau_i))(1+\gamma_i v(t-\tau_i))} - \frac{x_i v(1+\alpha_i x_i^*)(1+\gamma_i v^*)}{x_i^* v^*(1+\alpha_i x_i)(1+\gamma_i v)}\right.\\
&-\frac{y_i^* x_i(t-\tau_i)v(t-\tau_i)(1+\alpha_i x_i^*)(1+\gamma_i v^*)}{y_i x_i^* v^*(1+\alpha_i x_i(t-\tau_i))(1+\gamma_i v(t-\tau_i))} + \frac{y_i^* x_i v(1+\alpha_i x_i^*)(1+\gamma_i v^*)}{y_i x_i^* v^*(1+\alpha_i x_i)(1+\gamma_i v)}\\
&\left.\left.+\frac{y_i(t-\omega_i)}{y_i^*} - \frac{y_i}{y_i^*} - \frac{v^* y_i(t-\omega_i)}{v y_i^*} + \frac{v^* y_i}{v y_i^*}\right)\right]
\end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^{2}\frac{p_i}{a_i}\left[\frac{-d_i\beta_i'(x_i-x_i^*)^2}{\beta_i x_i(1+\alpha_i x_i^*)} - a_i y_i^*\frac{\gamma_i(v-v^*)^2}{v^*(1+\gamma_i v)(1+\gamma_i v^*)}\right.\\
&+a_i y_i^*\left(4-\frac{x_i^*(1+\alpha_i x_i)}{x_i(1+\alpha_i x_i^*)} - \frac{y_i^* x_i(t-\tau_i)v(t-\tau_i)(1+\alpha_i x_i^*)(1+\gamma_i v^*)}{y_i x_i^* v^*(1+\alpha_i x_i(t-\tau_i))(1+\gamma_i v(t-\tau_i))} - \frac{v^* y_i(t-\omega_i)}{v y_i^*}\right.\\
&\left.\left.-\frac{1+\gamma_i v}{1+\gamma_i v^*} + \log\frac{y_i(t-\omega_i)x_i(t-\tau_i)v(t-\tau_i)(1+\alpha_i x_i)(1+\gamma_i v)}{y_i x_i v(1+\alpha_i x_i(t-\tau_i))(1+\gamma_i v(t-\tau_i))}\right)\right]\\
&+\sum_{i=1}^{2}\frac{p_i}{a_i}a_i y_i^*\left[\frac{x_i(t-\tau_i)v(t-\tau_i)(1+\alpha_i x_i^*)(1+\gamma_i v^*)}{x_i^* v^*(1+\alpha_i x_i(t-\tau_i))(1+\gamma_i v(t-\tau_i))} - \frac{x_i v(1+\alpha_i x_i^*)(1+\gamma_i v^*)}{x_i^* v^*(1+\alpha_i x_i)(1+\gamma_i v)}\right.\\
&\left.-\log\frac{x_i(t-\tau_i)v(t-\tau_i)(1+\alpha_i x_i)(1+\gamma_i v)}{x_i v(1+\alpha_i x_i(t-\tau_i))(1+\gamma_i v(t-\tau_i))}\right] + \sum_{i=1}^{2}p_i y_i^*\left(\frac{y_i(t-\omega_i)}{y_i^*} - \frac{y_i}{y_i^*} - \log\frac{y_i(t-\omega_i)}{y_i}\right). \qquad (32)
\end{aligned}$$

Using the arithmetic-geometric inequality (23), the first terms of (32) satisfy:

$$\begin{aligned}
&\sum_{i=1}^{2}\frac{p_i}{a_i}\left[\frac{-d_i\beta_i'(x_i-x_i^*)^2}{\beta_i x_i(1+\alpha_i x_i^*)} - a_i y_i^*\frac{\gamma_i(v-v^*)^2}{v^*(1+\gamma_i v)(1+\gamma_i v^*)}\right.\\
&+a_i y_i^*\left(4-\frac{x_i^*(1+\alpha_i x_i)}{x_i(1+\alpha_i x_i^*)} - \frac{y_i^* x_i(t-\tau_i)v(t-\tau_i)(1+\alpha_i x_i^*)(1+\gamma_i v^*)}{y_i x_i^* v^*(1+\alpha_i x_i(t-\tau_i))(1+\gamma_i v(t-\tau_i))} - \frac{v^* y_i(t-\omega_i)}{v y_i^*}\right.\\
&\left.\left.-\frac{1+\gamma_i v}{1+\gamma_i v^*} + \log\frac{y_i(t-\omega_i)x_i(t-\tau_i)v(t-\tau_i)(1+\alpha_i x_i)(1+\gamma_i v)}{y_i x_i v(1+\alpha_i x_i(t-\tau_i))(1+\gamma_i v(t-\tau_i))}\right)\right] \le 0.
\end{aligned}$$

Define the following functionals:

$$\overline{W_i} = \int_0^{\tau_i} H\left(\frac{x_i(t-\theta)v(t-\theta)((1+\alpha_i x_i^*)(1+\gamma_i v^*)}{x_i^* v^*(1+\alpha_i x_i(t-\theta))(1+\gamma_i v(t-\theta))}\right)d\theta, \qquad i=1,2.$$

$$\widetilde{W_i} = \int_0^{\omega_i} H\left(\frac{y_i(t-\theta)}{y_i^*}\right)d\theta, \qquad i=1,2.$$

Therefore, we construct Lyapunov functional as follows:

$$W_1 = W_* + \sum_{i=1}^{2}p_i y_i^* \overline{W}_i + \sum_{i=1}^{2}p_i y_i^* \widetilde{W}_i.$$

Then, we can easily drive that

$$\frac{dW_1}{dt} = \sum_{i=1}^{2} \frac{p_i}{a_i} \left[\frac{-d_i \beta_i' (x_i - x_i^*)^2}{\beta_i x_i (1+\alpha_i x_i^*)} - a_i y_i^* \frac{\gamma_i (v-v^*)^2}{v^*(1+\gamma_i v)(1+\gamma_i v^*)} \right.$$
$$+ a_i y_i^* \left(4 - \frac{x_i^*(1+\alpha_i x_i)}{x_i(1+\alpha_i x_i^*)} - \frac{y_i^* x_i(t-\tau_i) v(t-\tau_i)(1+\alpha_i x_i^*)(1+\gamma_i v^*)}{y_i x_i^* v^* (1+\alpha_i x_i(t-\tau_i))(1+\gamma_i v(t-\tau_i))} - \frac{v^* y_i(t-\omega_i)}{v y_i^*} \right.$$
$$\left.\left. - \frac{1+\gamma_i v}{1+\gamma_i v^*} + \log \frac{y_i(t-\omega_i) x_i(t-\tau_i) v(t-\tau_i)(1+\alpha_i x_i)(1+\gamma_i v)}{y_i x_i v (1+\alpha_i x_i(t-\tau_i))(1+\gamma_i v(t-\tau_i))} \right) \right] \leq 0.$$

Clearly, if $R_0 > 1$, then E_* exists and $\frac{dW}{dt} \leq 0$ for all $x_i, y_i, v > 0, i = 1, 2$, and $\frac{dW}{dt} = 0$ if and only if $x_i = x_i^*$, $y_i = y_i^*$, and $v = v^*$, which is the infected steady state E_*, then E_* is GAS.

3 Model with distributed-time delays

In this section we propose an HIV dynamics model with two classes of target cells and Crowley-Martin functional response. Two types of distributed-time delays are accounted into the model. The proposed model can be seen as a generalization of several HIV mathematical models presented in the literature.

$$\dot{x} = \lambda_i - d_i x_i - \frac{\beta_i x_i v}{(1+\alpha_i x_i)(1+\gamma_i v)}, \qquad i = 1, 2 \tag{33}$$

$$\dot{y_i} = \int_0^{h_i} f_i(\tau) \frac{\beta_i' x_i(t-\tau) v(t-\tau)}{(1+\alpha_i x_i(t-\tau))(1+\gamma_i v(t-\tau))} d\tau - a_i y_i, \qquad i = 1, 2 \tag{34}$$

$$\dot{v} = \sum_{i=1}^{2} p_i \int_0^{l_i} g_i(\omega) y_i (t-\omega) \, d\omega - r v, \tag{35}$$

where $\beta_i' < \beta_i$ and $p_i < k_i, i = 1, 2$. All the variables and other parameters of the model have the same meanings as given in the previous section. To account for the time lag between viral contacting a target cell and the production of new virus particles, two distributed intracellular delays are introduced. It is assumed that the target cells are contacted by the virus particles at time $t - \tau$ become infected cells at time t, where τ is a random variable with a probability distribution $f_i(\tau)$ over the interval $[0, h_i]$ and h_i is limit superior of this delay. On the other hand, it is assumed that, a cell infected at time $t - \omega$ starts to yield new infectious virus at time t where ω is distributed according to a probability distribution $g_i(\omega)$ over the interval $[0, l_i]$ and l_i is limit superior of this delay. The probability distribution functions $f_i(\tau) : [0, h_i] \to \mathbb{R}_+$ and $g_i(\omega) : [0, l_i] \to \mathbb{R}_+$ are integral functions with

$$\int_0^{h_i} f_i(\tau) d\tau = \int_0^{l_i} g_i(\omega) d\omega = 1, \quad i = 1, 2.$$

We put

$$D_{f_i,h_i}[x_i v] = \int_0^{h_i} f_i(\tau) \frac{x_i(t-\tau) v(t-\tau)}{(1+\alpha_i x_i(t-\tau))(1+\gamma_i v(t-\tau))} d\tau$$

and

$$D_{g_i,l_i}[y_i] = \int_0^{l_i} g_i(\omega) y_i(t-\omega) d\tau.$$

Then system (33)-(35) can be rewritten as:

$$\dot{x_i} = \lambda_i - d_i x_i - \frac{\beta_i x_i v}{(1+\alpha_i x_i)(1+\gamma_i v)}, \qquad i = 1, 2 \tag{36}$$

$$\dot{y_i} = \beta_i' D_{f_i,h_i}[x_i v] - a_i y_i, \qquad i = 1, 2 \tag{37}$$

$$\dot{v} = \sum_{i=1}^{2} p_i D_{g_i,l_i}[y_i] - r v. \tag{38}$$

Initial conditions

The initial conditions for system (36)-(38) take the form

$$\begin{aligned}&x_1(\theta)=\varphi_1(\theta),\ x_2(\theta)=\varphi_2(\theta), y_1(\theta)=\varphi_3(\theta),\ y_2(\theta)=\varphi_4(\theta), v(\theta)=\varphi_5(\theta),\\ &\varphi_i(\theta)\geq 0,\ \theta\in[-\ell,0),\ \varphi_j(0)>0,\ j=1,...,5,\end{aligned}\tag{39}$$

where, $(\varphi_1(\theta),\varphi_2(\theta),...,\varphi_5(\theta))\in C$ and $C=C([-\ell,0],\mathbb{R}^5_+)$ is the Banach space of continuous functions mapping the interval $[-\ell,0]$ into $\mathbb{R}^5_+$, where $\ell=\max\{h_1,h_2,l_1,l_2\}$. By the fundamental theory of functional differential equations [13], system (36)-(38) has a unique solution $(x_1(t),x_2(t),y_1(t),y_2(t),v(t))$ satisfying initial conditions (39).

3.1 Non-negativity and boundedness of solutions

In the following, we establish the non-negativity and boundedness of solutions of (36)-(38) with initial conditions (39).

Proposition 2. Let $X(t)=(x_1(t),x_2(t),y_1(t),y_2(t),v(t))^T$ be any solution of (36)-(38) satisfying the initial conditions (39), then $X(t)$ is non-negative for $t\geq 0$ and ultimately bounded.

Proof. Similar to the proof of Proposition 1, we have $x_i(t)>0$, $i=1,2$ for all $t\geq 0$. Further, from Eqs. (37) and (38) we have

$$\begin{aligned}y_i(t)&=y_i(0)e^{-a_it}+\beta_i'\int_0^t e^{-a_i(t-\eta)}\int_0^{h_i} f_i(\tau)\frac{x_i(\eta-\tau)v(\eta-\tau)}{(1+\alpha_ix_i(\eta-\tau))(1+\gamma_iv(\eta-\tau))}d\tau d\eta,\quad i=1,2\\ v(t)&=v(0)e^{-rt}+\sum_{i=1}^{2}p_i\int_0^t e^{-r(t-\eta)}\int_0^{l_i} g_i(\omega)y_i(\eta-\omega)d\omega d\eta,\end{aligned}$$

confirming that $y_i(t)\geq 0$, $i=1,2$, and $v(t)\geq 0$ for all $t\in[0,\ell]$. By a recursive argument, we obtain $y_i(t)\geq 0$, $i=1,2$, and $v(t)\geq 0$ for all $t\geq 0$.

Now we show the boundedness of the solutions of (36)-(38). Eqs. (36) implies that $\limsup_{t\to\infty} x_i(t)\leq x_i^0$, where $x_i^0=\lambda_i/d_i$, and thus $x_i(t)$ is ultimately bounded. It follows that

$$\int_0^{h_i} f_i(\tau)x_i(t-\tau)d\tau\leq x_i^0.$$

Let $Y_i(t)=\frac{\beta_i'}{\beta_i}\int_0^{h_i} f_i(\tau)x_i(t-\tau)d\tau+y_i(t)$, $i=1,2$. Then

$$\begin{aligned}\dot{Y}_i(t)=&\frac{\beta_i'}{\beta_i}\int_0^{h_i} f_i(\tau)\left(\lambda_i-d_ix_i(t-\tau)-\frac{\beta_ix_i(t-\tau)v(t-\tau)}{(1+\alpha_ix_i(t-\tau))(1+\gamma_iv(t-\tau))}\right)d\tau\\ &+\int_0^{h_i} f_i(\tau)\frac{\beta_i'x_i(t-\tau)v(t-\tau)}{(1+\alpha_ix_i(t-\tau))(1+\gamma_iv(t-\tau))}d\tau-a_iy_i(t),\\ \leq&\frac{\beta_i'\lambda_i}{\beta_i}-\sigma_iY_i(t),\end{aligned}$$

where $\sigma_i=\min\{d_i,a_i\}$. Hence $\limsup_{t\to\infty} Y_i(t)\leq L_i$, where $L_i=\lambda_i\beta_i'/\beta_i\sigma_i$. Since $\int_0^{h_i} f_i(\tau)x_i(t-\tau)d\tau>0$, we get $\limsup_{t\to\infty} y_i(t)\leq L_i$. On the other hand,

$$\begin{aligned}\dot{v}(t)&\leq\sum_{i=1}^{2}p_iL_i\int_0^{l_i} g_i(\omega)d\omega-rv\\ &=\sum_{i=1}^{2}p_iL_i-rv,\end{aligned}$$

then $\lim_{t\to\infty}\sup v(t)\leq L^*$, where $L^*=\sum_{i=1}^{2}\frac{p_iL_i}{r}$. Therefore, $X(t)$ is ultimately bounded.□

3.2 Steady States

It is clear that, system (36)-(38) has an uninfected steady state $E_0(x_1^0, x_2^0, y_1^0, y_2^0, v^0)$, where $x_i^0 = \frac{\lambda_i}{d_i}$, $y_i^0 = 0$, $i = 1,2$ and $v^0 = 0$. The system can also has another steady state which is the infected steady state $E_*(x_1^*, x_2^*, y_1^*, y_2^*, v^*)$, with coordinates if exist, they satisfy Eqs. (8)-(10). The basic reproduction number for system (36)-(38) is also given by Eq. (11).

Lemma 2. If $R_0 > 1$, then there exists a positive steady state E_*.

The proof is the same as given in Lemma 1.

3.3 Global stability analysis

In this section, we study the global stability of the uninfected and infected steady states of system (36)-(38). The strategy of the proofs is to use suitable Lyapunov functionals which are similar in nature to those used in [16] and [28].

Define

$$\delta_{f_i h_i}(\tau) = \int_\tau^{h_i} f_i(\sigma)d\sigma, \quad \delta_{g_i,l_i}(\omega) = \int_\omega^{l_i} g_i(\sigma)d\sigma \quad i = 1,2, \tag{40}$$

implies that

$$\delta_{f_i h_i}(0) = 1, \delta_{f_i h_i}(h_i) = 0, \qquad \frac{d\delta_{f_i h_i}(\tau)}{d\tau} = -f_i(\tau), \tag{41}$$

$$\delta_{g_i,l_i}(0) = 1, \delta_{g_i,l_i}(l_i) = 0, \qquad \frac{d\delta_{g_i l_i}(\omega)}{d\tau} = -g_i(\omega). \tag{42}$$

Also, for a continuous function x , we have that

$$\frac{d}{dt}\int_0^h \delta_{f,h}(\tau)R(x(t-\tau))d\tau = \int_0^h f(\tau)\left[R(x(t)) - R\left(x\left(t-\tau\right)\right)\right]d\tau.$$

First we prove the global stability of the uninfected steady state E_0 employing the method of Lyapunov functional.

Theorem 3. Consider the system (36)-(38), if $R_0 \leq 1$ then E_0 is GAS.

Proof. Define a Lyapunov functional W_0 as follows:

$$W_0 = \sum_{i=1}^{2}\frac{p_i}{a_i}\left[\frac{\beta_i' x_i^0}{\beta_i(1+\alpha_i x_i^0)}\left(\frac{x_i}{x_i^0} - 1 - \log\frac{x_i}{x_i^0}\right) + y_i + \beta_i'\int_0^{h_i}\delta_{f_i,h_i}(\tau)\frac{x_i(t-\tau)v(t-\tau)}{(1+\alpha_i x_i(t-\tau))(1+\gamma_i v(t-\tau))}d\tau \right.$$
$$\left. + a_i\int_0^{l_i}\delta_{g_i,l_i}(\omega)y_i(t-\omega)d\omega\right] + v. \tag{43}$$

We note that W_0 is defined and continuous for all $x_i, y_i > 0, i = 1,2$ and $v > 0$. Also, the global minimum $W_0 = 0$ occurs at the uninfected steady state E_0. The time derivative of W_0 along the solution of (36)-(38) is given by

$$\begin{aligned}\frac{dW_0}{dt} &= \sum_{i=1}^{2}\frac{p_i}{a_i}\left[\frac{\beta_i'}{\beta_i(1+\alpha_i x_i^0)}\left(1 - \frac{x_i^0}{x_i}\right)\left(\lambda_i - d_i x_i - \frac{\beta_i x_i v}{(1+\alpha_i x_i)(1+\gamma_i v)}\right)\right.\\
&+\beta_i' D_{f_i h_i}[x_i v] - a_i y_i + \int_0^{h_i} f_i(\tau)\left(\frac{\beta_i' x_i v}{(1+\alpha_i x_i)(1+\gamma_i v)} - \frac{\beta_i' x_i(t-\tau)v(t-\tau)}{(1+\alpha_i x_i(t-\tau))(1+\gamma_i v(t-\tau))}\right)d\tau\\
&\left.+a_i\int_0^{l_i} g_i(\omega)(y_i - y_i(t-\omega))d\omega + a_i D_{g_i l_i}[y_i]\right] - rv\\
&= \sum_{i=1}^{2}\frac{p_i}{a_i}\left[\frac{-\beta_i'\lambda_i\left(x_i - x_i^0\right)^2}{\beta_i x_i x_i^0(1+a_i x_i^0)} - \frac{\beta_i' x_i v}{(1+\alpha_i x_i^0)\left(1+\alpha_i x_i\right)(1+\gamma_i v)}\right.\\
&\left.+\frac{\beta_i' x_i^0 v}{(1+\alpha_i x_i^0)\left(1+\alpha_i x_i\right)(1+\gamma_i v)} + \frac{\beta_i' x_i v}{(1+\alpha_i x_i)(1+\gamma_i v)}\right] - rv\end{aligned} \tag{44}$$

Collecting terms of (44) we obtain

$$\begin{aligned}
\frac{dW_0}{dt} &= \sum_{i=1}^{2} \frac{p_i}{a_i}\left[\frac{-\beta_i'\lambda_i\left(x_i-x_i^0\right)^2}{\beta_i x_i x_i^0(1+a_i x_i^0)} + \frac{\beta_i' x_i^0 v + \alpha_i x_i^0 \beta_i' x_i v}{\left(1+\alpha_i x_i^0\right)(1+\alpha_i x_i)(1+\gamma_i v)}\right] - rv \\
&= \sum_{i=1}^{2} \frac{p_i}{a_i}\left[\frac{-\beta_i'\lambda_i\left(x_i-x_i^0\right)^2}{\beta_i x_i x_i^0(1+a_i x_i^0)} + \frac{\beta_i' x_i^0 v}{\left(1+\alpha_i x_i^0\right)(1+\gamma_i v)}\right] - rv \\
&= -\sum_{i=1}^{2} \frac{p_i}{a_i}\frac{\beta_i'\lambda_i\left(x_i-x_i^0\right)^2}{\beta_i x_i x_i^0(1+a_i x_i^0)} + \sum_{i=1}^{2}\frac{p_i}{a_i}\frac{\beta_i' x_i^0 v}{\left(1+\alpha_i x_i^0\right)(1+\gamma_i v)} - rv \\
&= -\sum_{i=1}^{2} \frac{p_i\beta_i'\lambda_i\left(x_i-x_i^0\right)^2}{a_i\beta_i x_i x_i^0(1+a_i x_i^0)} + \sum_{i=1}^{2}\frac{rR_i(v+\gamma_i v^2-\gamma_i v^2)}{(1+\gamma_i v)} - rv \\
&= -\sum_{i=1}^{2} \frac{p_i\beta_i'\lambda_i\left(x_i-x_i^0\right)^2}{a_i\beta_i x_i x_i^0(1+a_i x_i^0)} + \sum_{i=1}^{2} rR_i v - \sum_{i=1}^{2}\frac{rR_i\gamma_i v^2}{(1+\gamma_i v)} - rv \\
&= -\sum_{i=1}^{2} \frac{p_i\beta_i'\lambda_i\left(x_i-x_i^0\right)^2}{a_i\beta_i x_i x_i^0(1+a_i x_i^0)} - \sum_{i=1}^{2}\frac{rR_i\gamma_i v^2}{(1+\gamma_i v)} + (R_0-1)rv. \qquad (45)
\end{aligned}$$

It can be seen that, if $R_0 \le 1$ then $\frac{dW_0}{dt} \le 0$ for all $x_i, v > 0$, $i = 1,2$. By Theorem 5.3.1 in [13], the solutions of system (36)-(38) limit to M, the largest invariant subset of $\left\{\frac{dW_0}{dt} = 0\right\}$. Clearly, it follows from (45) that $\frac{dW_0}{dt} = 0$ if and only if $x_i = x^0$, $i = 1,2$ and $v = 0$. Noting that M is invariant, for each element of M we have $v = 0$, then $\dot{v} = 0$. From Eq. (35) we drive that

$$0 = \dot{v} = \sum_{i=1}^{2} p_i \int_{\omega}^{l_i} g_i(\omega) y_i(t-\omega)d\omega.$$

Since $y_i\left(t-\theta\right) \ge 0$ for all $\theta \in [0,\ell]$, then $\sum_{i=1}^{2} p_i \int_{\omega}^{l_i} g_i(\omega)y_i(t-\omega)d\omega = 0$ if and only if $y_i(t-\omega) = 0$, $i = 1,2$. Hence $\frac{dW_0}{dt} = 0$ if and only if $x_i = x^0$, $y_i = 0$, $i = 1,2$ and $v = 0$. From LaSalle's invariance principle, E_0 is GAS.

Theorem 4. If $R_0 > 1$, then E_* is GAS.

Proof. First, we consider the system (26)-(28), and define Lyapunov functional W_* as follows:

$$W_* = \sum_{i=1}^{2}\frac{p_i}{a_i}\left[\frac{\beta_i'}{\beta_i}\left(x_i - x_i^* - \int_{x_i^*}^{x_i}\frac{x_i^*(1+\alpha_i\eta)}{\eta(1+\alpha_i x_i^*)}d\eta\right) + y_i^*\left(\frac{y_i}{y_i^*} - 1 - \log\frac{y_i}{y_i^*}\right)\right] + \left(\frac{v}{v^*} - 1 - \log\frac{v}{v^*}\right). \qquad (46)$$

By calculating the time derivative of W_* along (26)-(28) we get

$$\begin{aligned}
\nabla W_* \cdot G(X) &= \sum_{i=1}^{2}\frac{p_i}{a_i}\left[\frac{\beta_i'}{\beta_i}\left(1 - \frac{x_i^*(1+\alpha_i x_i)}{x_i(1+\alpha_i x_i^*)}\right)\left(\lambda_i - d_i x_i - \frac{\beta_i x_i v}{(1+\alpha_i x_i)(1+\gamma_i v)}\right)\right. \\
&\left.+\left(1-\frac{y_i^*}{y_i}\right)\left(\frac{\beta_i' x_i v}{(1+\alpha_i x_i)(1+\gamma_i v)} - a_i y_i\right)\right] + \left(1-\frac{v^*}{v}\right)\left(\sum_{i=1}^{2} p_i y_i - rv\right).
\end{aligned}$$

Similar to proof of Theorem 2, one can show

$$\begin{aligned}
\nabla W_* \cdot G(X) &= \sum_{i=1}^{2}\frac{p_i}{a_i}\left[\frac{-d_i\beta_i'(x_i-x_i^*)^2}{\beta_i x_i(1+\alpha_i x_i^*)} - a_i y_i^*\left(\frac{\gamma_i(v-v^*)^2}{v^*(1+\gamma_i v)(1+\gamma_i v^*)}\right)\right. \\
&\left.+a_i y_i^*\left(4 - \frac{x_i^*(1+\alpha_i x_i)}{x_i(1+\alpha_i x_i^*)} - \frac{y_i^*}{y_i}\frac{x_i v(1+\alpha_i x_i^*)(1+\gamma_i v^*)}{x_i^* v^*(1+\alpha_i x_i)(1+\gamma_i v)} - \frac{y_i v^*}{y_i^* v} - \frac{1+\gamma_i v}{1+\gamma_i v^*}\right)\right] \le 0.
\end{aligned}$$

Now, we compute the time derivative of W_* along the solution of the system (36)-(38) as:

$$\begin{aligned}
\frac{dW_*}{dt} &= \sum_{i=1}^{2}\frac{p_i}{a_i}\left[\frac{\beta_i'}{\beta_i}\left(1-\frac{x_i^*(1+\alpha_i x_i)}{x_i(1+\alpha_i x_i^*)}\right)\left(\lambda_i - d_i x_i - \frac{\beta_i x_i v}{(1+\alpha_i x_i)(1+\gamma_i v)}\right)\right.\\
&\left.+\left(1-\frac{y_i^*}{y_i}\right)\left(\beta_i' D_{f_i,h_i}[x_i v] - a_i y_i + \frac{\beta_i' x_i v}{(1+\alpha_i x_i)(1+\gamma_i v)} - \frac{\beta_i' x_i v}{(1+\alpha_i x_i)(1+\gamma_i v)}\right)\right]\\
&+\left(1-\frac{v^*}{v}\right)\left(\sum_{i=1}^{2} p_i D_{g_i,l_i}[y_i] - rv + \sum_{i=1}^{2} p_i y_i - \sum_{i=1}^{2} p_i y_i\right)\\
&= \left\{\sum_{i=1}^{2}\frac{p_i}{a_i}\left[\frac{\beta_i'}{\beta_i}\left(1-\frac{x_i^*(1+\alpha_i x_i)}{x_i(1+\alpha_i x_i^*)}\right)\left(\lambda_i - d_i x_i - \frac{\beta_i x_i v}{(1+\alpha_i x_i)(1+\gamma_i v)}\right)\right.\right.\\
&\left.\left.+\left(1-\frac{y_i^*}{y_i}\right)\left(\frac{\beta_i' x_i v}{(1+\alpha_i x_i)(1+\gamma_i v)} - a_i y_i\right)\right] + \left(1-\frac{v^*}{v}\right)\left(\sum_{i=1}^{2} p_i y_i - rv\right)\right\}\\
&+\sum_{i=1}^{2}\frac{p_i}{a_i}\left(1-\frac{y_i^*}{y_i}\right)\left(\beta_i' D_{f_i h_i}[x_i v] - \frac{\beta_i' x_i v}{(1+\alpha_i x_i)(1+\gamma_i v)}\right) + \left(1-\frac{v^*}{v}\right)\left(\sum_{i=1}^{2} p_i D_{g_i,l_i}[y_i] - \sum_{i=1}^{2} p_i y_i\right).
\end{aligned}$$

Using the infected steady state conditions, we get

$$\begin{aligned}
\frac{dW_*}{dt} &= \nabla W_* \cdot G(X) + \sum_{i=1}^{2}\frac{p_i}{a_i} a_i y_i^*\left[\left(1-\frac{y_i^*}{y_i}\right)\left(\frac{D_{f_i h_i}[x_i v](1+\alpha_i x_i^*)(1+\gamma_i v^*)}{x_i^* v^*} - \frac{x_i v(1+\alpha_i x_i^*)(1+\gamma_i v^*)}{x_i^* v^*(1+\alpha_i x_i)(1+\gamma_i v)}\right)\right.\\
&\left.+\left(1-\frac{v^*}{v}\right)\left(\frac{D_{g_i,l_i}[y_i]}{y_i^*} - \frac{y_i}{y_i^*}\right)\right]\\
&= \sum_{i=1}^{2}\frac{p_i}{a_i}\left[\frac{-d_i\beta_i'(x_i-x_i^*)^2}{\beta_i x_i(1+\alpha_i x_i^*)} - a_i y_i^*\left(\frac{\gamma_i(v-v^*)^2}{v^*(1+\gamma_i v)(1+\gamma_i v^*)}\right)\right.\\
&+a_i y_i^*\left(4 - \frac{x_i^*(1+\alpha_i x_i)}{x_i(1+\alpha_i x_i^*)} - \frac{y_i^*}{y_i}\frac{x_i v(1+\alpha_i x_i^*)(1+\gamma_i v^*)}{x_i^* v^*(1+\alpha_i x_i)(1+\gamma_i v)} - \frac{y_i v^*}{y_i^* v} - \frac{1+\gamma_i v}{1+\gamma_i v^*}\right)\\
&+ a_i y_i^*\left(\frac{D_{f_i h_i}[x_i v](1+\alpha_i x_i^*)(1+\gamma_i v^*)}{x_i^* v^*} - \frac{x_i v(1+\alpha_i x_i^*)(1+\gamma_i v^*)}{x_i^* v^*(1+\alpha_i x_i)(1+\gamma_i v)}\right.\\
&-\frac{y_i^* D_{f_i h_i}[x_i v](1+\alpha_i x_i^*)(1+\gamma_i v^*)}{y_i x_i^* v^*} + \frac{y_i^* x_i v(1+\alpha_i x_i^*)(1+\gamma_i v^*)}{y_i x_i^* v^*(1+\alpha_i x_i)(1+\gamma_i v)}\\
&\left.\left.+\frac{D_{g_i,l_i}[y_i]}{y_i^*} - \frac{y_i}{y_i^*} - \frac{v^* D_{g_i,l_i}[y_i]}{v y_i^*} + \frac{v^* y_i}{v y_i^*}\right)\right]\\
&= \sum_{i=1}^{2}\frac{p_i}{a_i}\left[\frac{-d_i\beta_i'(x_i-x_i^*)^2}{\beta_i x_i(1+\alpha_i x_i^*)} - a_i y_i^*\left(\frac{\gamma_i(v-v^*)^2}{v^*(1+\gamma_i v)(1+\gamma_i v^*)}\right)\right.\\
&+a_i y_i^*\int_0^{h_i}\int_0^{l_i} f_i(\tau) g_i(\omega)\left(4 - \frac{x_i^*(1+\alpha_i x_i)}{x_i(1+\alpha_i x_i^*)} - \frac{y_i^* x_i(t-\tau)v(t-\tau)(1+\alpha_i x_i^*)(1+\gamma_i v^*)}{y_i x_i^* v^*(1+\alpha_i x_i(t-\tau))(1+\gamma_i v(t-\tau))} - \frac{v^* y_i(t-\omega)}{v y_i^*}\right.\\
&\left.\left.-\frac{1+\gamma_i v}{1+\gamma_i v^*} + \log\frac{y_i(t-\omega)x_i(t-\tau)v(t-\tau)(1+\alpha_i x_i)(1+\gamma_i v)}{y_i x_i v(1+\alpha_i x_i(t-\tau))(1+\gamma_i v(t-\tau))}\right)d\omega d\tau\right]\\
&+\sum_{i=1}^{2}\frac{p_i}{a_i} a_i y_i^*\left[\int_0^{h_i} f(\tau)\left(\frac{x_i(t-\tau)v(t-\tau)(1+\alpha_i x_i^*)(1+\gamma_i v^*)}{x_i^* v^*(1+\alpha_i x_i(t-\tau))(1+\gamma_i v(t-\tau))} - \frac{x_i v(1+\alpha_i x_i^*)(1+\gamma_i v^*)}{x_i^* v^*(1+\alpha_i x_i)(1+\gamma_i v)}\right.\right.\\
&\left.\left.-\log\frac{x_i(t-\tau)v(t-\tau)(1+\alpha_i x_i)(1+\gamma_i v)}{x_i v(1+\alpha_i x_i(t-\tau))(1+\gamma_i v(t-\tau))}\right)d\tau\right]\\
&+\sum_{i=1}^{2} p_i y_i^*\int_0^{l_i} g(\omega)\left(\frac{y_i(t-\omega)}{y_i^*} - \frac{y_i}{y_i^*} - \log\frac{y_i(t-\omega)}{y_i}\right)d\omega. \qquad (47)
\end{aligned}$$

We define the following functionals:

$$\overline{W}_i = \int_0^{h_i}\delta_{f_i,h_i}(\tau) H\left(\frac{x_i(t-\tau)v(t-\tau)((1+\alpha_i x_i^*)(1+\gamma_i v^*)}{x_i^* v^*(1+\alpha_i x_i(t-\tau))(1+\gamma_i v(t-\tau))}\right)d\tau, \qquad i=1,2$$

$$\widetilde{W}_i = \int_0^{l_i} \delta_{g_i,l_i}(\omega) H\left(\frac{y_i(t-\omega)}{y_i^*}\right) d\omega, \qquad i = 1,2.$$

Therefore, we construct Lyapunov functional as follows:

$$W_1 = W_* + \sum_{i=1}^{2} p_i y_i^* \overline{W}_i + \sum_{i=1}^{2} p_i y_i^* \widetilde{W}_i$$

$$\begin{aligned}
\frac{dW_1}{dt} &= \sum_{i=1}^{2} \frac{p_i}{a_i} \left[\frac{-d_i \beta_i' (x_i - x_i^*)^2}{\beta_i x_i (1+\alpha_i x_i^*)} - a_i y_i^* \left(\frac{\gamma_i (v - v^*)^2}{v^* (1+\gamma_i v)(1+\gamma_i v^*)} \right) \right. \\
&+ a_i y_i^* \int_0^{h_i} \int_0^{l_i} f_i(\tau) g_i(\omega) \left(4 - \frac{x_i^*(1+\alpha_i x_i)}{x_i(1+\alpha_i x_i^*)} - \frac{y_i^* x_i(t-\tau) v(t-\tau)(1+\alpha_i x_i^*)(1+\gamma_i v^*)}{y_i x_i^* v^* (1+\alpha_i x_i(t-\tau))(1+\gamma_i v(t-\tau))} - \frac{v^* y_i(t-\omega)}{v y_i^*} \right. \\
&\left. \left. - \frac{1+\gamma_i v}{1+\gamma_i v^*} + \log \frac{y_i(t-\omega) x_i(t-\tau) v(t-\tau)(1+\alpha_i x_i)(1+\gamma_i v)}{y_i x_i v (1+\alpha_i x_i(t-\tau))(1+\gamma_i v(t-\tau))} \right) d\omega d\tau \right] \le 0.
\end{aligned}$$

It is clear that $\frac{dW_1}{dt} \le 0$ for all $x_i, y_i > 0, i = 1,2$ and $v > 0$, and $\frac{dW_1}{dt} = 0$ at the infected state E_*, then $E_*$is GAS.

4 Conclusion

In this paper, we have proposed two HIV dynamics models describing the interaction of the HIV with two classes of target cells, $CD4^+$ T cells and macrophages taking into account the Crowley-Martin functional response. We have incorporated two types of discrete delays and two types of distributed delays in the first and second model, respectively. The global stability of the uninfected and infected steady states of the models has been established by using suitable Lyapunov functionals and LaSalle invariant principle. We have proven that, if the basic reproduction number R_0 is less than or equal unity, then the uninfected steady state is GAS and if $R_0 > 1$, then the infected steady state exists and it is GAS.

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Accelerated Newton-GPSS methods for systems of nonlinear equations☆

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Abstract: By utilising generalized positive-definite and skew-Hermitian splitting (GPSS) iteration as the inner solver of inexact Newton method, a class of inexact Newton-GPSS methods for solving systems of nonlinear equations with positive Jacobian matrices are proposed. We discuss the local and semilocal convergence properties under some proper assumptions. Moreover, an accelerated Newton-GPSS method is established and its convergence behavior is analyzed. Numerical results demonstrate the robustness of our methods.

Key words: system of nonlinear equations, generalized positive-definite and skew-Hermitian splitting, inexact Newton method, convergence property.
Mathematics Subject Classfications (2010): 65H10, 65F10, 65N22.

1. Introduction

Consider the solution of large sparse systems of nonlinear equations:

$$F(x) = 0, \tag{1}$$

where $F : \mathbb{D} \subset \mathbb{C}^n \to \mathbb{C}^n$ is a continuously differentiable function defined on an open convex subset of complex linear space $\mathbb{C}^n$. The Jacobian matrix $F'(x)$ is sparse, non-Hermitian, and positive definite. This kind of nonlinear equations can be derived in many practical problems, see [1–3].

As it is well known, the most popular iterative method for solving (1) is the Newton method [2, 3], which can be written as:

$$x^{(k+1)} = x^{(k)} - F'(x^{(k)})^{-1} F(x^{(k)}), \quad k = 0, 1, 2, \ldots,$$

where $x^{(0)} \in \mathbb{D}$ is a given initial vector. Notice that we must solve the Newton equation

$$F'(x^{(k)}) s^{(k)} = -F(x^{(k)}), \quad \text{with} \quad x^{(k+1)} := x^{(k)} + s^{(k)}, \tag{2}$$

at the k-th iteration step.

When the scale of the problem become large, we often choose iterative methods [4, 5] to solve the Newton equation (2), which results in the following inexact Newton method [6–8] for solving (1):

$$x^{(k+1)} = x^{(k)} + s^{(k)}, \quad \text{with} \quad F'(x^{(k)}) s^{(k)} = -F(x^{(k)}) + r^{(k)},$$

where $r^{(k)}$ is a residual yielded by the inner iteration. Newton-Krylov subspace methods [9, 10], which use the Krylov subspace methods for solving the Newton equation (2) , have been widely used.

To solve (2) by iterative methods, we need efficient splittings of the coefficient matrix [4]. In [11], Bai, Golub and Ng first presented a Hermitian and skew-Hermitian splitting (HSS) iterative method for non-Hermitian positive definite linear systems. Subsequently, some HSS-based iterative methods [12–18] were further studied to improve the robustness of the HSS method. Furthermore, a generalized positive-definite and skew-Hermitian splitting (GPSS) scheme [19] was proposed by Cao *et al.* for solving non-Hermitian positive definite linear systems. Theoretical analysis shows that the GPSS iterative method preserves all properties of both positive-definite and skew-Hermitian splitting (PSS) [12] and generalized HSS (GHSS) methods [16].

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Recently, Bai and Guo established the Newton-HSS method [20] which use the HSS method as the inner solver to solve (1); see also [21–25]. Numerical results have shown that the Newton-HSS method considerably outperforms the Newton-Krylov subspace methods from aspects of iteration steps and CPU time. Subsequently, Guo *et al.* [22] analyzed the semilocal convergence of the above Newton-HSS method. Moreover, Yang *et al.* in [26] and [27] established Newton-MHSS and Newton-PSS methods, respectively, for solving systems of nonlinear equations.

In this paper, the Newton-GPSS method which use the GPSS iteration as the inner solver of inexact Newton method, is presented for solving (1). We discuss the local and semilocal convergence properties under some proper assumptions. Moreover, based on the successive-overrelaxation (SOR) acceleration, an accelerated Newton-GPSS method is established and its convergence behavior is analyzed. Finally, numerical results illustrate that our two methods considerably outperform the Newton-HSS method from aspects of iteration steps and CPU time.

This paper is organized as follows. In Section 2, we introduce the Newton-GPSS method. In Section 3, the local and semilocal convergence properties of the Newton-GPSS method are discussed under some proper assumptions. In Section 4, we establish an accelerated Newton-GPSS method and analyze its convergence behavior. Numerical examples are given in Section 5 to demonstrate the effectiveness of our methods with comparison to the Newton-HSS method. Finally, in Section 6, some short conclusions are given.

2. The Newton-GPSS methods

First, let us review some HSS-based iterative methods([11, 12, 16, 19]) for solving linear system

$$Ax = b, \quad A \in \mathbb{C}^{n\times n}, \quad x, b \in \mathbb{C}^n, \tag{3}$$

where $A \in \mathbb{C}^{n\times n}$ is a non-Hermitian positive definite matrix and $b \in \mathbb{C}^n$ is a given vector.

Since the matrix A naturally possesses the Hermitian and skew-Hermitian splitting (HSS)

$$A = H + S,$$

where

$$H = \frac{1}{2}(A + A^*) \quad \text{and} \quad S = \frac{1}{2}(A - A^*),$$

Bai *et al.* used the following HSS iteration method [11] for solving (3).

Algorithm 1. *(The HSS iteration method)*

Given an initial guess $x^{(0)} \in \mathbb{C}^n$, compute $x^{(\ell+1)}$ for $\ell = 0, 1, 2, \ldots$ using the following iteration scheme until $\{x^{(\ell)}\}$ satisfies the stopping criterion:

$$\begin{cases} (\alpha I + H)x^{(\ell+\frac{1}{2})} = (\alpha I - S)x^{(\ell)} + b, \\ (\alpha I + S)x^{(\ell+1)} = (\alpha I - H)x^{(\ell+\frac{1}{2})} + b, \end{cases} \tag{4}$$

where α is a given positive constant and I denotes the identity matrix.

To improve the robustness of the HSS method, Bai *et al.* proposed a positive-definite and skew-Hermitian splitting (PSS) iterative method [12] for solving (3). Since A possesses a splitting of the form

$$A = P + S,$$

where $P \in \mathbb{C}^{n\times n}$ and $S \in \mathbb{C}^{n\times n}$ are positive definite matrix and skew-Hermitian matrix, respectively, the PSS iterative method can be obtained as follows:

Algorithm 2. *(The PSS iteration method)*

Given an initial guess $x^{(0)} \in \mathbb{C}^n$, compute $x^{(\ell+1)}$ for $\ell = 0, 1, 2, \ldots$ using the following iteration scheme until $\{x^{(\ell)}\}$ satisfies the stopping criterion:

$$\begin{cases} (\alpha I + P)x^{(\ell+\frac{1}{2})} = (\alpha I - S)x^{(\ell)} + b, \\ (\alpha I + S)x^{(\ell+1)} = (\alpha I - P)x^{(\ell+\frac{1}{2})} + b, \end{cases} \tag{5}$$

where α is a given positive constant and I denotes the identity matrix.

Hereafter, Benzi proposed a generalized HSS (GHSS) scheme [16] for solving (3). Because we can split H into the sum of two Hermitian positive semidefinite matrices:

$$H = G + K,$$

where K is of simple form (e.g., diagonal or block diagonal with blocks of small size), we get the following splitting

$$A = G + (S + K).$$

Then the GHSS iteration method was proposed as follows:

Algorithm 3. *(The GHSS iteration method)*

Given an initial guess $x^{(0)} \in \mathbb{C}^n$, compute $x^{(\ell+1)}$ for $\ell = 0,1,2,\ldots$ using the following iteration scheme until $\{x^{(\ell)}\}$ satisfies the stopping criterion:

$$\begin{cases} (\alpha I + G)x^{(\ell+\frac{1}{2})} = (\alpha I - S - K)x^{(\ell)} + b, \\ (\alpha I + S + K)x^{(\ell+1)} = (\alpha I - G)x^{(\ell+\frac{1}{2})} + b, \end{cases} \tag{6}$$

where α is a given positive constant and I denotes the identity matrix.

Recently, motivated by the PSS and the GHSS method, Cao *et al.* proposed a generalized positive-definite and skew-Hermitian splitting (GPSS) scheme [19] for solving (3). In fact, A can be split into

$$A = P_1 + P_2, \tag{7}$$

where P_1 and P_2 are positive definite matrices. The corresponding alternating iterative scheme, called the GPSS iterative method, can be described as follows:

Algorithm 4. *(The GPSS iteration method)*

Given an initial guess $x^{(0)} \in \mathbb{C}^n$, compute $x^{(\ell+1)}$ for $\ell = 0,1,2,\ldots$ using the following iteration scheme until $\{x^{(\ell)}\}$ satisfies the stopping criterion:

$$\begin{cases} (\alpha I + P_1)x^{(\ell+\frac{1}{2})} = (\alpha I - P_2)x^{(\ell)} + b, \\ (\alpha I + P_2)x^{(\ell+1)} = (\alpha I - P_1)x^{(\ell+\frac{1}{2})} + b, \end{cases} \tag{8}$$

where α is a given positive constant and I denotes the identity matrix.

Two typical choices of splitting (7) can be

$$P_1 = D + 2L_G,\ P_2 = K + L_G^* - L_G + S, \tag{9}$$

or

$$P_1 = D + 2L_G^*,\ P_2 = K + L_G - L_G^* + S, \tag{10}$$

where D and L_G are the diagonal matrix and the strictly lower triangular matrix of G, respectively. Other practical choices of P_1 and P_2 can be also obtained from practical problems.

Cao *et al.* in [19] shows that the GPSS iterative method preserves all properties of both PSS and GHSS method. One advantage of the GPSS scheme consists in the fact that the solution of systems with coefficient matrix $\alpha I + P_2$ by inner iterations is made easier, since this matrix is more diagonally dominant than $\alpha I + S$ used in the PSS iteration.

The above GPSS iteration method can be equivalently rewritten as

$$x^{(\ell+1)} = \Gamma(\alpha)x^{(\ell)} + Q(\alpha)b = \Gamma(\alpha)^{\ell+1}x^{(0)} + \sum_{j=0}^{\ell}\Gamma(\alpha)^j Q(\alpha)b, \quad \ell = 0,1,2\ldots, \tag{11}$$

where

$$\Gamma(\alpha) = (\alpha I + P_2)^{-1}(\alpha I - P_1)(\alpha I + P_1)^{-1}(\alpha I - P_2),\ \ Q(\alpha) = 2\alpha(\alpha I + P_2)^{-1}(\alpha I + P_1)^{-1}. \tag{12}$$

Here, $\Gamma(\alpha)$ is the iteration matrix of the GPSS method. The unconditional convergence property of the GPSS iteration can be described in the following theorem:

Theorem 1. *[19] Let $A \in \mathbb{C}^{n\times n}$ be a positive definite matrix and $A = P_1 + P_2$, where P_1 and P_2 are also positive definite matrices. Let $\rho(\Gamma(\alpha))$ be the spectral radius of the GPSS iteration matrix. Then it holds that*

$$\rho(\Gamma(\alpha)) \le \|(\alpha I - P_1)(\alpha I + P_1)^{-1}\|_2 \|(\alpha I - P_2)(\alpha I + P_2)^{-1}\|_2 < 1. \tag{13}$$

i.e., the GPSS iteration unconditionally converges to the exact solution of (3).

Now, by making use of the Newton iteration (2) as the outer iteration and the GPSS iteration as the inner iteration, we can establish the following Newton-GPSS method for solving (1):

Algorithm 5. *(The Newton-GPSS method)*

Let $F : \mathbb{D} \subset \mathbb{C}^n \to \mathbb{C}^n$ be a continuously differentiable function with the positive definite Jacobian matrix $F'(x)$ at any $x \in \mathbb{D}$. Let $F'(x) = P_1(x) + P_2(x)$, where $P_1(x)$ and $P_2(x)$ are also positive-definite matrices.

1. Given an initial guess $x^{(0)} \in \mathbb{D}$, and positive constants α and tol, *and a positive integer sequence $\{\ell_k\}_{k=0}^{\infty}$.*

2. For $k = 0, 1, \ldots$ until $\|F(x^{(k)})\|_2 \le \mathrm{tol}\|F(x^{(0)})\|_2$ do:

2.1. Set $s^{(k,0)} := \mathbf{0}$.

2.2. For $\ell = 0, 1, 2, \ldots, \ell_k - 1$, apply the GPSS method:

$$\begin{cases} (\alpha I + P_1(x^{(k)}))s^{(k,\ell+\frac{1}{2})} = (\alpha I - P_2(x^{(k)}))s^{(k,\ell)} - F(x^{(k)}), \\ (\alpha I + P_2(x^{(k)}))s^{(k,\ell+1)} = (\alpha I - P_1(x^{(k)}))s^{(k,\ell+\frac{1}{2})} - F(x^{(k)}), \end{cases} \tag{14}$$

and obtain $s^{(k,\ell_k)}$ such that

$$\|F(x^{(k)}) + F'(x^{(k)})s^{(k,\ell_k)}\|_2 \le \eta_k \|F(x^{(k)})\|_2 \quad \text{for some } \eta_k \in [0,1).$$

2.3. Set

$$x^{(k+1)} = x^{(k)} + s^{(k,\ell_k)}.$$

Due to (11), the Newton-GPSS method can be rewritten as

$$x^{(k+1)} = x^{(k)} - \sum_{j=0}^{\ell_k - 1} \Gamma(\alpha; x^{(k)})^j Q(\alpha; x^{(k)}) F(x^{(k)}), \quad k = 0, 1, 2 \ldots, \tag{15}$$

where

$$\begin{aligned} \Gamma(\alpha; x) &= (\alpha I + P_2(x))^{-1}(\alpha I - P_1(x))(\alpha I + P_1(x))^{-1}(\alpha I - P_2(x)), \\ Q(\alpha; x) &= 2\alpha(\alpha I + P_2(x))^{-1}(\alpha I + P_1(x))^{-1}. \end{aligned} \tag{16}$$

The Jacobian matrix $F'(x)$ has the splitting

$$F'(x) = M(\alpha; x) - N(\alpha; x),$$

where

$$M(\alpha; x) = \frac{1}{2\alpha}(\alpha I + P_1(x))(\alpha I + P_2(x)), \;\; N(\alpha; x) = \frac{1}{2\alpha}(\alpha I - P_1(x))(\alpha I - P_2(x)). \tag{17}$$

Then it holds that

$$\Gamma(\alpha; x) = M(\alpha; x)^{-1} N(\alpha; x), \;\; M(\alpha; x) = Q(\alpha; x)^{-1}, \;\; F'(x)^{-1} = (I - \Gamma(\alpha; x))^{-1} Q(\alpha; x). \tag{18}$$

Hence, the Newton-GPSS method can be equivalently express as

$$x^{(k+1)} = x^{(k)} - (I - \Gamma(\alpha; x^{(k)})^{\ell_k}) F'(x^{(k)})^{-1} F(x^{(k)}) = x^{(k)} - F'(x^{(k)})^{-1}(F(x^{(k)}) - r(\alpha; x^{(k)}, \ell_k)), \;\; k = 0, 1, 2 \ldots, \tag{19}$$

where

$$r(\alpha; x, \ell) := F'(x)\Gamma(\alpha; x)^{\ell} F'(x)^{-1} F(x).$$

3. Local and semilocal convergence properties of the Newton-GPSS method

For any $x \in \mathbb{C}^n$ and $X \in \mathbb{C}^{n\times n}$, the vector norm and the induced matrix norm can be defined by

$$\|x\| := \|(\alpha I + P_2(x^\star))x\|_2, \ \ \|X\| := \|(\alpha I + P_2(x^\star))X(\alpha I + P_2(x^\star))^{-1}\|_2,$$

where $x^\star \in \mathbb{D}$ is the zero point of (1).

Obviously, we can obtain the following estimate from (16) and Theorem 1:

$$\|\Gamma(\alpha; x^\star)\| = \|(\alpha I - P_1(x^\star))(\alpha I + P_1(x^\star))^{-1}(\alpha I - P_2(x^\star))(\alpha I + P_2(x^\star))^{-1}\|_2 < 1.$$

In Theorem 11.1.5 of [3], the authors give a local convergence property about an inexact Newton method which uses a general splitting iteration scheme as the inner solver. As a special case, we can immediately obtain the following local convergence theorem of the Newton-GPSS method.

Theorem 2. *Let $F : \mathbb{D} \subset \mathbb{C}^n \to \mathbb{C}^n$ be G-differentiable on an open neighborhood $\mathbb{N}^\star \subset \mathbb{D}$ of a point $x^\star \in \mathbb{D}$ which satisfies $F(x^\star) = 0$, and $F'(x)$ is continuous, positive definite, non-Hermitian on $\mathbb{N}^\star$. Then there exists an open neighborhood $\mathbb{N} \subset \mathbb{N}^\star$ of $x^\star$ such that for any $x^{(0)} \in \mathbb{N}$ and any sequence of positive integers ℓ_k, $k = 0,1,2\ldots$, the iteration sequence $\{x^{(k)}\}_{k=0}^{\infty}$ generated by the Newton-GPSS method is well-defined and convergent to $x^\star$. Moreover, it holds that*

$$\lim_{k\to\infty}\sup \|x^{(k)} - x^\star\|^{\frac{1}{k}} \le \rho(\Gamma(\alpha; x^\star))^{\ell_0}, \quad \text{with} \quad \ell_0 = \lim_{k\to\infty}\inf \ell_k;$$

in particular, if $\lim_{k\to\infty} \ell_k = +\infty$, then the rate of convergence is R-superlinear, i.e.,

$$\lim_{k\to\infty}\sup \|x^{(k)} - x^\star\|^{\frac{1}{k}} = 0.$$

Proof. We can obtain straightforward from Theorem 11.1.5 of [3]. □

Next we establish the following more exact local convergence property for the Newton-GPSS method.

Theorem 3. *Let $F : \mathbb{D} \subset \mathbb{C}^n \to \mathbb{C}^n$ be G-differentiable on an open neighborhood $\mathbb{N}^\star \subset \mathbb{D}$ of a point $x^\star \in \mathbb{D}$ which satisfies $F(x^\star) = 0$, and $F'(x)$ is continuous, positive definite, non-Hermitian on $\mathbb{N}^\star$. In addition, denote by $\mathbb{N}(x^\star; r) \subset \mathbb{N}^\star$ an open ball centered at $x^\star$ with radius r and assume the following conditions hold for all $x \in \mathbb{N}(x^\star; r)$:*

(i) (The bounded condition) there exist positive constants β and γ such that

$$\max\{\|P_1(x^\star)\|, \|P_2(x^\star)\|\} \le \beta, \ \ \|F'(x^\star)^{-1}\| \le \gamma. \tag{20}$$

(ii) (The Lipschitz condition) there exist nonnegative constants L_1, L_2 such that

$$\|P_1(x) - P_1(x^\star)\| \le L_1\|x - x^\star\|, \ \ \|P_2(x) - P_2(x^\star)\| \le L_2\|x - x^\star\|. \tag{21}$$

Here $r \in (0, r_0)$, $r_0 := \min\{r_1, r_2\}$ and

$$r_1 = \frac{\alpha+\beta}{L}\left(\sqrt{\frac{2\tau\alpha\theta}{\gamma(2+\tau\theta)(\alpha+\beta)^2} + 1} - 1\right), \quad r_2 = \frac{1 - 2\beta\gamma[(\tau+1)\theta]^{\ell_0}}{3\gamma L},$$

where $L := L_1 + L_2$, $\ell_0 = \lim_{k\to\infty}\inf \ell_k$ satisfying

$$\ell_0 > -\frac{\ln(2\beta\gamma)}{\ln((\tau+1)\theta)}, \tag{22}$$

$\tau \in (0, (1-\theta)/\theta)$ is a prescribed positive constant and

$$\theta \equiv \theta(\alpha; x^\star) = \|\Gamma(\alpha; x^\star)\| < 1. \tag{23}$$

Then, for any $x^{(0)} \in \mathbb{N}(x^\star, r)$ and any sequence $\{\ell_k\}_{k=0}^{\infty}$ of positive integers, the iteration sequence $\{x^{(k)}\}_{k=0}^{\infty}$ generated by the Newton-GPSS method is well-defined and convergent to $x^\star$.

Proof. The proof is essentially analogous to the proofs of Theorem 3.2 in [20], with only replacing H and S by P_1 and P_2, respectively. □

In the following, we give the following a Kantorovich-type semilocal convergence theorem for the above Newton-GHSS method.

Theorem 4. *Let $F:\mathbb{D}\subset\mathbb{C}^n\to\mathbb{C}^n$ be G-differentiable on an open neighborhood $\mathbb{N}^{(0)}\subset\mathbb{D}$ of a initial approximation $x^{(0)}\in\mathbb{D}$, and $F'(x)$ is continuous, positive definite, non-Hermitian for any $x\in\mathbb{D}$. In addition, assume the following conditions hold:*

(i) (The bounded condition) there exist positive constants $\bar\beta$, $\bar\gamma$ and δ such that

$$\max\{\|P_1(x^{(0)})\|,\|P_2(x^{(0)})\|\}\le\bar\beta,\ \ \|F'(x^{(0)})^{-1}\|\le\bar\gamma,\ \ \|F(x^{(0)})\|\le\delta. \tag{24}$$

(ii) (The Lipschitz condition) there exist nonnegative constants $\bar L_1$, $\bar L_2$ such that for all $x,y\in\mathbb{B}(x^{(0)},\bar r)\subset\mathbb{N}^{(0)}$,

$$\|P_1(x)-P_1(y)\|\le\bar L_1\|x-y\|,\ \ \|P_2(x)-P_2(y)\|\le\bar L_2\|x-y\|. \tag{25}$$

Here $\bar L:=\bar L_1+\bar L_2$, $\bar r:=\min\{\bar r_1,\bar r_2\}$, and

$$\bar r_1=\frac{\alpha+\bar\beta}{\bar L}\left(\sqrt{\frac{2\bar\tau\alpha\bar\theta}{\bar\gamma(2+\bar\tau\bar\theta)(\alpha+\bar\beta)^2}+1}-1\right),\quad \bar r_2=\frac{1-\eta-\sqrt{(1-\eta)^2-4a\bar\gamma\delta}}{a},$$

where $\eta:=\max_k\{\eta_k\}<1$, $\bar\tau\in(0,(1-\bar\theta)/\bar\theta)$ is a prescribed positive constant,

$$\bar\theta\equiv\bar\theta(\alpha;x^{(0)})=\|\Gamma(\alpha;x^{(0)})\|<1, \tag{26}$$

and

$$a=\frac{\bar\gamma\bar L(1+\eta)}{1+2\bar\gamma^2\delta\bar L\eta},\qquad \ell_0=\liminf_{k\to\infty}\ell_k>\frac{\ln\eta}{\ln((\bar\tau+1)\bar\theta)}. \tag{27}$$

If the above constants satisfy the following condition

$$\eta+2\delta\bar\gamma^2\bar L(1+\eta^2)\le1, \tag{28}$$

then the iteration sequence $\{x^{(k)}\}_{k=0}^{\infty}$ generated by the Newton-GPSS method is well-defined and converges to $x^\star$, which satisfies $F(x^\star)=0$.

Proof. Analogously to Theorem 3.2 in [22], we can also prove easily by replacing H and S by P_1 and P_2, respectively. □

4. The accelerated Newton-GPSS method

Bai *et al.* [13] proposed a successive-overrelaxation (SOR) acceleration scheme for the HSS iteration. Based on the idea of [13], we can present the SOR acceleration scheme for the Newton-GPSS method .

Algorithm 6. *(The accelerated Newton-GPSS method)*

Let $F:\mathbb{D}\subset\mathbb{C}^n\to\mathbb{C}^n$ be a continuously differentiable function with the positive-definite Jacobian matrix $F'(x)$ at any $x\in\mathbb{D}$. Let $F'(x)=P_1(x)+P_2(x)$, where $P_1(x)$ and $P_2(x)$ are also positive-definite matrices.

1. Given an initial guess $x^{(0)}\in\mathbb{D}$, and positive constants α and tol, *and a positive integer sequence $\{\ell_k\}_{k=0}^{\infty}$.*

2. For $k=0,1,\ldots$ until $\|F(x^{(k)})\|_2\le\mathrm{tol}\|F(x^{(0)})\|_2$ do:

2.1. Set $s_*^{(k,0)}:=\mathbf{0}$, $s^{(k,0)}:=\mathbf{0}$.

2.2. For $\ell=0,1,2,\ldots,\ell_k-1$, apply the accelerated GPSS method:

$$\begin{cases}(\alpha I+P_1(x^{(k)}))s_*^{(k,\ell+\frac12)}=(\alpha I-P_2(x^{(k)}))s^{(k,\ell)}-F(x^{(k)}),\\ s_*^{(k,\ell+1)}=(1-\omega)s_*^{(k,\ell)}+\omega s_*^{(k,\ell+\frac12)},\\ (\alpha I+P_2(x^{(k)}))s^{(k,\ell+\frac12)}=(\alpha I-P_1(x^{(k)}))s_*^{(k,\ell+1)}-F(x^{(k)}),\\ s^{(k,\ell+1)}=(1-\omega)s^{(k,\ell)}+\omega s^{(k,\ell+\frac12)},\end{cases} \tag{29}$$

where ω is the relaxation parameter, and obtain $s^{(k,\ell_k)}$ such that

$$\|F(x^{(k)})+F'(x^{(k)})s^{(k,\ell_k)}\|_2 \leq \eta_k \|F(x^{(k)})\|_2 \quad \text{for some } \eta_k \in [0,1).$$

2.3. Set

$$x^{(k+1)} = x^{(k)} + s^{(k,\ell_k)}.$$

Next we show the local convergence of the accelerated Newton-GPSS method. First the accelerated GPSS method (29) can be rewritten as

$$z^{\ell+1} = \Upsilon_\omega(\alpha) z^\ell + \Theta_\omega(\alpha),$$

where $z^\ell = \begin{pmatrix} s_*^{(k,\ell)} \\ s^{(k,\ell)} \end{pmatrix}$ and $\Upsilon_\omega(\alpha) = \begin{pmatrix} (1-\omega)I & \omega(\alpha I + P_1(x^{(k)}))^{-1}(\alpha I - P_2(x^{(k)})) \\ \omega(1-\omega)(\alpha I + P_2(x^{(k)}))^{-1}(\alpha I - P_1(x^{(k)})) & (1-\omega)I + \omega^2 \Gamma(\alpha; x^{(k)}) \end{pmatrix}$.
Then the following lemma are given to show that there exists a functional relationship between the eigenvalues of $\Upsilon_\omega(\alpha)$ and $\Gamma(\alpha; x^{(k)})$.

Lemma 1. *Suppose that $\omega \neq 0$, if λ is a non-zero eigenvalue of $\Upsilon_\omega(\alpha)$ and if μ satisfies*

$$(\lambda + \omega - 1)^2 = \omega^2 \lambda \mu, \tag{30}$$

then μ is an eigenvalue of $\Gamma(\alpha; x^{(k)})$. Conversely, if μ is an eigenvalue of $\Gamma(\alpha; x^{(k)})$ and λ satisfies (30)*, then λ is an eigenvalue of $\Upsilon_\omega(\alpha)$.*

Proof. Suppose that λ is an eigenvalue of $\Upsilon_\omega(\alpha)$ and $(x^T, y^T)^T$ is a corresponding eigenvector, from

$$\Upsilon_\omega(\alpha) \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix},$$

we get that

$$(1-\omega)x + \omega(\alpha I + P_1(x^{(k)}))^{-1}(\alpha I - P_2(x^{(k)}))y = \lambda x \tag{31}$$

and

$$\omega(1-\omega)(\alpha I + P_2(x^{(k)}))^{-1}(\alpha I - P_1(x^{(k)}))x + (1-\omega)y + \omega^2 \Gamma(\alpha; x^{(k)})y = \lambda y. \tag{32}$$

then it follows from (31) that

$$(\lambda + \omega - 1)x = \omega(\alpha I + P_1(x^{(k)}))^{-1}(\alpha I - P_2(x^{(k)}))y.$$

As a result of (32) and the definition of $\Gamma(\alpha; x)$ in (16), we obtain

$$\omega^2 \lambda \Gamma(\alpha; x^{(k)}) y = (\lambda + \omega - 1)^2 y.$$

Evidently, if μ satisfies (30), then μ is an eigenvalue of $\Gamma(\alpha; x^{(k)})$. Reversing the process, we can get the proof of the second assertion. □

Then we can obtain the convergence property of the accelerated GPSS method (29) straightforwardly from Lemma 1.

Theorem 5. *Let all the symbols be defined as those in Algorithm 6, and let the conditions of Theorem 2 be satisfied.*

(i) When all of the eigenvalues of $\Gamma(\alpha; x^{(k)})$ are real, $\rho(\Upsilon_\omega(\alpha)) < 1$ if and only if

$$0 < \omega < 2.$$

(ii) When some of the eigenvalues of $\Gamma(\alpha; x^{(k)})$ is complex, if for some positive number $t \in (0,1)$ and each eigenvalue μ of $\Gamma(\alpha; x^{(k)})$, $\kappa + \imath\nu$ is the square root of μ and the point (κ, ν) lies in the interior of the ellipse

$$\kappa^2 + \frac{\nu^2}{t^2} = 1$$

and ω satisfies

$$0 < \omega < \frac{2}{1+t},$$

then $\rho(\Upsilon_\omega(\alpha)) < 1$.

Proof. We known that $\rho(\Gamma(\alpha;x^{(k)}))<1$ from Section 3. Then (i) and (ii) can be proved immediately from Theorems 2.2 and 4.1 in Chapter 6 of [5], respectively. □

Remark 1. *From Theorem 11.1.5 in [3], we see that when the relaxation parameter ω satisfies the assumptions for two different cases (i) and (ii), the accelerated Newton-GPSS method is locally convergent.*

5. Numerical examples

Consider the two-dimensional nonlinear convection-diffusion equation [20, 22]:

$$\begin{cases} -\left(u_{xx}+u_{yy}\right)+q(u_x+u_y)=-e^u, & (x,y)\in\Omega, \\ u(x,y)=0, & (x,y)\in\partial\Omega, \end{cases} \tag{33}$$

where $\Omega=(0,1)\times(0,1)$, with $\partial\Omega$ its boundary, and q is a positive constant for measuring magnitudes of the convective terms.

Applying five-point finite-difference scheme to the diffusive term and the central difference scheme to the convective term, respectively, we get the system of nonlinear equations (1) of the form

$$F(u)=Mu+h^2\Psi(u)=0, \tag{34}$$

where $h=\frac{1}{N+1}$ is the equidistant step-size with N being a prescribed positive integer, $M=A_N\otimes I_N+I_N\otimes A_N$ and $\Psi(u)=(e^{u_1},e^{u_2},\dots,e^{u_n})^T$ with the tridiagonal matrix $A_N=\text{tridiag}(-1-qh/2,2,-1+qh/2)$, $\otimes$ means the Kronecker product symbol, and $n=N\times N$.

From [19], there exist many choices of GPSS splitting of Jacobian matrix $F'(u^{(k)})$, we first use the following special splitting

$$F'(u^{(k)})=G(u^{(k)})+\left(K(u^{(k)})+S(u^{(k)})\right),$$

where $K(u^{(k)})$ is the corresponding Jacobian matrix of nonlinear term $h^2\Psi(u)$, $G(u^{(k)})=H(u^{(k)})-K(u^{(k)})$ with $H(u^{(k)})=(F'(u^{(k)})+F'(u^{(k)})^*)/2$ is the Hermitian part of $F'(u^{(k)})$, $S(u^{(k)})=(F'(u^{(k)})-F'(u^{(k)})^*)/2$ is the skew-Hermitian part of $F'(u^{(k)})$. Obviously, $K(u^{(k)})$ is a nonnegative diagonal matrix and hence it is positive definite; $G(u^{(k)})$ is symmetric positive definite. Then the choices of $P_1(u^{(k)})$ and $P_2(u^{(k)})$ can be according to (9).

Consequently, we can use the Algorithm 5-6 to solve (34). In actual computations, the initial guess is chosen to be $u^{(0)}=\mathbf{0}$, the stopping criterion for the outer Newton iteration is set to be $\frac{\|F(u^{(k)})\|_2}{\|F(u^{(0)})\|_2}\le 10^{-6}$, and that for the inner GPSS and HSS iteration is set to be

$$\frac{\|F'(u^{(k)})s^{(k,\ell_k)}+F(u^{(k)})\|_2}{\|F(u^{(k)})\|_2}\le\eta.$$

In [20], numerical examples have shown that the Newton-HSS method outperforms the Newton-USOR, the Newton-GMRES and the Newton-GCG methods. So in this paper, we just compare our methods with Newton-HSS method. In the implementations, we adopt the numerical optimal parameters α for the Newton-HSS and Newton-GPSS methods, which yield the least CPU times for these iteration methods, respectively; see Table 1. We use accelerated Newton-GPSS method to solve the problem and adopt the same parameters α as the Newton-GPSS method by convention. Moreover, we adopt the SOR parameter $\omega=0.8$ in all cases.

Table 1: The optimal values α for Newton-HSS and Newton-GPSS when $q=1000$

	η	0.1	0.2	0.4
Newton-HSS	$N=30$	1.1	1.1	1.4
	$N=40$	1.4	1.2	1.3
Newton-GPSS	$N=30$	5.9	6.1	5.3
	$N=40$	5.0	4.5	4.7

In Tables 2-3, we give the numerical results about the three methods, corresponding to the problem parameter $N = 30$, 40 and the inner tolerance $\eta = 0.1$, 0.2, 0.4, respectively. The accelerated Newton-GPSS method is compared with the Newton-GPSS and Newton-HSS methods for different tolerances η in the sense of the outer and inner iteration steps (denoted as $\mathrm{IT}_{\mathrm{out}}$ and $\mathrm{IT}_{\mathrm{int}}$, respectively), the condition numbers (denoted as COND) and the total CPU time (in seconds, denoted as CPU). Here $\mathrm{IT}_{\mathrm{int}}$ denotes the average inner iteration steps at each outer Newton iterate, and COND denotes the average condition numbers of inner iteration coefficient matrices at each outer Newton iterate.

Table 2: Numerical results of inexact Newton methods for $N = 30, q = 1000$

	η	0.1	0.2	0.4
Newton-HSS	$\mathrm{IT}_{\mathrm{int}}$	13.2	9.3	6.2
	$\mathrm{IT}_{\mathrm{out}}$	6	8	13
	COND	58.3587	58.3587	45.8574
	CPU	1.6570	1.5310	1.6720
Newton-GPSS	$\mathrm{IT}_{\mathrm{int}}$	7.2	5.1	3.1
	$\mathrm{IT}_{\mathrm{out}}$	6	8	13
	COND	10.2514	9.9186	11.4013
	CPU	0.3440	0.3440	0.3590
accelerated Newton-GPSS	$\mathrm{IT}_{\mathrm{int}}$	3.0	2.1	1.4
	$\mathrm{IT}_{\mathrm{out}}$	5	7	11
	COND	10.2514	9.9186	11.4013
	CPU	0.1410	0.1560	0.1720

Table 3: Numerical results of inexact Newton methods for $N = 40, q = 1000$

	η	0.1	0.2	0.4
Newton-HSS	$\mathrm{IT}_{\mathrm{int}}$	15.2	11.5	6.9
	$\mathrm{IT}_{\mathrm{out}}$	6	8	13
	COND	34.7554	40.5435	37.4267
	CPU	4.8440	5.1100	4.9840
Newton-GPSS	$\mathrm{IT}_{\mathrm{int}}$	7.2	5.1	3.2
	$\mathrm{IT}_{\mathrm{out}}$	6	8	13
	COND	8.9846	9.9710	9.5511
	CPU	0.7030	0.6720	0.7030
accelerated Newton-GPSS	$\mathrm{IT}_{\mathrm{int}}$	3.8	2.4	1.3
	$\mathrm{IT}_{\mathrm{out}}$	5	8	12
	COND	8.9846	9.9710	9.5511
	CPU	0.3440	0.3600	0.3440

From the results in Tables 2-3, we see that the inner iteration coefficient matrices of the accelerated Newton-GPSS method and the Newton-GPSS method are considerably better conditioned than those of Newton-HSS. Furthermore, we find that the accelerated Newton-GPSS method uses the least iteration steps and CPU times comparing with the other two methods. Moreover, the Newton-GPSS method is also superior to the Newton-HSS method both in iteration steps and CPU times.

6. Conclusions

In this paper, we establish the Newton-GPSS method which uses the GPSS iteration as the inner iteration to solve the system of nonlinear equations (1). We also have discussed its local and semilocal convergence under some proper assumptions. Moreover, we propose an accelerated Newton-GPSS method and analyze its convergence behavior. Numerical results illustrate that our two methods considerably outperform the Newton-HSS method from aspects of iteration steps and CPU time. Finally, it should be mentioned that the choice of the optimal relaxation parameter ω of the accelerated Newton-GPSS method are interesting topics and we will study in future.

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Common fixed points of a pair of Hardy Rogers Type Mappings on a Closed Ball in Ordered Partial Metric Spaces

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Abstract: Common fixed point results for mappings satisfying locally contractive conditions on a closed ball in a 0-complete ordered partial metric space have been established for two, three and four mappings. Instead of monotone mapping, the notion of dominated mappings is applied. We have used weaker contractive conditions and weaker restrictions to obtain unique fixed points. An example is given which shows that how this result can be used when the corresponding results cannot. Our results generalize, extend and improve several well-known conventional results.

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1 Introduction and Preliminaries

Let $T : X \longrightarrow X$ be a mapping. A point $x \in X$ is called a fixed point of T if $x = Tx$. Fixed points results of mappings satisfying certain contractive condition on the entire domain has been at the centre of rigorous research activity, for example (see [3, 7]).

Ran and Reurings [12] proved an analogue of Banach's fixed point theorem in metric space endowed with a partial order and gave applications to matrix equations. Subsequently, Nieto et. al. [11] extended the result in [12] for nondecreasing mappings and applied it to obtain a unique solution for a 1st order ordinary differential equation with periodic boundary conditions.

Partial metric spaces have applications in theoretical computer science (see [10]). [2] used the idea of partial metric space and partial order and gave some fixed point theorems for contractive condition on ordered partial metric spaces. Romaguera [13] has given the idea of 0-complete partial metric space.

From the application point of view the situation is not yet completely satisfactory because it frequently happens that a mapping T is a contraction not on the entire space X but merely on a subset Y of X. However, if Y is closed then by imposing a subtle restriction, one can establish the existence of a fixed point of T. Arshad et. al. [4] proved a result concerning the existence of fixed points of a mapping satisfying a contractive conditions on closed ball in a complete dislocated metric space. Other results on closed ball can be seen in [6, 5].

Consistent with [1, 2, 3, 8, 9, 10, 13], the following definitions and results will be needed in the sequel.

Definition 1.1. **[10]** Let X be a nonempty set. If for any $x, y, z \in X$, mapping $p : X \times X \to R^+$ satisfies

(P_1) $p(x, x) = p(y, y) = p(x, y)$ if and only if $x = y$,

(P_2) $p(x, x) \leq p(x, y)$,

(P_3) $p(x, y) = p(y, x)$,

(P_4) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

Then it is said to be a partial metric on X and the pair (X, p) is called a partial metric space.

Each partial metric p on X induces a T_0 topology τ_p on X which has as a base the family of open balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$. Also $\overline{B(x_0, r)} = \{y \in X : p(x, y) \leq p(x, x) + \varepsilon\}$ is a closed ball in (X, p).

It is clear that if $p(x, y) = 0$, then from P_1 and P_2, $x = y$. But if $x = y$, then $p(x, y)$ may not be 0.

Example 1.2. [10] If $X = [0, \infty)$ then, $p(x, y) = \max\{x, y\}$ for all $x, y \in X$, defines a partial metric p on X.

Definition 1.3. [13] Let (X, p) be a partial metric space, then, a sequence $\{x_n\}$ in (X, p) is called 0-Cauchy if $\lim_{n,m\to\infty} p(x_n, x_m) = 0$. The space (X, p) is called 0-complete if every 0-Cauchy sequence in X converges to a point $x \in X$ such that $p(x, x) = 0$.

If (X, p) is a partial metric space, then $p_s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$, $x, y \in X$, is a metric on X.

Lemma 1.4. [13] Let (X, p) be a partial metric space. Then,

(a) Every 0-Cauchy sequence in (X, p) is Cauchy in (X, p_s).

(b) If (X, p) is complete, then it is 0-complete.

Romaguera [13] has given an example which proves that converse assertions of (c) and (d) do not hold. It is easy to see that every closed subset of a 0-complete partial metric space is 0-complete

Definition 1.5. **[2]** Let X be a nonempty set. Then $(X, \preceq, p)$ is called an ordered partial metric space if:

(i) p is a partial metric on X and (ii) $\preceq$ is a partial order on X.

Definition 1.6. Let $(X, \preceq)$ be a partial ordered set. Then $x, y \in X$ are called comparable if $x \preceq y$ or $y \preceq x$ holds.

Definition 1.7. **[1]** Let $(X, \preceq)$ be a partially ordered set. A self mapping f on X is called dominated if $fx \preceq x$ for each x in X.

Example 1.8. **[1]** Let $X = [0, 1]$ be endowed with the usual ordering and $f : X \to X$ be defined by $fx = x^n$ for some $n \in \mathbb{N}$. Since $fx = x^n \leq x$ for all $x \in X$, therefore f is a dominated map.

Definition 1.9. Let X be a non empty set and $T, f : X \to X$. A point $y \in X$ is called point of coincidence of T and f if there exists a point $x \in X$ such that $y = Tx = fx$. The mappings T, f are said to be weakly compatible if they commute at their coincidence point (i. e. $Tfx = fTx$ whenever $Tx = fx$).

We require the following lemmas for subsequent use:

Lemma 1.10. [8] Let X be a non empty set and $f : X \to X$ a function. Then there exists a subset $E \subset X$ such that $fE = fX$ and $f : E \to X$ is one to one.

Lemma 1.11. [3] Let X be a non empty set and the mappings $S, T, f : X \to X$ have a unique point of coincidence v in X. If (S, f) and (T, f) are weakly compatible, then S, T, f have a unique common fixed point.

Theorem 1.12 [9] Let (X, d) be a complete metric space, $S : X \to X$ be a mapping, $r > 0$ and x_0 be an arbitrary point in X. Suppose there exists $k \in [0, 1)$ with

$$d(Sx, Sy) \leq kd(x, y), \text{ for all } x, y \in Y = \overline{B(x_0, r)}$$

and $d(x_0, Sx_0) < (1 - k)r$. Then there exists a unique point x^* in $\overline{B(x_0, r)}$ such that $x^* = Sx^*$.

2 Fixed Points of Hardy-Rogers Mapping

Theorem 2.1. Let $(X, \preceq, p)$ be a 0-complete ordered partial metric space, $x_0 \in X$, $r > 0$ and $S, T : X \to X$ be two dominated mappings. Suppose that there exists $a, b, c \in [0, 1)$ such that $a + 2b + 2c < 1$ and

$$p(Sx, Ty) \leq ap(x, y) + b[p(x, Sx) + p(y, Ty)] + c[p(y, Sx) + p(x, Ty)], \quad (2.1)$$

for all comparable elements x, y in $\overline{B(x_0, r)}$.

$$\text{and } p(x_0, Sx_0) \leq (1 - \lambda)[r + p(x_0, x_0)], \quad (2.2)$$

where $\lambda = \dfrac{a + b + c}{1 - b - c}$. Then there exists a point x^* in $\overline{B(x_0, r)}$ such that $p(x^*, x^*) = 0$. If, for a nonincreasing sequence $\{x_n\}$ in $\overline{B(x_0, r)}$, $\{x_n\} \to u$ implies that $u \preceq x_n$, then $x^* = Sx^* = Tx^*$.

Proof. Choose a point x_1 in X such that $x_1 = Sx_0$. As $Sx_0 \preceq x_0$ so $x_1 \preceq x_0$ and let $x_2 = Tx_1$. Now $Tx_1 \preceq x_1$ gives $x_2 \preceq x_1$, Continuing this process and having chosen x_n in X such that,

$$x_{2k+1} = Sx_{2k} \text{ and } x_{2k+2} = Tx_{2k+1}, \text{ where } k = 0, 1, 2, \ldots.$$

where $x_{2k+1} = Sx_{2k} \preceq x_{2k}$. We will prove that $x_n \in \overline{B(x_0, r)}$ for all $n \in N$, by mathematical induction. By using inequality (2.2), we have,

$$p(x_0, x_1) \leq r + p(x_0, x_0).$$

Therefore $x_1 \in \overline{B(x_0, r)}$. Let $x_2, ..., x_j \in \overline{B(x_0, r)}$ for some $j \in N$. If $j = 2k+1$, then, $x_{2k+1} = Sx_{2k} \preceq x_{2k}$, where $k = 0, 1, 2, \ldots \frac{j-1}{2}$, so using inequality (2.1), we obtain,

$$\begin{aligned} p(x_{2k+1}, x_{2k+2}) &= p(Sx_{2k}, Tx_{2k+1}) \\ &\leq a[p(x_{2k}, x_{2k+1})] + b[p(x_{2k}, x_{2k+1}) + p(x_{2k+1}, x_{2k+2})] + \\ &\quad c[p(x_{2k}, x_{2k+1}) + p(x_{2k+1}, x_{2k+2})], \end{aligned}$$

which implies that,

$$p(x_{2k+1}, x_{2k+2}) \leq \lambda p(x_{2k}, x_{2k+1}) \leq ... \leq \lambda^{2k+1} p(x_0, x_1). \tag{2.3}$$

If $j = 2k+2$, then as x_1, x_2 $..., x_j \in \overline{B(x_0, r)}$ and $x_{2k+2} \preceq x_{2k+1}$, $(k = 0, 1, 2, \ldots, \frac{j-2}{2})$, we obtain,

$$p(x_{2k+2}, x_{2k+3}) \leq \lambda^{2k+2} p(x_0, x_1). \tag{2.4}$$

Thus from inequality (2.3) and (2.4), we have

$$p(x_j, x_{j+1}) \leq \lambda^j p(x_0, x_1) \text{ for some } j \in N. \tag{2.5}$$

Now,

$$\begin{aligned} p(x_0, x_{j+1}) &\leq p(x_0, x_1) + ... + p(x_j, x_{j+1}) - [p(x_1, x_1) + ... + p(x_j, x_j)] \\ &\leq p(x_0, x_1)[1 + ... + \lambda^{j-1} + \lambda^j], \quad \text{(by 2.5)} \\ p(x_0, x_{j+1}) &\leq (1-\lambda)[r + p(x_0, x_0)] \frac{(1-\lambda^{j+1})}{1-\lambda}. \end{aligned}$$

Thus $x_{j+1} \in \overline{B(x_0, r)}$. Hence $x_n \in \overline{B(x_0, r)}$ for all $n \in N$. Also $x_{n+1} \preceq x_n$ for all $n \in N$. It implies that,

$$p(x_n, x_{n+1}) \leq \lambda^n p(x_0, x_1) \text{ for all } n \in N. \tag{2.6}$$

So we have,

$$\begin{aligned} p(x_{n+i}, x_n) &\leq p(x_{n+i}, x_{n+i-1}) + ... + p(x_{n+1}, x_n) \\ &\leq \lambda^{n+i-1} p(x_0, x_1) + ... + \lambda^n p(x_0, x_1) \\ &\leq \lambda^n p(x_0, x_1) \frac{(1-\lambda^i)}{1-\lambda} \longrightarrow 0 \text{ as } n \to \infty. \end{aligned}$$

Hence the sequence $\{x_n\}$ is a 0-Cauchy sequence in $(\overline{B(x_0, r)}, p)$. As $\overline{B(x_0, r)}$ is closed so is 0-complete. Therefore there exists a point $x^* \in \overline{B(x_0, r)}$ with

$$p(x^*, x^*) = \lim_{n \to \infty} p(x_n, x^*) = 0. \tag{2.7}$$

Now,

$$p(x^*, Sx^*) \leq p(x^*, x_{2n+2}) + p(x_{2n+2}, Sx^*) - p(x_{2n+2}, x_{2n+2}).$$

On taking limit as $n \to \infty$ and by assumptions $x^* \preceq x_n$ as $x_n \longrightarrow x^*$, therefore,

$$\begin{aligned} p(x^*, Sx^*) &\leq \lim_{n\to\infty}[p(x^*, x_{2n+2}) + ap(x_{2n+1}, x^*) + b\{p(x_{2n+1}, Tx_{2n+1}) \\ &\quad +p(x^*, Sx^*)\} + c\{p(x_{2n+1}, Sx^*) + p(x^*, Tx_{2n+1})\}] \\ &\leq \lim_{n\to\infty}[p(x^*, x_{2n+2}) + ap(x_{2n+1}, x^*) + b\{p(x_{2n+1}, x_{2n+2}) \\ &\quad +p(x^*, Sx^*)\} + c\{p(x_{2n+1}, x^*) + p(x^*, Sx^*) + p(x^*, x_{2n+2})\}]. \end{aligned}$$

By using inequality (2.6) and (2.7) we obtain,

$$(1 - b - c)p(x^*, Sx^*) \leq 0,$$

which implies that $x^* = Sx^*$. Similarly, from,

$$p\left(x^*, Tx^*\right) \leq p(x^*, x_{2n+1}) + p(x_{2n+1}, Tx^*) - p(x_{2n+1}, x_{2n+1}),$$

we can obtain $x^* = Tx^*$. Hence S and T have a common fixed point in $\overline{B(x_0, r)}$. ■

Example 2.2. Let $X = [0, +\infty) \cap Q$ be endowed with order, $x \preceq y$ if $p(x, x) \leq p(y, y)$ and let $p : X \times X \to R^+$ be the 0-complete ordered partial metric on X defined by $p(x, y) = \max\{x, y\}$. Define

$$Sx = \begin{cases} \dfrac{x}{16} \text{ if } x \in [0, 1] \cap Q \\ x - \dfrac{1}{6} \text{ if } x \in (1, \infty) \cap Q \end{cases} \text{ and } Tx = \begin{cases} \dfrac{5x}{17} \text{ if } x \in [0, 1] \cap Q \\ x - \dfrac{1}{7} \text{ if } x \in (1, \infty) \cap Q \end{cases}.$$

Clearly, S and T are dominated mappings. Take, $a = \dfrac{1}{5}$, $b = \frac{1}{10}$, $c = \frac{1}{15}$, $x_0 = \dfrac{1}{2}$, $r = \dfrac{1}{2}$, then $\overline{B(x_0, r)} = [0, 1] \cap X$. We have, $p(x_0, x_0) = \max\{\dfrac{1}{2}, \dfrac{1}{2}\} = \dfrac{1}{2}$, $\lambda = \dfrac{a + b + c}{1 - b - c} = \dfrac{11}{25}$ with

$$(1 - \lambda)[r + p(x_0, x_0)] = \frac{14}{25}$$

and

$$p(x_0, Sx_0) = p(\frac{1}{2}, \frac{1}{32}) = \frac{1}{2} < (1 - \lambda)[r + p(x_0, x_0)].$$

Also if, $x, y \in (1, \infty) \cap Q$, then,

$$\begin{aligned} p(Sx, Ty) &= \max\{x - \frac{1}{6}, y - \frac{1}{7}\} \\ &\geq \frac{1}{5}\max\{x, y\} + \frac{1}{10}[\max\{x, x - \frac{1}{6}\} + \max\{y, y - \frac{1}{7}\}] \\ &\quad + \frac{1}{15}[\max\{x, y - \frac{1}{7}\} + \max\{y, x - \frac{1}{6}\}] \\ &\geq ap(x, y) + b[p(x, Sx) + p(y, Ty)] + c[p(y, Sx) + p(x, Ty)] \end{aligned}$$

So the contractive condition does not hold on X.

$$\begin{aligned}
\text{Now if, } x,y &\in \overline{B(x_0,r)}, \text{ then } p(Sx,Ty)=\max\{\frac{x}{16},\frac{5y}{17}\} \\
&\le \frac{1}{5}\max\{x,y\}+\frac{1}{10}[\max\{x,\frac{x}{16}\}+\max\{y,\frac{5y}{17}\}] \\
&\quad +\frac{1}{15}[\max\{x,\frac{5y}{17}\}+\max\{y,\frac{x}{16}\}] \\
&= ap(x,y)+b[p(x,Sx)+p(y,Ty)]+c[p(y,Sx)+p(x,Ty)]
\end{aligned}$$

Therefore, all the conditions of Theorem 2.1 are satisfied. Moreover, 0 is the common fixed point of S and T and $p(0,0)=0$.

Theorem 2.3. Let $(X,\preceq,p)$ be a 0-complete ordered partial metric space, $x_0\in X$, $r>0$ and $S:X\to X$ be two dominated mapping. Suppose that there exists $a,b,c\in[0,1)$ such that $a+2b+2c<1$ and

$$p(Sx,Sy)\le ap(x,y)+b[p(x,Sx)+p(y,Sy)]+c[p(y,Sx)+p(x,Sy)],$$

for all comparable elements x,y in $\overline{B(x_0,r)}$.

$$\text{and } p(x_0,Sx_0)\le(1-\lambda)[r+p(x_0,x_0)],$$

where $\lambda=\dfrac{a+b+c}{1-b-c}$. Then there exists a point x^* in $\overline{B(x_0,r)}$ such that $p(x^*,x^*)=0$. If, for a nonincreasing sequence $\{x_n\}$ in $\overline{B(x_0,r)}$, $\{x_n\}\to u$ implies that $u\preceq x_n$, then $x^*=Sx^*$.

Proof. In Theorem 2.1 take $T=S$ to get fixed point $x^*\in\overline{B(x_0,r)}$ such that $x^*=Sx^*$. ■

In Theorem 2.1,the condition "for a nonincreasing sequence $\{x_n\}\to u$ implies that $u\preceq x_n$", is imposed to restrict the condition (2.1) only for comparable elements. However, the following result relax this restriction but impose the condition (2.1) for all elements in $\overline{B(x_0,r)}$.

Theorem 2.4. Let (X,p) be a 0-complete partial metric space, $x_0\in X$, $r>0$ and $S,T:X\to X$ be two dominated mappings. Suppose that there exists $a,b,c\in[0,1)$ such that $a+2b+2c<1$ and

$$p(Sx,Ty)\le ap(x,y)+b[p(x,Sx)+p(y,Ty)]+c[p(y,Sx)+p(x,Ty)],$$

for all elements x,y in $\overline{B(x_0,r)}$.

$$\text{and } p(x_0,Sx_0)\le(1-\lambda)[r+p(x_0,x_0)],$$

where $\lambda=\dfrac{a+b+c}{1-b-c}$. Then there exists a unique point x^* in $\overline{B(x_0,r)}$ such that $x^*=Sx^*=Tx^*$ and $p(x^*,x^*)=0$. Moreover, S and T have no fixed point other than x^*.

Proof. By following similar arguments of Theorem 2.1, we can obtain a point x^* in $\overline{B(x_0,r)}$ such that $x^* = Sx^* = Tx^*$. Let $y = Ty$. Then y is the fixed point of T and it may not be the fixed point of S. Then,

$$p(x^*, y) = p(Sx^*, Ty) \leq (a + b + 2c)p(x^*, y).$$

This shows that $x^* = y$. Hence T has no fixed point other than x^*. Similarly, S has no fixed point other than x^*. ■

In Theorem 2.1, the condition (2.2) is imposed to restrict the condition (2.1) only for x, y in $\overline{B(x_0,r)}$ and Example 2.2 explains the utility of this restriction. However, the following result relax the condition (2.2) but impose the condition (2.1) for all comparable elements in the whole space X. Moreover, we introduce a weaker restriction to obtain unique common fixed point.

Theorem 2.5. Let $(X, \preceq, p)$ be a 0-complete ordered partial metric space, $x_0 \in X$ and $S, T : X \to X$ be two dominated mappings. Suppose that there exists there exists $a, b, c \in [0, 1)$ such that $a + 2b + 2c < 1$ and

$$p(Sx, Ty) \leq ap(x, y) + b[p(x, Sx) + p(y, Ty)] + c[p(y, Sx) + p(x, Ty)],$$

for all comparable elements x, y in X. If, for a nonincreasing sequence $\{x_n\}$ in X, $\{x_n\} \to u$ implies that $u \preceq x_n$. Then there exists a point x^* in X such that $x^* = Sx^* = Tx^*$ and $p(x^*, x^*) = 0$. Moreover, the point x^* is unique if, for any two points x, y in X there exists a point $z_0 \in X$ such that $z_0 \preceq x^*$ and $z_0 \preceq y$.

Proof. By following similar arguments of Theorem 2.1, we can obtain a point x^* in X such that $x^* = Sx^* = Tx^*$. By Theorem 2.4, x^* is unique common fixed point for all comparable elements. Now if x^* and y are not comparable such that $y = Sy = Ty$. Then there exists a point $z_0 \in X$ such that $z_0 \preceq x^*$ and $z_0 \preceq y$. Choose a point z_1 in X such that $z_1 = Tz_0$. As $Tz_0 \preceq z_0$ so $z_1 \preceq z_0$ and let $z_2 = Sz_1$. Now $Sz_1 \preceq z_1$ gives $z_2 \preceq z_1$, continuing this process and having chosen z_n in X such that

$$z_{2i+1} = Tz_{2i},\ z_{2i+2} = Sz_{2i+1} \text{ and } z_{2i+1} = Tz_{2i} \preceq z_{2i} \text{ where } i = 0, 1, 2, \ldots.$$

It follows that $z_{n+1} \preceq z_n \preceq ... \preceq z_0 \preceq x^*$. Following similar arguments as we have used to prove inequality (2.6), we have,

$$p(z_n, z_{n+1}) \leq \lambda^n p(z_0, z_1) \text{ for all } n \in N. \tag{2.8}$$

As $z_0 \preceq x^*$ and $z_0 \preceq y$, it follows that $z_n \preceq Tx^*$ and $z_n \preceq Ty$ for all $n \in N$.

Then, for $i \in N$,

$$\begin{aligned}
p(Tx^*, Sz_{2i-1}) &\leq ap(x^*, z_{2i-1}) + b[p(x^*, Tx^*) + p(z_{2i-1}, Sz_{2i-1})] \\
&\quad + c[p(x^*, Sz_{2i-1}) + p(z_{2i-1}, Tx^*)], \\
(1-c)p(x^*, Sz_{2i-1}) &\leq (a+c)p(x^*, z_{2i-1}) + bp(z_{2i-1}, z_{2i}), \\
p(x^*, Sz_{2i-1}) &\leq \delta p(x^*, z_{2i-1}) + \mu p(z_{2i-1}, z_{2i}), \\
(\text{where } \delta &= \frac{a+c}{1-c} \text{ and } \mu = \frac{b}{1-c}) \\
p(x^*, Sz_{2i-1}) &\leq \delta^2 p(x^*, z_{2i-2}) + \delta\mu p(z_{2i-2}, z_{2i-1}) + \mu p(z_{2i-1}, z_{2i}) \\
&\vdots \\
&\leq \delta^{2i} p(x^*, z_0) + \delta^{2i-1}\mu p(z_0, z_1) + \cdots \\
&\quad + \delta\mu p(z_{2i-2}, z_{2i-1}) + \mu p(z_{2i-1}, z_{2i}).
\end{aligned}$$

On taking limit as $i \to \infty$ and by inequality (2.8), we have,

$$p(x^*, Sz_{2i-1}) = 0. \tag{2.9}$$

Similarly,

$$p(Sz_{2i-1}, y) \longrightarrow 0 \text{ as } n \to \infty. \tag{2.10}$$

Now by using inequality (2.9) and (2.10), we have

$$p(x^*, y) \leq p(x^*, Sz_{2i-1}) + p(Sz_{2i-1}, y) - p(Sz_{2i-1}, Sz_{2i-1}) \longrightarrow 0 \text{ as } n \to \infty.$$

So, $x^* = y$. ■

Now we can apply our Theorem 2.5 to obtain unique common fixed point and point of coincidence of three mappings in 0-complete ordered partial metric space. One can easily prove this result by using the technique given in the proof of Theorem 2.7 [4].

Theorem 2.6. Let $(X, \preceq, p)$ be a ordered partial metric space and S, T self mapping and f be a dominated mapping on X such that $SX \cup TX \subset fX$ with $Tx, Sx \preceq fx$. Assume that the following conditions holds for $a, b, c \in [0, 1)$ such that $a + 2b + 2c < 1$:

$$\begin{aligned}
p(Sx, Ty) &\leq ap(fx, fy) + b[p(fx, Sx) + p(fy, Ty)] \\
&\quad + c[p(fy, Sx) + p(fx, Ty)],
\end{aligned}$$

for all comparable elements fx, $fy \in fX$.

If for a nonincreasing sequence $\{x_n\}$ in fX, $\{x_n\} \to u$ implies that $u \preceq x_n$. Also for any two points z and x in fX there exists a point $y \in fX$ such that $y \preceq z$, $y \preceq x$. If fX is 0-complete subspace of X, then S, T and f have a unique common point of coincidence fz in fX and $p(fz, fz) = 0$. Moreover, if (S, f) and (T, f) are weakly compatible, then fz is a unique common fixed point of S, T and f in fX.

In a similar way, we can apply our Theorem 2.4 to obtain unique common fixed point and point of coincidence results for three mappings on closed ball in complete partial metric space.

In the following theorem we can use Theorem 2.4 to establish the existence of a unique common fixed point and point of coincidence of four mappings on closed ball in 0-complete partial metric space. One can easily prove this result by using the technique given in the proof of Theorem 2.8 [4].

Theorem 2.7. Let (X, p) be a partial metric space and S, T, g and f be self mappings on X such that SX, $TX \subset fX = gX$. Assume that the following condition holds:

$$\begin{aligned} p\left(Sx, Ty\right) &\leq ap(fx, gy) + b[p(fx, Sx) + p(gy, Ty)] \\ &\quad + c[p(gy, Sx) + p(fx, Ty)], \end{aligned}$$

for all elements fx, $fy \in \overline{B(fx_0, r)} \subseteq fX$; with $a, b, c \in [0, 1)$ such that $a + 2b + 2c < 1$ and for $r > 0$,

$$p(fx_0, Sx_0) \leq (1 - \lambda)[r + p(fx_0, fx_0)],$$

where $\lambda = \dfrac{a+b+c}{1-b-c}$. If fX is 0-complete subspace of X, then S, T, f and g have a unique common point of coincidence fz in fX and $p(fz, fz) = 0$. Moreover, if (S, f) and (T, g) are weakly compatible, then fz is a unique common fixed point of S, T and f in fX.

Remark 2.8 We can obtain the metric version of all theorems which are still not present in the literature.

Competing interests

The authors declare that they have no competing interests.

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Generalized vague soft set and its lattice structures

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Abstract: Molodtsov initiated the soft set theory, which has been successful used as an effective mathematical tool for dealing with vagueness and uncertainties. In this paper, we propose a new soft set model, called generalized vague soft set, which is an extension of the generalized fuzzy soft set. We also define some basic operations and discuss their some properties on generalized vague fuzzy soft set. Based on the proposed intersection and union operations, we further investigate the lattice structures of generalized vague soft set and obtain two lattice structures, and it is proved that the two lattices are bounded contributive lattices.

Keywords: Soft set; vague set; generalized fuzzy soft set; generalized vague soft set; lattice

1 Introduction

A lot of practical and complicated problems in social science, economics, medical science, engineering, environmental science etc. have various uncertainties. In general, there are theories such as theory of probability, fuzzy sets, rough sets, vague sets and interval mathematics which can be considered as mathematical tools for dealing with uncertainties. Unfortunately, all these theories have their inherent difficulties. To overcome these difficulties, Molodsov [1] proposed the soft sets theory, which is a completely new approach for modeling vagueness and uncertainty.

At present, work on the soft set theory is progressing rapidly and many important results have been achieved in theory and application. Maji and Biswas et al. [2] defined some algebraic operations on soft set theory and verified that De Morgan' s laws hold in soft set theory. Ali et al. [3, 4] gave some new operations on soft sets, discussed their basic properties and studied semiring (hemiring) structures of soft sets. Aktas and Cagman [5] compared soft sets to the related concepts of fuzzy sets and rough sets, proposed the concept of soft groups and investigated their basic properties. some authors also applied soft sets to other algebra structures such as ordered semigroups [6], rings [7], semirings [8], BCK/BCI-algebras [9], d-algebras [10], and BL-algebras [11].

At the same time, the study on fuzzy extension of soft set theory has also received much attention by many researchers. Maji et al. [12] introduced the notions of fuzzy soft set. Majumdar and Samanta [13] further generalized the concept of fuzzy soft sets and derived the generalized fuzzy soft set model. Yang et al. [14] extended fuzzy soft sets to the interval-valued fuzzy soft sets. Xu et al. [15] proposed the concept of vague soft set by combining the vague set and soft set. In this paper, we combine the generalized fuzzy soft sets [13] and vague sets [16] and obtain a more generalized soft set model called generalized vague soft set, which can be viewed as a vague extension of the generalized fuzzy soft set theory [13] or a generalization of the vague soft set theory [15].

The rest of this paper is organized as follows. The following section briefly reviews some basic notions of soft sets. In section 3, the concept and some operations of generalized vague soft sets are given and some of their properties are investigated. In section 4, we discuss the lattice structures of generalized vague soft sets. Finally, the conclusion is given in section 5.

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2 Preliminary

In the current section we will briefly recall the notions of soft sets, fuzzy soft sets and generalized fuzzy soft sets. See especially [1, 12, 13, 16] for further details and background. Throughout this work, U refers to an initial universe, E is a set of parameters, the pair (U, E) is the soft universe on U with E, $A, B, C \subseteq E$ and α, β, γ are vague subsets of A, B, C respectively.

Definition 2.1. *[1] Let $P(U)$ is the power set of U. Then pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \to P(U)$.*

Definition 2.2. *[12] Let $\mathcal{P}(U)$ denote the set of all fuzzy subsets of U. Then a pair $(\widetilde{F}, A)$ is called a fuzzy soft set over U, where $\widetilde{F}$ is a mapping given by $\widetilde{F} : A \to \mathcal{P}(U)$.*

Definition 2.3. *[13] Let U be an initial universal set, E be a set of parameters. Let $F : E \to \mathcal{P}(U)$ and μ be a fuzzy subset of E, i.e. $\mu : E \to [0, 1]$. Let $F_\mu : E \to \mathcal{P}(U) \times [0, 1]$ be a function defined as $F_\mu(e) = (F(e), \mu(e))$, where $F(e) \in \mathcal{P}(U)$. Then F_μ is called a generalized fuzzy soft set over the soft universe (U, E).*

Definition 2.4. *[16] A vague set X in the universe $U = \{h_1, h_2, \cdots, h_n\}$ can be expressed by $X = \{h_i/[t_X(h_i), 1 - f_X(h_i)]|h_i \in U\}$, and the condition $0 \le t_X(h_i) \le 1 - f_X(h_i)$ should hold for any $h_i \in U$, where $t_X(h_i)$ is called membership degree of element h_i to the vague set X, while $f_X(h_i)$ is the degree of nonmembership of that element h_i to set X.*

Definition 2.5. *[16] Let X and Y be two vague sets on universe U. Then some operations of vague sets are given as follows:*

$$
\begin{aligned}
&X \cup Y = \{h_i/[t_X(h_i) \vee t_Y(h_i), 1 - f_X(h_i \wedge f_Y(h_i)])|h_i \in U\},\\
&X \cap Y = \{h_i/t_X(h_i) \wedge Y(h_i), 1 - f_X(h_i) \vee f_Y(h_i)]|h_i \in U\},\\
&X^c = \{h_i/[f_X(h_i), 1 - t_X(h_i)]|h_i \in U\}\\
&X \subseteq Y \Leftrightarrow t_X(h_i) \le t_Y(h_i) \ and \ f_X(h_i) \ge f_Y(h_i) \ for \ all \ h_i \in U.
\end{aligned}
$$

3 generalized vague soft set

It is well known that, by combining interval-valued fuzzy set and soft set, Yang et al. defined interval-valued fuzzy soft set in [14]. Motivated by this ideal, we first introduce the definition of generalized vague soft set based on the generalized fuzzy soft set and vague set.

Definition 3.1. *Let $A \subseteq E$, $\widetilde{F} : A \to \mathcal{F}(U)$ and α be a vague sets of A, i.e. $\alpha : A \to [0, 1]^2$, where $\mathcal{F}(U)$ is the set of all vague subsets of U. Let $\widetilde{F}_\alpha : A \to \mathcal{F}(U) \times [0, 1]^2$ be a function, defined as $\widetilde{F}_\alpha(e) = (\widetilde{F}(e) = \{h/\mu_{\widetilde{F}(e)}(h)|h \in U\}, \alpha(e))$, where $\mu_{\widetilde{F}(e)}(h) = [t_{\widetilde{F}(e)}(h), 1 - f_{\widetilde{F}(e)}(h)]$ is vague value is called the degree of membership an element h to $\widetilde{F}(e)$, and $\alpha(e) = [t_{\alpha(e)}, 1 - f_{\alpha(e)}]$ is called the degree of possibility of such belongness. Then $\widetilde{F}_\alpha$ is called generalized vague soft set $(inshortGVSset)$ over the soft universe (U, E).*

Here for each parameter e, $\widetilde{F}_\alpha(e)$ indicates not only the degree of belongingness of elements of U in $\widetilde{F}(e)$, but also the degree of preference of such belongingness which is represented by $\alpha(e)$.

To illustrate this idea, let us consider the following example.

Example 3.2. *Let $U = \{u_1, u_2, u_3\}$ be a set of mobile telephones and $A = \{e_1, e_2, e_3\} \subseteq E$ be a set of parameters. The $e_i (i = 1, 2, 3)$ stand for the parameters "expensive", "beautiful" and "multifunctional", respectively. Let $\widetilde{F}_\alpha : A \to \mathcal{P}(U) \times [0, 1]^2$ be a function given as follows:*

$$
\begin{aligned}
\widetilde{F}_\alpha(e_1) =&(\{h_1/[0.4, 0.9], h_2/[0.6, 0.8], h_3/[0.2, 0.6]\}, [0.4, 0.7]),\\
\widetilde{F}_\alpha(e_2) =&(\{h_1/[0.5, 0.8], h_2/[0.3, 0.7], h_3/[0.5, 0.9]\}, [0.6, 0.8]),\\
\widetilde{F}_\alpha(e_3) =&(\{h_1/[0.2, 0.6], h_2/[0.5, 0.9], h_3/[0.1, 0.8]\}, [0.3, 0.9]).
\end{aligned}
$$

Then $\widetilde{F}_\alpha$ is a GVS set.

Clearly, if $t_{\alpha(e)} + f_{\alpha(e)} = 1$ and $t_{\widetilde{F}(e)}(h) + f_{\widetilde{F}(e)}(h) = 1$ for all $e \in A$ and $h \in U$, then generalized vague soft set reduces to generalized fuzzy soft set [13]. If we do not consider the degree of preference e belonging to A, then generalized vague soft set reduces to vague soft set [15].

Definition 3.3. *Let U be an initial universe, E be a set of parameters and $A, B \subseteq E$. Let $\widetilde{F}_\alpha$ and $\widetilde{G}_\beta$ be two GVS sets, We say $\widetilde{F}_\alpha$ is a GVS subset of $\widetilde{G}_\beta$ if*
(1) $A \subseteq B$;
(2) $\widetilde{F}(e)$ is a vague subset of $\widetilde{G}(e)$ for all $e \in A$, i.e. $t_{\widetilde{F}(e)}(h) \leq t_{\widetilde{G}(e)}(h)$ and $f_{\widetilde{F}(e)}(h) \geq f_{\widetilde{G}(e)}(h)$ for all $h \in U$ and $e \in A$;
(3) α is a vague subset of β, i.e. $t_{\alpha(e)} \leq t_{\beta(e)}$ and $f_{\alpha(e)} \geq f_{\beta(e)}$ for all $e \in A$.

In this case, the above relationship is denoted by $\widetilde{F}_\alpha \Subset \widetilde{G}_\beta$. And $\widetilde{G}_\beta$ is said to be a GVS superset of $\widetilde{F}_\alpha$.

Definition 3.4. *Let $\widetilde{F}_\alpha$ and $\widetilde{G}_\beta$ be two GVS sets. Then we say $\widetilde{F}_\alpha$ and $\widetilde{G}_\beta$ to be GVS equal, denoted by $\widetilde{F}_\alpha = \widetilde{G}_\beta$, if and only if $\widetilde{F}_\alpha \Subset \widetilde{G}_\beta$ and $\widetilde{G}_\beta \Subset \widetilde{F}_\alpha$.*

Definition 3.5. *The complement of a GVS set $\widetilde{F}_\alpha$ over soft universe (U, E) is denoted by $\widetilde{F}^c_\alpha$ and is defined by $\widetilde{F}^c_\alpha = (\widetilde{F}^c, \alpha^c)$, where $\widetilde{F}^c : A \to \mathcal{F}(U)$ is a mapping given by $\widetilde{F}^c(e) = \{(h/\mu_{\widetilde{F}^c(e)}(h))|h \in U\}$ and $\alpha^c : A \to [0,1]^2$ is a mapping given by $\alpha^c(e) = \alpha^c(e) = [f_{\alpha(e)}, 1 - t_{\alpha(e)}]$ for all $e \in A$, where $\mu_{\widetilde{F}^c(e)}(h) = [f_{\widetilde{F}(e)}(h), 1 - t_{\widetilde{F}(e)}(h)]$.*

Example 3.6. (*continued*) *The complement of $\widetilde{F}_\alpha$ is following as:*

$$\begin{aligned}
\widetilde{F}^r_\alpha(e_1) =& (\{h_1/[0.1, 0.6], h_2/[0.2, 0.4], h_3/[0.4, 0.8]\}, [0.3, 0.6]),\\
\widetilde{F}^r_\alpha(e_2) =& (\{h_1/[0.2, 0.5], h_2/[0.3, 0.7], h_3/[0.1, 0.5]\}, [0.2, 0.4]),\\
\widetilde{F}^r_\alpha(e_2) =& (\{h_1/[0.4, 0.6], h_2/[0.1, 0.5], h_3/[0.2, 0.9]\}, [0.1, 0.7]).
\end{aligned}$$

Definition 3.7. *Let $A \subseteq E$, a GVS set $\widetilde{F}_\alpha$ over soft universe (U, E) is said to be relative absolute GVS set, denoted by $\widetilde{\Omega}_A$, if $t_{\widetilde{F}(e)}(h) = 1$, $f_{\widetilde{F}(e)}(h) = 0$ and $t_{\alpha(e)} = 1$, $f_{\alpha(e)} = 0$ for all $h \in U$ and $e \in A$.*

Definition 3.8. *Let $A \subseteq E$, a GVS set $\widetilde{F}_\alpha$ over soft universe (U, E) is said to be relative null GVS set denoted by $\widetilde{\Phi}_A$, if $t_{\widetilde{F}(e)}(h) = 0$, $f_{\widetilde{F}(e)}(h) = 1$ and $t_{\alpha(e)} = 0$, $f_{\alpha(e)} = 1$ for all $h \in U$ and $e \in A$.*

Definition 3.9. *Let $A, B \subseteq E$. we define a mapping $\widetilde{H}_\gamma : A \cup B \to \mathcal{F}(U) \times [0,1]^2$ such that for all $e \in A \cup B \neq \emptyset$,*

$$\widetilde{H}_\gamma(e) = \begin{cases} (\widetilde{F}_\alpha(e), [t_{\alpha(e)}, 1 - f_{\alpha(e)}]), & \text{if } e \in A - B,\\ (\{h/[t_{\widetilde{G}(e)}(h), 1 - f_{\widetilde{G}(e)}(h)]|h \in U\}, [t_{\beta(e)}, 1 - f_{\beta(e)}]), & \text{if } e \in B - A,\\ (\{h/[t_{\widetilde{H}(e)}(h), 1 - f_{\widetilde{H}(e)}(h)]|h \in U\}, [t_{\gamma(e)}, 1 - f_{\gamma(e)}]), & \text{if } e \in A \cap B. \end{cases}$$

(1) If $t_{\widetilde{H}(e)}(h) = t_{\widetilde{F}(e)}(h) \vee t_{\widetilde{G}(e)}(h)$, $f_{\widetilde{H}(e)}(h) = f_{\widetilde{F}(e)}(h) \wedge f_{\widetilde{G}(e)}(h)$, $t_{\gamma(e)} = t_{\alpha(e)} \vee t_{\beta(e)}$ and $f_{\gamma(e)} = f_{\alpha(e)} \wedge t_{\beta(e)}$, then $\widetilde{H}_\gamma$ is called the extended union of $\widetilde{F}_\alpha$ and $\widetilde{G}_\beta$, denoted by $\widetilde{F}_\alpha \widetilde{\cup} \widetilde{G}_\beta$.
(2) If $t_{\widetilde{H}(e)}(h) = t_{\widetilde{F}(e)}(h) \wedge t_{\widetilde{G}(e)}(h)$, $f_{\widetilde{H}(e)}(h) = f_{\widetilde{F}(e)}(h) \vee f_{\widetilde{G}(e)}(h)$, $t_{\gamma(e)} = t_{\alpha(e)} \wedge t_{\beta(e)}$ and $f_{\gamma(e)} = f_{\alpha(e)} \vee t_{\beta(e)}$, then $\widetilde{H}_\gamma$ is called the extended intersection of $\widetilde{F}_\alpha$ and $\widetilde{G}_\beta$, denoted by $\widetilde{F}_\alpha \widetilde{\cap} \widetilde{G}_\beta$.

If $A \cup B = \emptyset$, then $\widetilde{F}_\alpha \widetilde{\cup} \widetilde{G}_\beta = \widetilde{\Phi}_\emptyset$ and $\widetilde{F}_\alpha \widetilde{\cap} \widetilde{G}_\beta = \widetilde{\Phi}_\emptyset$.

Definition 3.10. *Let $A, B \subseteq E$. we define a mapping $\widetilde{H}_\gamma : A \cup B \to \mathcal{F}(U) \times [0,1]^2$ such that for all $e \in A \cap B \neq \emptyset$,*

$$\widetilde{H}_\gamma(e) = (\{h/[t_{\widetilde{H}(e)}(h), 1 - f_{\widetilde{H}(e)}(h)]|h \in U\}, [t_{\gamma(e)}, 1 - f_{\gamma(e)}]). \ .$$

(1) If $t_{\widetilde{H}(e)}(h) = t_{\widetilde{F}(e)}(h) \vee t_{\widetilde{G}(e)}(h)$, $f_{\widetilde{H}(e)}(h) = f_{\widetilde{F}(e)}(h) \wedge f_{\widetilde{G}(e)}(h)$, $t_{\gamma(e)} = t_{\alpha(e)} \vee t_{\beta(e)}$ and $f_{\gamma(e)} = f_{\alpha(e)} \wedge t_{\beta(e)}$, then $\widetilde{H}_\gamma$ is called the strict union of $\widetilde{F}_\alpha$ and $\widetilde{G}_\beta$, denoted by $\widetilde{F}_\alpha \Cup \widetilde{G}_\beta$.
(2) If $t_{\widetilde{H}(e)}(h) = t_{\widetilde{F}(e)}(h) \wedge t_{\widetilde{G}(e)}(h)$, $f_{\widetilde{H}(e)}(h) = f_{\widetilde{F}(e)}(h) \vee f_{\widetilde{G}(e)}(h)$, $t_{\gamma(e)} = t_{\alpha(e)} \wedge t_{\beta(e)}$ and $f_{\gamma(e)} = f_{\alpha(e)} \vee t_{\beta(e)}$, then $\widetilde{H}_\gamma$ is called the strict intersection of $\widetilde{F}_\alpha$ and $\widetilde{G}_\beta$, denoted by $\widetilde{F}_\alpha \Cap \widetilde{G}_\beta$.

If $A \cap B = \emptyset$, then $\widetilde{F}_\alpha \Cup \widetilde{G}_\beta = \widetilde{\Phi}_\emptyset$ and $\widetilde{F}_\alpha \Cap \widetilde{G}_\beta = \widetilde{\Phi}_\emptyset$.

Based on the above definitions, we can obtain the following properties.

Proposition 3.11. *Let $A \subseteq E$ and $\widetilde{F}_\alpha$ be a GVS set over (U, E). Then*
(1) $\widetilde{F}_\alpha \Cap \widetilde{\Omega}_E = \widetilde{F}_\alpha = \widetilde{F}_\alpha \widetilde{\cap}\, \widetilde{\Omega}_A$.
(2) $\widetilde{F}_\alpha \widetilde{\cup}\, \widetilde{\Omega}_A = \widetilde{\Omega}_A = \widetilde{F}_\alpha \Cup \widetilde{\Omega}_E$.
(3) $\widetilde{F}_\alpha \Cap \widetilde{\Phi}_E = \widetilde{\Phi}_A = \widetilde{F}_\alpha \widetilde{\cap} \widetilde{\Phi}_A$.
(4) $\widetilde{F}_\alpha \widetilde{\cup} \widetilde{\Phi}_A = \widetilde{F}_\alpha = \widetilde{F}_\alpha \Cup \widetilde{\Phi}_E$.
(5) $\widetilde{F}_\alpha \Cap \widetilde{\Phi}_\emptyset = \widetilde{\Phi}_\emptyset = \widetilde{F}_\alpha \Cup \widetilde{\Phi}_\emptyset$.
(6) $\widetilde{F}_\alpha \widetilde{\cup} \widetilde{\Phi}_\emptyset = \widetilde{F}_\alpha = \widetilde{F}_\alpha \widetilde{\cap} \widetilde{\Phi}_\emptyset$.

Theorem 3.12. *Let $\widetilde{F}_\alpha$, $\widetilde{G}_\beta$ and $\widetilde{H}_\gamma$ be three GVS sets. Then*
(1) $\widetilde{F}_\alpha \Cap \widetilde{F}_\alpha = \widetilde{F}_\alpha$.
(2) $\widetilde{F}_\alpha \Cap \widetilde{G}_\beta = \widetilde{G}_\beta \Cap \widetilde{F}_\alpha$.
(3) $\widetilde{F}_\alpha \Cap (\widetilde{G}_\beta \Cap \widetilde{H}_\gamma) = (\widetilde{F}_\alpha \Cap \widetilde{G}_\beta) \Cap \widetilde{H}_\gamma$.

Proof. (1) and (2) are trivial. We only prove (3). Assume that the parameter sets of GVS set $\widetilde{J}_\delta$ and $\widetilde{K}_\eta$ are denoted by M and N respectively. Let $\widetilde{F}_\alpha \Cap (\widetilde{G}_\beta \Cap \widetilde{H}_\gamma) = \widetilde{J}_\delta$ and $(\widetilde{F}_\alpha \Cap \widetilde{G}_\beta) \Cap \widetilde{H}_\gamma = \widetilde{K}_\eta$, where $M = N = A \cap B \cap C$. For each $e \in M, h \in U$, $\widetilde{J}(e) = \widetilde{F}(e) \cap (\widetilde{G}(e) \cap \widetilde{H}(e)) = (\widetilde{F}(e) \cap \widetilde{G}(e)) \cap \widetilde{H}(e) = \widetilde{F}(e) \cap \widetilde{G}(e) \cap \widetilde{H}(e) = \widetilde{K}(e)$, $\delta(e) = \alpha(e) \wedge \beta(e) \wedge \gamma(e) = \eta(e)$. Thus $\widetilde{J}_\delta = \widetilde{K}_\eta$, that is $\widetilde{F}_\alpha \Cap (\widetilde{G}_\beta \Cap \widetilde{H}_\gamma) = (\widetilde{F}_\alpha \Cap \widetilde{G}_\beta) \Cap \widetilde{H}_\gamma$. □

The following results can be obtained similarly.

Theorem 3.13. *Let $\widetilde{F}_\alpha$, $\widetilde{G}_\beta$ and $\widetilde{H}_\gamma$ be three GVS sets. Then*
(1) $\widetilde{F}_\alpha \widetilde{\cup}\, \widetilde{F}_\alpha = \widetilde{F}_\alpha$, $\widetilde{F}_\alpha \widetilde{\cap} \widetilde{F}_\alpha = \widetilde{F}_\alpha$, $\widetilde{F}_\alpha \Cup \widetilde{F}_\alpha = \widetilde{F}_\alpha$.
(2) $\widetilde{F}_\alpha \widetilde{\cup}\, \widetilde{G}_\beta = \widetilde{G}_\beta \widetilde{\cup}\, \widetilde{F}_\alpha$, $\widetilde{F}_\alpha \widetilde{\cap} \widetilde{G}_\beta = \widetilde{G}_\beta \widetilde{\cap} \widetilde{F}_\alpha$, $\widetilde{F}_\alpha \Cup \widetilde{G}_\beta = \widetilde{G}_\beta \Cup \widetilde{F}_\alpha$.
(3) $\widetilde{F}_\alpha \widetilde{\cup} (\widetilde{G}_\beta \widetilde{\cup}\, \widetilde{H}_\gamma) = (\widetilde{F}_\alpha \widetilde{\cup}\, \widetilde{G}_\beta) \widetilde{\cup}\, \widetilde{H}_\gamma$, $\widetilde{F}_\alpha \widetilde{\cap} (\widetilde{G}_\beta \widetilde{\cap} \widetilde{H}_\gamma) = (\widetilde{F}_\alpha \widetilde{\cap} \widetilde{G}_\beta) \widetilde{\cap} \widetilde{H}_\gamma$, $\widetilde{F}_\alpha \Cup (\widetilde{G}_\beta \Cup \widetilde{H}_\gamma) = (\widetilde{F}_\alpha \Cup \widetilde{G}_\beta) \Cup \widetilde{H}_\gamma$.

Remark 3.14. *Theorem 3.12 and Theorem 3.13 show that the operations $\Cap$, $\widetilde{\cup}$, $\widetilde{\cap}$ and $\Cup$ are idempotent, commutative and associative.*

Theorem 3.15. *Let $\widetilde{F}_\alpha$ and $\widetilde{G}_\beta$ be two GVS sets. Then the following holds.*
(1) $(\widetilde{F}_\alpha \Cap \widetilde{G}_\beta)^c = (\widetilde{F}_\alpha)^c \Cup (\widetilde{G}_\beta)^c$,
(2) $(\widetilde{F}_\alpha \Cup \widetilde{G}_\beta)^c = (\widetilde{F}_\alpha)^c \Cap (\widetilde{G}_\beta)^c$,
(3) $(\widetilde{F}_\alpha \widetilde{\cap} \widetilde{G}_\beta)^c = (\widetilde{F}_\alpha)^c \widetilde{\cup} (\widetilde{G}_\beta)^c$,
(4) $(\widetilde{F}_\alpha \widetilde{\cup} \widetilde{G}_\beta)^c = (\widetilde{F}_\alpha)^c \widetilde{\cap} (\widetilde{G}_\beta)^c$,

Proof. Because of (2),(3) and (4) are similar to (1), in the following, we only prove (1).

It is clear when $A \cap B = \emptyset$. Suppose that $\widetilde{F}_\alpha \Cap \widetilde{G}_\beta = \widetilde{H}_\gamma$, then $C = A \cap B \neq \emptyset$, and for all $e \in C$ and $h \in U$, we have

$$\begin{aligned}
\mu_{\widetilde{H}(e)}(h) &= [t_{\widetilde{F}(e)}(h) \wedge t_{\widetilde{G}(e)}(h)), (1 - f_{\widetilde{F}(e)}(h)) \wedge (1 - f_{\widetilde{G}(e)}(h))] \\
&= [t_{\widetilde{F}(e)}(h) \wedge t_{\widetilde{G}(e)}(h)), 1 - (f_{\widetilde{F}(e)}(h) \vee f_{\widetilde{G}(e)}(h))], \\
\gamma(e) &= [t_\alpha(e) \wedge t_\beta(e), (1 - f_\alpha(e)) \wedge (1 - f_\beta(e))] \\
&= [t_\alpha(e) \wedge t_\beta(e), 1 - (f_\alpha(e) \vee f_\beta(e))].
\end{aligned}$$

Then $(\widetilde{F}_\alpha \Cap \widetilde{G}_\beta)^c = \widetilde{H}^c_\gamma = (\{(h/\mu_{\widetilde{F}^c(e)}(h)) | h \in U\}, \gamma^c(e))$, where

$$\begin{aligned}
\mu_{\widetilde{H}^c(e)}(h) &= [f_{\widetilde{F}(e)}(h) \vee f_{\widetilde{G}(e)}(h)), 1 - t_{\widetilde{F}(e)}(h) \wedge t_{\widetilde{G}(e)}(h)], \\
\gamma^c(e) &= [f_\alpha(e) \vee f_\beta(e)), 1 - t_\alpha(e) \wedge t_\beta(e)].
\end{aligned}$$

On the other hand, since

$$\begin{aligned}
\mu_{\widetilde{F}^c(e)}(h) &= [f_{\widetilde{F}(e)}(h), 1 - t_{\widetilde{F}(e)}(h)],\ \alpha^c(e) = [f_\alpha(e), 1 - t_\alpha(e)], \\
\mu_{\widetilde{G}^c(e)}(h) &= [f_{\widetilde{G}(e)}(h), 1 - t_{\widetilde{G}(e)}(h)],\ \beta^c(e) = [f_\beta(e), 1 - t_\beta(e)].
\end{aligned}$$

And let $\widetilde{F}^c_\alpha \Cup \widetilde{G}^c_\beta = \widetilde{J}_\delta$ (the parameters set of $\widetilde{J}_\delta$ is denoted D). Then $D = A \cap B$, and for each $e \in D$, we have

$$\begin{aligned}
\mu_{\widetilde{J}(e)}(h) &= \mu_{\widetilde{F}^c(e)}(h) \vee \mu_{\widetilde{G}^c(e)}(h)\\
&= [f_{\widetilde{F}(e)}(h) \vee f_{\widetilde{G}(e)}(h), (1 - t_{\widetilde{F}(e)}(h)) \vee (1 - t_{\widetilde{G}(e)}(h))]\\
&= [f_{\widetilde{F}(e)}(h) \vee f_{\widetilde{G}(e)}(h), 1 - t_{\widetilde{F}(e)}(h) \wedge t_{\widetilde{G}(e)}(h)]\\
&= \mu_{\widetilde{H}^c(e)}(h),\\
\delta(e) &= \alpha^c(e) \vee \beta^c(e)\\
&= [f_\alpha(e) \vee f_\beta(e), (1 - t_\alpha(e)) \vee (1 - t_\beta(e))]\\
&= [f_\alpha(e) \vee f_\beta(e), 1 - t_\alpha(e) \wedge t_\beta(e)]\\
&= \gamma^c(e).
\end{aligned}$$

Therefore, $\widetilde{H}^c_\gamma$ and $\widetilde{J}_\delta$ are the same GVS sets. Thus, $(\widetilde{F}_\alpha \Cap \widetilde{G}_\beta)^c = (\widetilde{F}_\alpha)^c \Cup (\widetilde{G}_\beta)^c$. □

4 The lattice structures of GVS sets

In this section, we will discuss the lattice structures of GVS sets. For convenience, the set of all GVS sets over soft universe (U, E) denotes by $GVS(U, E)$, i.e., $GVS(U, E) = \{\widetilde{F}_\alpha | A \subseteq E, F : A \to \mathcal{F}(U), \alpha : A \to [0, 1]^2\}$. The following theorem shows that the distribution law with respect to operations $\widetilde{\cup}$ and $\Cap$ holds.

Theorem 4.1. *Let $A, B, C \in E$ and $\widetilde{F}_\alpha$, $\widetilde{G}_\beta$ and $\widetilde{H}_\gamma$ be three GVS sets. Then*
(1) $\widetilde{F}_\alpha \Cap (\widetilde{G}_\beta \widetilde{\cup} \widetilde{H}_\gamma) = (\widetilde{F}_\alpha \Cap \widetilde{G}_\beta) \widetilde{\cup} (\widetilde{F}_\alpha \Cap \widetilde{H}_\gamma)$.
(2) $\widetilde{F}_\alpha \widetilde{\cup} (\widetilde{G}_\beta \Cap \widetilde{H}_\gamma) = (\widetilde{F}_\alpha \widetilde{\cup} \widetilde{G}_\beta) \Cap (\widetilde{F}_\alpha \widetilde{\cup} \widetilde{H}_\gamma)$.

Proof. (1) Assume that the parameter sets of two GVS sets $\widetilde{J}_\delta$ and $\widetilde{K}_\eta$ are denoted by M and N respectively. Let $\widetilde{F}_\alpha \Cap (\widetilde{G}_\beta \widetilde{\cup} \widetilde{H}_\gamma) = \widetilde{J}_\delta$ and $(\widetilde{F}_\alpha \Cap \widetilde{G}_\beta) \widetilde{\cup} (\widetilde{F}_\alpha \Cap \widetilde{H}_\gamma) = \widetilde{K}_\eta$. Then $M = A \cap (B \cup C) = (A \cap B) \cup (A \cap C) = N$. For each $e \in M$, it follows that $e \in A$ and $e \in B \cup C$.
(i) if $e \in A, e \notin B, e \in C$, then $\widetilde{J}(e) = \widetilde{F}(e) \cap \widetilde{H}(e) = \widetilde{K}(e)$, and $\delta(e) = \alpha(e) \wedge \gamma(e) = \eta(e)$.
(ii) if $e \in A, e \in B, e \notin C$, then $\widetilde{J}(e) = \widetilde{F}(e) \cap \widetilde{G}(e) = \widetilde{K}(e)$, and $\delta(e) = \alpha(e) \wedge \beta(e) = \eta(e)$.
(iii) if $e \in A, e \in B, e \in C$, then $\widetilde{J}(e) = \widetilde{F}(e) \cap (\widetilde{G}(e) \cup \widetilde{H}(e)) = (\widetilde{F}(e) \cap \widetilde{G}(e) \cup (\widetilde{F}(e) \cap \widetilde{H}_\gamma) = \widetilde{K}(e)$, $\delta(e) = \alpha(e) \wedge (\beta(e) \vee \gamma(e)) = (\alpha(e) \wedge \beta(e)) \vee (\alpha(e) \wedge \gamma(e)) = \eta(e)$.
Thus $\widetilde{J}_\delta = \widetilde{K}_\eta$, that is $\widetilde{F}_\alpha \Cap (\widetilde{G}_\beta \widetilde{\cup} \widetilde{H}_\gamma) = (\widetilde{F}_\alpha \Cap \widetilde{G}_\beta) \widetilde{\cup} (\widetilde{F}_\alpha \Cap \widetilde{H}_\gamma)$.
(2) The proof is similar to that of (1). □

Theorem 4.2. (1) *For all $\widetilde{F}_\alpha, \widetilde{G}_\beta \in GVS(U, E)$, if the ordering relation in $GVS(U, E)$ is defined as $\widetilde{F}_\alpha \leq_1 \widetilde{G}_\beta \Leftrightarrow A \subseteq B, \widetilde{F}(e) \subseteq \widetilde{G}(e)$ and $\alpha(e) \leq \beta(e)$ for all $e \in A$, then $(GVS(U, E), \leq_1)$ is a partially ordered set ($POSET for short$).*
(2) $(GVS(U, E), \widetilde{\cup}, \Cap)$ *is a bounded lattice.*

Proof. (1) (Reflexivity): Let $\widetilde{F}_\alpha \in GVS(U, E)$. It is clear that $\widetilde{F}_\alpha \leq_1 \widetilde{F}_\alpha$.
(Antisymmetry): Let $\widetilde{F}_\alpha \leq_1 \widetilde{G}_\beta$ and $\widetilde{G}_\beta \leq_1 \widetilde{F}_\alpha$, then $A \subseteq B$, $B \subseteq A$, $\widetilde{F}(e) \subseteq \widetilde{G}(e)$, $\widetilde{G}(e) \subseteq \widetilde{F}(e)$, $\alpha(e) \leq \beta(e)$ and $\beta(e) \leq \alpha(e)$ for all $e \in A$. It follows that $A = B$, $\widetilde{F}(e) = \widetilde{G}(e)$ and $\alpha(e) = \beta(e)$. Thus $\widetilde{F}_\alpha = \widetilde{G}_\beta$.
(Transitivity): Let $\widetilde{F}_\alpha, \widetilde{G}_\beta, \widetilde{H}_\gamma \in GVS(U, E)$, $\widetilde{F}_\alpha \leq_1 \widetilde{G}_\beta$ and $\widetilde{G}_\beta \leq_1 \widetilde{H}_\gamma$, then $A \subseteq B \subseteq C$, $\widetilde{F}(e) \subseteq \widetilde{G}(e) \subseteq \widetilde{H}(e)$ and $\alpha(e) \leq \beta(e) \leq \gamma(e)$ for all $e \in A$. Thus $\widetilde{F}_\alpha \leq_1 \widetilde{H}_\gamma$.
(2) Let Let $\widetilde{F}_\alpha, \widetilde{G}_\beta, \widetilde{H}_\gamma \in GVS(U, E)$ and $\widetilde{F}_\alpha \widetilde{\cup} \widetilde{G}_\beta = \widetilde{H}_\gamma$. Then $C = A \cup B$ and for all $e \in C$,

$$\widetilde{H}_\gamma(e) = \begin{cases} (\widetilde{F}(e), \alpha(e)), & \text{if } e \in A - B,\\ (\widetilde{G}(e), \beta(e)), & \text{if } e \in B - A,\\ (\widetilde{H}(e), \gamma(e)), & \text{if } e \in A \cap B.\end{cases}$$

Then $A \subseteq C$. i) If $e \in A$ and $e \notin B$, then $\widetilde{F}(e) \subseteq \widetilde{F}(e) = \widetilde{H}(e)$ and $\alpha(e) \leq \alpha(e) = \gamma(e)$; ii) If $e \in A$ and $e \in B$, then $\widetilde{F}(e) \subseteq \widetilde{F}(e) \cup \widetilde{G}(e) = \widetilde{H}(e)$ and $\alpha(e) \leq \alpha(e) \vee \beta(e) = \gamma(e)$. Thus $\widetilde{F}_\alpha \leq_1 \widetilde{H}_\gamma$. Similarly, we have $\widetilde{G}_\beta \leq_1 \widetilde{H}_\gamma$. Therefore $\widetilde{F}_\alpha \widetilde{\cup} \widetilde{G}_\beta$ is a upper bound of $\widetilde{F}_\alpha$ and $\widetilde{G}_\beta$.

Suppose that $\widetilde{J}_\delta$ (the parameters set of $\widetilde{J}_\delta$ is denoted D) is another upper bound of $\widetilde{F}_\alpha$ and $\widetilde{G}_\beta$, then $A \in D$, $B \in D$, $\widetilde{F}(e) \subseteq \widetilde{J}(e)$, $\widetilde{G}(e) \subseteq \widetilde{J}(e)$, $\alpha(e) \leq \delta(e)$ and $\beta(e) \leq \delta(e)$ for all $e \in A$. It implies $A \cup B \in D$, $\widetilde{F}(e) \cup \widetilde{G}(e) \subseteq \widetilde{J}(e)$ and $\alpha(e) \vee \beta(e) \leq \delta(e)$ for all $e \in A$. Thus $\widetilde{F}_\alpha \widetilde{\cup} \widetilde{G}_\beta \leq_1 \widetilde{J}_\delta$. Therefore, for all $\widetilde{F}_\alpha, \widetilde{G}_\beta \in GVS(U,E)$, we have $\widetilde{F}_\alpha \widetilde{\cup} \widetilde{G}_\beta$ is the supremum of $\widetilde{F}_\alpha$ and $\widetilde{G}_\beta$. Similarly, we can prove that $\widetilde{F}_\alpha \Cap \widetilde{G}_\beta$ is the infimum of $\widetilde{F}_\alpha$ and $\widetilde{G}_\beta$. It is obvious that $\widetilde{\Omega}_E$ and $\widetilde{\Phi}_\emptyset$ are the maximum and minimum element of $(GVS(U,E), \widetilde{\cup}, \Cap)$, respectively. □

Corollary 4.3. *$(GVS(U,E), \widetilde{\cup}, \Cap)$ is a bounded distributive lattice.*

Now we consider GVS sets over a definite parameter set. Let $A \subseteq E$ and $GVS_A = \{\widetilde{F}_\alpha | F : A \to \mathcal{F}(U), \alpha : A \to [0,1]^2\}$ denote the set of all GVS sets over (U,A). It is easy to verify that $\widetilde{F}_\alpha \widetilde{\cup} \widetilde{G}_\alpha \in GVS_A$ and $\widetilde{F}_\alpha \Cap \widetilde{G}_\alpha \in GVS_A$ for all $\widetilde{F}_\alpha, \widetilde{G}_\alpha \in GVS_A$. Thus, the following proposition holds.

Proposition 4.4. *$(GVS_A, \widetilde{\cup}, \Cap)$ is a sublattice of $(GVS(U,E), \widetilde{\cup}, \Cap)$.*

For operations $\Cup$ and $\widetilde{\cap}$, we can obtain similar results as follows.

Theorem 4.5. *Let $\widetilde{F}_\alpha$, $\widetilde{G}_\beta$ and $\widetilde{H}_\gamma$ be three GVS sets over (U,E). Then the following holds:*
(1) $\widetilde{F}_\alpha \widetilde{\cap} (\widetilde{G}_\beta \Cup \widetilde{H}_\gamma) = (\widetilde{F}_\alpha \widetilde{\cap} \widetilde{G}_\beta) \Cup (\widetilde{F}_\alpha \widetilde{\cap} \widetilde{H}_\gamma)$.
(2) $\widetilde{F}_\alpha \Cup (\widetilde{G}_\beta \widetilde{\cap} \widetilde{H}_\gamma) = (\widetilde{F}_\alpha \Cup \widetilde{G}_\beta) \widetilde{\cap} (\widetilde{F}_\alpha \Cup \widetilde{H}_\gamma)$.

Theorem 4.6. (1) *For all $\widetilde{F}_\alpha, \widetilde{G}_\beta \in GVS(U,E)$, if the ordering relation in $GVS(U,E)$ is defined as $\widetilde{F}_\alpha \leq_2 \widetilde{G}_\beta \Leftrightarrow B \subseteq A, \widetilde{F}(e) \subseteq \widetilde{G}(e)$ and $\alpha(e) \leq \beta(e)$ for all $e \in A$, then $(GVS(U,E), \leq_2)$ is a $POSET$.*
(2) *$(GVS(U,E), \Cup, \widetilde{\cap})$ is a bounded distributive lattice.*

Proposition 4.7. *$(GVS_A, \Cup, \widetilde{\cap})$ is a sublattice of $(GVS(U,E), \Cup, \widetilde{\cap})$.*

However, It is worth noting that the lattice structure $(GVS(U,E), \widetilde{\cup}, \Cap)$ is different from that of $(GVS(U,E), \Cup, \widetilde{\cap})$.

Remark 4.8. *In general, $(GVS(U,E), \Cap, \Cup)$ and $(GVS(U,E), \widetilde{\cap}, \widetilde{\cup})$ are not lattice. Because the absorptive laws with respect to two pair operations $\Cap$ and $\Cup$, $\widetilde{\cap}$ and $\widetilde{\cup}$ may not necessarily hold, i.e. $(\widetilde{F}_\alpha \Cup \widetilde{G}_\beta) \Cap \widetilde{F}_\alpha = \widetilde{F}_\alpha$ and $(\widetilde{F}_\alpha \widetilde{\cap} \widetilde{G}_\beta) \widetilde{\cup} \widetilde{F}_\alpha = \widetilde{F}_\alpha$ do not hold in general.*

To illustrate the above Remark, we give an example as follows.

Example 4.9. *Let $U = \{h_1, h_2, h_3\}$ be the universe and $E = \{e_1, e_2, e_3\}$ be the set of parameters. Let $\widetilde{F}_\alpha$ and $\widetilde{G}_\beta$ be two GVS sets, where $A = \{e_1, e_2\}$, $B = \{e_2, e_3\}$, and they are given as follows:*

$$
\begin{aligned}
\widetilde{F}_\alpha(e_1) &= (\{h_1/[0.1,0.2], h_2/[0.8,0.9], h_3/[0.3,0.7]\}, [0.1,0.2]),\\
\widetilde{F}_\alpha(e_2) &= (\{h_1/[0.6,0.8], h_2/[0.1,0.3], h_3/[0.4,0.8]\}, [0.3,0.5]),\\
\widetilde{G}_\beta(e_2) &= (\{h_1/[0.5,0.8], h_2/[0.2,0.5], h_3/[0.5,0.6]\}, [0.4,0.8]),\\
\widetilde{G}_\beta(e_3) &= (\{h_1/[0.3,0.7], h_2/[0.4,0.8], h_3/[0.3,0.8]\}, [0.3,0.6]).
\end{aligned}
$$

Let the parameters set of a GVS set $\widetilde{J}_\delta$ is denoted C and suppose that $\widetilde{H}_\gamma = (\widetilde{F}_\alpha \Cup \widetilde{G}_\beta) \Cap \widetilde{F}_\alpha$. Then $C = A \cap B = \{e_2\} \neq A$. So $\widetilde{H}_\gamma \neq \widetilde{F}_\alpha$, i.e. $(\widetilde{F}_\alpha \Cup \widetilde{G}_\beta) \Cap \widetilde{F}_\alpha \neq \widetilde{F}_\alpha$.

Again suppose that the parameters set of a GVS set $\widetilde{J}_\delta$ is denoted D, and $\widetilde{J}_\delta = (\widetilde{F}_\alpha \widetilde{\cap} \widetilde{G}_\beta) \widetilde{\cup} \widetilde{F}_\alpha$. Then $D = A \cup B = \{e_1, e_2, e_3\} \neq A$, Hence $\widetilde{J}_\delta \neq \widetilde{F}_\alpha$, i.e. $(\widetilde{F}_\alpha \widetilde{\cap} \widetilde{G}_\beta) \widetilde{\cup} \widetilde{F}_\alpha \neq \widetilde{F}_\alpha$.

5 Conclusion

The researches on theories and application of soft sets have received more and more attention from many scholars. In order to extend the rang of application of soft set, in this paper, we have proposed the notion of generalized vague soft set by combining generalized fuzzy soft set and vague set, which is a generalization of the generalized fuzzy soft set and vague soft set. Some basic operations and some desirable properties of these operations have been presented. Furthermore, the lattice structure of generalized vague soft set have been investigated in detail.

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Some Inequalities of Hermite-Hadamard Type for Functions Whose Third Derivatives Are (α, m)-Convex

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Abstract

In the paper, the authors establish a new integral identity and, by this identity and Hölder's inequality, discover some new Hermite-Hadamard type integral inequalities for functions whose third derivatives are (α, m)-convex.

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1 Introduction

It is common knowledge in mathematical analysis that a function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex on an interval I if

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y) \tag{1.1}$$

for all $x, y \in I$ and $\lambda \in [0,1]$. For such a kind of convex functions on I with $a, b \in I$ and $a < b$, the double inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \le \frac{f(a)+f(b)}{2} \tag{1.2}$$

holds. This inequality is known in the literature as Hermite-Hadamard inequality for convex functions. Many classical inequalities in real numbers can be derived from Hermite-Hadamard inequality (1.2).

The above traditionally defined convex functions can be generalized as follows.

Definition 1.1 ([11])**.** Let $f : [0, b] \to \mathbb{R}$ for $b > 0$ and $m \in [0,1]$. We say that f is m-convex on $[0, b]$ if

$$f(tx + m(1-t)y) \le tf(x) + m(1-t)f(y) \tag{1.3}$$

holds for all $x, y \in [0, b]$ and $t \in [0,1]$.

1

Definition 1.2 ([7]). Let $f : [0, b] \to \mathbb{R}$ for $b > 0$ and $(\alpha, m) \in [0, 1]^2$. We say that $f(x)$ is (α, m)-convex on $[0, b]$ if

$$f(\lambda x + m(1-\lambda)y) \le \lambda^\alpha f(x) + m(1-\lambda^\alpha)f(y) \tag{1.4}$$

holds for all $x, y \in [0, b]$ and $\lambda \in [0, 1]$.

For the above defined convex functions, we have the following Hermite-Hadamard type inequalities.

Theorem 1.1 ([4]). *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable on I° and $a, b \in I^\circ$ with $a < b$.*

1. *If $|f'(x)|$ is convex function on $[a, b]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{b-a}{8}\big(|f'(a)| + |f'(b)|\big). \tag{1.5}$$

2. *If $|f'(x)|^{p/(p-1)}$ is convex function on $[a, b]$ for $p > 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{b-a}{2(p+1)^{1/p}}\left[\frac{|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)}}{2}\right]^{(p-1)/p}. \tag{1.6}$$

Theorem 1.2 ([6, Theorem 2.2]). *Let $I \supset \mathbb{R}_0 = [0, \infty)$ be an open real interval and let $f : I \to \mathbb{R}$ be a differentiable function on I such that $f' \in L([a, b])$ for $0 \le a < b < \infty$. If $|f'(x)|^q$ is m-convex on $[a, b]$ for some given numbers $m \in (0, 1]$ and $q \ge 1$, then*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \\ &\quad \le \frac{b-a}{4}\min\left\{\left(\frac{|f'(a)|^q + m|f'(b/m)|^q}{2}\right)^{1/q}, \left(\frac{m|f'(a/m)|^q + |f'(b)|^q}{2}\right)^{1/q}\right\}.\end{aligned} \tag{1.7}$$

Theorem 1.3 ([6, Theorem 3.1]). *Let $I \supset \mathbb{R}_0$ be an open real interval and let $f : I \to \mathbb{R}$ be a differentiable function on I such that $f' \in L([a, b])$ for $0 \le a < b < \infty$. If $|f'(x)|^q$ is (α, m)-convex on $[a, b]$ for some given numbers $m, \alpha \in (0, 1]$ and $q \ge 1$, then*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \\ &\quad \le \frac{b-a}{2^{2-1/q}}\min\left\{\left[v_1|f'(a)|^q + v_2 m\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{1/q}, \left[v_2 m\left|f'\left(\frac{a}{m}\right)\right|^q + v_1|f'(b)|^q\right]^{1/q}\right\},\end{aligned} \tag{1.8}$$

where

$$v_1 = \frac{1}{(\alpha+1)(\alpha+2)}\left(\alpha + \frac{1}{2^\alpha}\right) \quad \textit{and} \quad v_2 = \frac{1}{(\alpha+1)(\alpha+2)}\left(\frac{\alpha^2+\alpha+2}{2} - \frac{1}{2^\alpha}\right). \tag{1.9}$$

Theorem 1.4 ([1, Theorem 2]). *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, such that $f^{(3)} \in L[a, b]$ for $a, b \in I^\circ$ and $a < b$. If $\left|f^{(3)}(x)\right|$ is quasi-convex on $[a, b]$, then*

$$\begin{aligned}&\left|\int_a^b f(x)\,\mathrm{d}x - \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right| \\ &\quad \le \frac{(b-a)^4}{1152}\left[\max\left\{\left|f^{(3)}(a)\right|, \left|f^{(3)}\left(\frac{a+b}{2}\right)\right|\right\} + \max\left\{\left|f^{(3)}\left(\frac{a+b}{2}\right)\right|, \left|f^{(3)}(b)\right|\right\}\right].\end{aligned} \tag{1.10}$$

In recent years, some other kinds of Hermite-Hadamard type inequalities were generated in, for example, [2, 3, 9, 10, 12, 13, 14] and related references therein. For more systematic information, please refer to monographs [5, 7, 8, 9] and related references therein.

In this paper, we will establish a new integral identity and, by this identity and Hölder's inequality, discover some new Hermite-Hadamard type integral inequalities for functions whose third derivatives are (α, m)-convex.

2 A lemma

For establishing some new integral inequalities of Hermite-Hadamard type for functions whose third derivatives are (α, m)-convex, we need the following lemma.

Lemma 2.1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a three times differentiable mapping on I° and $a, b \in I^\circ$ with $a < b$. If $f^{(3)} \in L_1([a, b])$, then*

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x - f\left(\frac{a+b}{2}\right) - \frac{(b-a)^2}{24} f''\left(\frac{a+b}{2}\right) \\ = \frac{(b-a)^3}{96}\left[\int_0^1 t^3 f^{(3)}\left(\frac{t}{2}a + \frac{2-t}{2}b\right)\mathrm{d}t - \int_0^1 t^3 f^{(3)}\left(\frac{2-t}{2}a + \frac{t}{2}b\right)\mathrm{d}t\right]. \end{aligned} \tag{2.1}$$

Proof. Integrating by parts and changing variable of definite integral yield

$$\begin{aligned} &\int_0^1 t^3 f^{(3)}\left(\frac{t}{2}a + \frac{2-t}{2}b\right)\mathrm{d}t \\ &= -\frac{2}{b-a} f''\left(\frac{a+b}{2}\right) + \frac{6}{b-a}\int_0^1 t^2 f''\left(\frac{t}{2}a + \frac{2-t}{2}b\right)\mathrm{d}t \\ &= -\frac{2}{b-a} f''\left(\frac{a+b}{2}\right) - \frac{12}{(b-a)^2} f'\left(\frac{a+b}{2}\right) + \frac{24}{(b-a)^2}\int_0^1 t f'\left(\frac{t}{2}a + \frac{2-t}{2}b\right)\mathrm{d}t \\ &= -\frac{2}{b-a} f''\left(\frac{a+b}{2}\right) - \frac{12}{(b-a)^2} f'\left(\frac{a+b}{2}\right) \\ &\quad - \frac{48}{(b-a)^3} f\left(\frac{a+b}{2}\right) + \frac{48}{(b-a)^3}\int_0^1 f\left(\frac{t}{2}a + \frac{2-t}{2}b\right)\mathrm{d}t \end{aligned}$$

and

$$\begin{aligned} &\int_0^1 t^3 f^{(3)}\left(\frac{2-t}{2}a + \frac{t}{2}b\right)\mathrm{d}t \\ &= \frac{2}{b-a} f''\left(\frac{a+b}{2}\right) - \frac{6}{b-a}\int_0^1 t^2 f''\left(\frac{2-t}{2}a + \frac{t}{2}b\right)\mathrm{d}t \\ &= \frac{2}{b-a} f''\left(\frac{a+b}{2}\right) - \frac{12}{(b-a)^2} f'\left(\frac{a+b}{2}\right) + \frac{24}{(b-a)^2}\int_0^1 t f'\left(\frac{2-t}{2}a + \frac{t}{2}b\right)\mathrm{d}t \\ &= \frac{2}{b-a} f''\left(\frac{a+b}{2}\right) - \frac{12}{(b-a)^2} f'\left(\frac{a+b}{2}\right) \\ &\quad + \frac{48}{(b-a)^3} f\left(\frac{a+b}{2}\right) - \frac{48}{(b-a)^3}\int_0^1 f\left(\frac{2-t}{2}a + \frac{t}{2}b\right)\mathrm{d}t. \end{aligned}$$

Lemma 2.1 is thus proved. □

3

3 Hermite-Hadamard type inequalities for (α,m)-convex functions

Now we are in a position to establish some integral inequalities of Hermite-Hadamard type for functions whose third derivatives are (α,m)-convex.

Theorem 3.1. *Let $f:\mathbb{R}_0\to\mathbb{R}$ be three times differentiable and $f^{(3)}\in L_1\big(\big[a,\frac{b}{m}\big]\big)$ for $0\le a<b$. If $\big|f^{(3)}\big|^q$ is (α,m)-convex on $\big[a,\frac{b}{m}\big]$ for $q\ge1$ and $(\alpha,m)\in[0,1]\times(0,1]$, then*

$$\begin{aligned}\bigg|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x-f\bigg(\frac{a+b}{2}\bigg)-\frac{(b-a)^2}{24}f''\bigg(\frac{a+b}{2}\bigg)\bigg|&\\ \le\frac{(b-a)^3}{384}\bigg[\frac{1}{(\alpha+4)2^\alpha}\bigg]^{1/q}\bigg\{\bigg[4\big|f^{(3)}(a)\big|^q+m((\alpha+4)2^\alpha-4)\bigg|f^{(3)}\bigg(\frac{b}{m}\bigg)\bigg|^q\bigg]^{1/q}&\\ +\bigg[m((\alpha+4)2^\alpha-4)\bigg|f^{(3)}\bigg(\frac{a}{m}\bigg)\bigg|^q+4\big|f^{(3)}(b)\big|^q\bigg]^{1/q}\bigg\}.&\qquad(3.1)\end{aligned}$$

Proof. Because $\big|f^{(3)}\big|^q$ is (α,m)-convex on $\big[a,\frac{b}{m}\big]$, by Lemma 2.1 and Hölder's inequality, we have

$$\begin{aligned}&\bigg|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x-f\bigg(\frac{a+b}{2}\bigg)-\frac{(b-a)^2}{24}f''\bigg(\frac{a+b}{2}\bigg)\bigg|\\ &\le\frac{(b-a)^3}{96}\bigg\{\int_0^1 t^3\bigg|f^{(3)}\bigg(\frac{t}{2}a+\frac{2-t}{2}b\bigg)\bigg|\,\mathrm{d}t+\int_0^1 t^3\bigg|f^{(3)}\bigg(\frac{2-t}{2}a+\frac{t}{2}b\bigg)\bigg|\,\mathrm{d}t\bigg\}\\ &\le\frac{(b-a)^3}{96}\bigg(\int_0^1 t^3\,\mathrm{d}t\bigg)^{1-1/q}\bigg\{\bigg[\int_0^1 t^3\bigg|f^{(3)}\bigg(\frac{t}{2}a+\frac{2-t}{2}b\bigg)\bigg|^q\,\mathrm{d}t\bigg]^{1/q}\\ &\quad+\bigg[\int_0^1 t^3\bigg|f^{(3)}\bigg(\frac{2-t}{2}a+\frac{t}{2}b\bigg)\bigg|^q\,\mathrm{d}t\bigg]^{1/q}\bigg\}\\ &\le\frac{(b-a)^3}{96}\bigg(\frac{1}{4}\bigg)^{1-1/q}\bigg\{\bigg[\int_0^1 t^3\bigg(\frac{t}{2}\bigg)^\alpha\big|f^{(3)}(a)\big|^q\,\mathrm{d}t+m\int_0^1 t^3\bigg(1-\bigg(\frac{t}{2}\bigg)^\alpha\bigg)\big|f^{(3)}(b/m)\big|^q\,\mathrm{d}t\bigg]^{1/q}\\ &\quad+\bigg[\int_0^1 t^3\bigg(\frac{t}{2}\bigg)^\alpha\big|f^{(3)}(b)\big|^q\,\mathrm{d}t+m\int_0^1 t^3\bigg(1-\bigg(\frac{t}{2}\bigg)^\alpha\bigg)\bigg|f^{(3)}\bigg(\frac{a}{m}\bigg)\bigg|^q\,\mathrm{d}t\bigg]^{1/q}\bigg\}\\ &=\frac{(b-a)^3}{384}\bigg[\frac{1}{(\alpha+4)2^\alpha}\bigg]^{1/q}\bigg\{\bigg[4\big|f^{(3)}(a)\big|^q+m((\alpha+4)2^\alpha-4)\bigg|f^{(3)}\bigg(\frac{b}{m}\bigg)\bigg|^q\bigg]^{1/q}\\ &\quad+\bigg[m((\alpha+4)2^\alpha-4)\bigg|f^{(3)}\bigg(\frac{a}{m}\bigg)\bigg|^q+4\big|f^{(3)}(b)\big|^q\bigg]^{1/q}\bigg\}.\end{aligned}$$

The proof of Theorem 3.1 is completed. □

Corollary 3.1. *Under the conditions of Theorem 3.1,*

1. *if $q=1$, we have*

$$\begin{aligned}&\bigg|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x-f\bigg(\frac{a+b}{2}\bigg)-\frac{(b-a)^2}{24}f''\bigg(\frac{a+b}{2}\bigg)\bigg|\\ &\le\frac{(b-a)^3}{3(\alpha+4)2^{\alpha+7}}\bigg\{4\big[\big|f^{(3)}(a)\big|+\big|f^{(3)}(b)\big|\big]+m((\alpha+4)2^\alpha-4)\bigg[\bigg|f^{(3)}\bigg(\frac{a}{m}\bigg)\bigg|+\bigg|f^{(3)}\bigg(\frac{b}{m}\bigg)\bigg|\bigg]\bigg\};\end{aligned}$$

2. if $q=\alpha=1$, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x - f\left(\frac{a+b}{2}\right) - \frac{(b-a)^2}{24} f''\left(\frac{a+b}{2}\right)\right| \le \frac{(b-a)^3}{1920}\left\{2\left[\left|f^{(3)}(a)\right| + \left|f^{(3)}(b)\right|\right] + 3m\left[\left|f^{(3)}\left(\frac{a}{m}\right)\right| + \left|f^{(3)}\left(\frac{b}{m}\right)\right|\right]\right\};$$

3. if $q=m=\alpha=1$, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x - f\left(\frac{a+b}{2}\right) - \frac{(b-a)^2}{24} f''\left(\frac{a+b}{2}\right)\right| \le \frac{(b-a)^3}{384}\left[\left|f^{(3)}(a)\right| + \left|f^{(3)}(b)\right|\right].$$

Theorem 3.2. *Let $f:\mathbb{R}_0\to\mathbb{R}$ be three times differentiable and $f^{(3)}\in L_1\left(\left[a,\frac{b}{m}\right]\right)$ for $0\le a<b$. If $\left|f^{(3)}\right|^q$ is (α,m)-convex on $\left[a,\frac{b}{m}\right]$ for $q>1$ and $(\alpha,m)\in[0,1]\times(0,1]$, then*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x - f\left(\frac{a+b}{2}\right) - \frac{(b-a)^2}{24} f''\left(\frac{a+b}{2}\right)\right| \\
&\le \frac{(b-a)^3}{96}\left[\frac{1}{(3q+1)(3q+\alpha+1)2^\alpha}\right]^{1/q} \\
&\times\left\{\left[(3q+1)\left|f^{(3)}(a)\right|^q + m\big((3q+\alpha+1)2^\alpha-(3q+1)\big)\left|f^{(3)}\left(\frac{b}{m}\right)\right|^q\right]^{1/q}\right. \\
&\left.+\left[m\big((3q+\alpha+1)2^\alpha-(3q+1)\big)\left|f^{(3)}\left(\frac{a}{m}\right)\right|^q + (3q+1)\left|f^{(3)}(b)\right|^q\right]^{1/q}\right\}.
\end{aligned}\tag{3.2}$$

Proof. By Lemma 2.1, Hölder's inequality, and (α,m)-convexity of $\left|f^{(3)}\right|^q$ on $\left[a,\frac{b}{m}\right]$, we have

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x - f\left(\frac{a+b}{2}\right) - \frac{(b-a)^2}{24} f''\left(\frac{a+b}{2}\right)\right| \\
&\le \frac{(b-a)^3}{96}\left\{\int_0^1 t^3\left|f^{(3)}\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|\mathrm{d}t + \int_0^1 t^3\left|f^{(3)}\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|\mathrm{d}t\right\} \\
&\le \frac{(b-a)^3}{96}\left\{\left[\int_0^1 t^{3q}\left|f^{(3)}\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^q\mathrm{d}t\right]^{1/q} + \left[\int_0^1 t^{3q}\left|f^{(3)}\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|^q\mathrm{d}t\right]^{1/q}\right\} \\
&\le \frac{(b-a)^3}{96}\left\{\left[\int_0^1 t^{3q}\left(\frac{t}{2}\right)^\alpha\left|f^{(3)}(a)\right|^q\mathrm{d}t + m\int_0^1 t^{3q}\left(1-\left(\frac{t}{2}\right)^\alpha\right)\left|f^{(3)}\left(\frac{b}{m}\right)\right|^q\mathrm{d}t\right]^{1/q}\right. \\
&\left.+\left[\int_0^1 t^{3q}\left(\frac{t}{2}\right)^\alpha\left|f^{(3)}(b)\right|^q\mathrm{d}t + m\int_0^1 t^{3q}\left(1-\left(\frac{t}{2}\right)^\alpha\right)\left|f^{(3)}\left(\frac{a}{m}\right)\right|^q\mathrm{d}t\right]^{1/q}\right\} \\
&= \frac{(b-a)^3}{96}\left[\frac{1}{(3q+1)(3q+\alpha+1)2^\alpha}\right]^{1/q} \\
&\times\left\{\left[(3q+1)\left|f^{(3)}(a)\right|^q + m\big((3q+\alpha+1)2^\alpha-(3q+1)\big)\left|f^{(3)}\left(\frac{b}{m}\right)\right|^q\right]^{1/q}\right. \\
&\left.+\left[m\big((3q+\alpha+1)2^\alpha-(3q+1)\big)\left|f^{(3)}\left(\frac{a}{m}\right)\right|^q + (3q+1)\left|f^{(3)}(b)\right|^q\right]^{1/q}\right\}.
\end{aligned}$$

The proof of Theorem 3.2 is complete. □

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Corollary 3.2. *Under the conditions of Theorem 3.2,*

1. *if* $m=1$, *we have*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x - f\left(\frac{a+b}{2}\right) - \frac{(b-a)^2}{24}f''\left(\frac{a+b}{2}\right)\right|\\ &\le \frac{(b-a)^3}{96}\left[\frac{1}{(3q+1)(3q+\alpha+1)2^\alpha}\right]^{1/q}\\ &\quad\times\left\{\left[(3q+1)\left|f^{(3)}(a)\right|^q + \left((3q+\alpha+1)2^\alpha-(3q+1)\right)\left|f^{(3)}(b)\right|^q\right]^{1/q}\right.\\ &\quad\left.+\left[\left((3q+\alpha+1)2^\alpha-(3q+1)\right)\left|f^{(3)}(a)\right|^q + (3q+1)\left|f^{(3)}(b)\right|^q\right]^{1/q}\right\};\end{aligned}$$

2. *if* $\alpha=1$, *we have*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x - f\left(\frac{a+b}{2}\right) - \frac{(b-a)^2}{24}f''\left(\frac{a+b}{2}\right)\right|\\ &\le \frac{(b-a)^3}{96}\left[\frac{1}{2(3q+1)(3q+2)}\right]^{1/q}\left\{\left[(3q+1)\left|f^{(3)}(a)\right|^q + 3m(q+1)\left|f^{(3)}\left(\frac{b}{m}\right)\right|^q\right]^{1/q}\right.\\ &\quad\left.+\left[3m(q+1)\left|f^{(3)}\left(\frac{a}{m}\right)\right|^q + (3q+1)\left|f^{(3)}(b)\right|^q\right]^{1/q}\right\};\end{aligned}$$

3. *if* $m=\alpha=1$, *we have*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x - f\left(\frac{a+b}{2}\right) - \frac{(b-a)^2}{24}f''\left(\frac{a+b}{2}\right)\right| \le \frac{(b-a)^3}{96}\left[\frac{1}{2(3q+1)(3q+2)}\right]^{1/q}\\ &\times\left\{\left[(3q+1)\left|f^{(3)}(a)\right|^q+3(q+1)\left|f^{(3)}(b)\right|^q\right]^{1/q}+\left[3(q+1)\left|f^{(3)}(a)\right|^q+(3q+1)\left|f^{(3)}(b)\right|^q\right]^{1/q}\right\}.\end{aligned}$$

Theorem 3.3. *Let* $f:\mathbb{R}_0\to\mathbb{R}$ *be three times differentiable and* $f^{(3)}\in L_1\left(\left[a,\frac{b}{m}\right]\right)$ *for* $0\le a<b$. *If* $\left|f^{(3)}\right|^q$ *is* (α,m)*-convex on* $\left[a,\frac{b}{m}\right]$ *for* $q>1$ *and* $(\alpha,m)\in[0,1]\times(0,1]$, *then*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x - f\left(\frac{a+b}{2}\right) - \frac{(b-a)^2}{24}f''\left(\frac{a+b}{2}\right)\right| \le \frac{(b-a)^3}{96}\left(\frac{q-1}{4q-1}\right)^{1-1/q}\\ &\quad\times\left[\frac{1}{(\alpha+1)2^\alpha}\right]^{1/q}\left\{\left[\left|f^{(3)}(a)\right|^q + m((\alpha+1)2^\alpha-1)\left|f^{(3)}\left(\frac{b}{m}\right)\right|^q\right]^{1/q}\right.\\ &\quad\left.+\left[m((\alpha+1)2^\alpha-1)\left|f^{(3)}\left(\frac{a}{m}\right)\right|^q + \left|f^{(3)}(b)\right|^q\right]^{1/q}\right\}. \quad (3.3)\end{aligned}$$

Proof. Since $|f^{(3)}|^q$ is (α,m)-convex on $\left[a,\frac{b}{m}\right]$, using Lemma 2.1 and Hölder's inequality, we have

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x - f\left(\frac{a+b}{2}\right) - \frac{(b-a)^2}{24}f''\left(\frac{a+b}{2}\right)\right|\\ &\le \frac{(b-a)^3}{96}\left\{\int_0^1 t^3\left|f^{(3)}\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|\mathrm{d}t + \int_0^1 t^3\left|f^{(3)}\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|\mathrm{d}t\right\}\\ &\le \frac{(b-a)^3}{96}\left(\int_0^1 t^{3q/(q-1)}\,\mathrm{d}t\right)^{1-1/q}\end{aligned}$$

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$$
\begin{aligned}
&\times\left\{\left[\int_0^1\left|f^{(3)}\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^q\,\mathrm{d}t\right]^{1/q}+\left[\int_0^1\left|f^{(3)}\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|^q\,\mathrm{d}t\right]^{1/q}\right\}\\
&\le\frac{(b-a)^3}{96}\left(\frac{q-1}{4q-1}\right)^{1-1/q}\left\{\left[\int_0^1\left(\frac{t}{2}\right)^\alpha\left|f^{(3)}(a)\right|^q\,\mathrm{d}t+m\int_0^1\left(1-\left(\frac{t}{2}\right)^\alpha\right)\left|f^{(3)}\left(\frac{b}{m}\right)\right|^q\,\mathrm{d}t\right]^{1/q}\right.\\
&\quad\left.+\left[\int_0^1\left(\frac{t}{2}\right)^\alpha\left|f^{(3)}(b)\right|^q\,\mathrm{d}t+m\int_0^1\left(1-\left(\frac{t}{2}\right)^\alpha\right)\left|f^{(3)}\left(\frac{a}{m}\right)\right|^q\,\mathrm{d}t\right]^{1/q}\right\}\\
&=\frac{(b-a)^3}{96}\left(\frac{q-1}{4q-1}\right)^{1-1/q}\left[\frac{1}{(\alpha+1)2^\alpha}\right]^{1/q}\left\{\left[\left|f^{(3)}(a)\right|^q+m((\alpha+1)2^\alpha-1)\left|f^{(3)}\left(\frac{b}{m}\right)\right|^q\right]^{1/q}\right.\\
&\quad\left.+\left[m((\alpha+1)2^\alpha-1)\left|f^{(3)}\left(\frac{a}{m}\right)\right|^q+\left|f^{(3)}(b)\right|^q\right]^{1/q}\right\}.
\end{aligned}
$$

The proof of Theorem 3.3 is complete. □

Corollary 3.3. *Under the conditions of Theorem 3.3, if $\alpha=m=1$, we have*

$$
\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x-f\left(\frac{a+b}{2}\right)-\frac{(b-a)^2}{24}f''\left(\frac{a+b}{2}\right)\right|\\
&\quad\le\frac{(b-a)^3}{96}\left(\frac{q-1}{4q-1}\right)^{1-1/q}\left\{\left[\frac{\left|f^{(3)}(a)\right|^q+3\left|f^{(3)}(b)\right|^q}{4}\right]^{1/q}+\left[\frac{3\left|f^{(3)}(a)\right|^q+\left|f^{(3)}(b)\right|^q}{4}\right]^{1/q}\right\}.
\end{aligned}
$$

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FUNCTIONAL INEQUALITIES IN β-HOMOGENEOUS F-SPACES

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ABSTRACT. In this paper, we prove the Hyers-Ulam stability problem for the following function inequalities

$$\left\|\sum_{i=1}^{N} f(x_i)\right\| \le \left\|Kf\left(\frac{\sum_{i=1}^{N} x_i}{K}\right)\right\|,$$

$$\left\|\sum_{i=1}^{N} f(x_i)\right\| \le \left\|Kf\left(\frac{\sum_{i=1}^{m} x_i}{K}\right) + Cf\left(\frac{\sum_{i=m+1}^{N} x_i}{C}\right)\right\|$$

in β-homogeneous F-spaces.

1. Introduction and preliminaries

We recall some basic facts concerning β-homogeneous F-spaces.

Definition 1.1. Let X be a linear space. A nonnegative valued function $\|\cdot\|$ is an F-norm if it satisfies the following conditions:

(FN_1) $\|x\| = 0$ if and only if $x = 0$;

(FN_2) $\|\lambda x\| = \|x\|$ for all $x \in X$ and all λ with $|\lambda| = 1$;

(FN_3) $\|x + y\| \le \|x\| + \|y\|$ for all $x, y \in X$;

(FN_4) $\|\lambda_n x\| \to 0$ provided $\lambda_n \to 0$;

(FN_5) $\|\lambda x_n\| \to 0$ provided $\|x_n\| \to 0$.

Then $(X, \|\cdot\|)$ is called an F^*-space. An F-space is a complete F^*-space.

An F-norm is called β-homogeneous ($\beta > 0$) if $\|tx\| = |t|^\beta \|x\|$ for all $x \in X$ and all $t \in \mathbb{R}$ (see [19]).

The stability problem of functional equations originated from a question of Ulam [22] concerning the stability of group homomorphisms. Hyers [11] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [15] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [9] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach. The stability problems for several functional equations or inequations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [2]–[8], [10], [12]–[14], [16]–[18], [20, 21, 24]).

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In this paper, we consider the following functional inequalities

$$\left\|\sum_{i=1}^{N} f(x_i)\right\| \leq \left\|Kf\left(\frac{\sum_{i=1}^{N} x_i}{K}\right)\right\|, \tag{1.1}$$

$$\left\|\sum_{i=1}^{N} f(x_i)\right\| \leq \left\|Kf\left(\frac{\sum_{i=1}^{m} x_i}{K}\right) + Cf\left(\frac{\sum_{i=m+1}^{N} x_i}{C}\right)\right\| \tag{1.2}$$

in β-homogeneous F-spaces, where N, m are positive integers with $N > 2$, and K, C are nonzero integers with $|K| < N$ and $|K + C| < N$.

Throughout this paper, assume that X is a β-homogeneous F^*-space, and Y is a β-homogeneous F-space.

2. Hyers-Ulam stability of the functional inequality (1.1)

Let K be a fixed integer with $|K| < N$.

Proposition 2.1. *Let $f : X \to Y$ be a mapping such that*

$$\left\|\sum_{i=1}^{N} f(x_i)\right\| \leq \left\|Kf\left(\frac{\sum_{i=1}^{N} x_i}{K}\right)\right\|, \quad \forall x_i \in X. \tag{2.1}$$

Then f is additive.

Proof. Letting $x_i = 0$ in (2.1) for all $i = 1, 2, \cdots, N$, we get

$$\|Nf(0)\| \leq \|Kf(0)\| \Rightarrow |N|^{\beta}\|f(0)\| \leq |K|^{\beta}\|f(0)\|.$$

Since $|K| < |N|$, $f(0) = 0$.

Letting $x_2 = -x_1$ and $x_3 = x_4 = \cdots = x_N = 0$ in (2.1), we get

$$\|f(x_1) + f(-x_1)\| \leq \|Kf(0)\| = |K|^{\beta}\|f(0)\| = 0, \quad \forall x_1 \in X.$$

Thus $f(-x) = -f(x)$ for any $x \in X$.

Letting $x_3 = -x_1 - x_2$ and $x_4 = x_5 = \cdots = x_N = 0$, we get

$$\begin{aligned}\|f(x_1) + f(x_2) - f(x_1 + x_2)\| &= \|f(x_1) + f(x_2) + f(-x_1 - x_2)\| \\ &\leq \|Kf(0)\| = |K|^{\beta}\|f(0)\| = 0, \quad \forall x_1, x_2 \in X.\end{aligned}$$

So

$$f(x + y) = f(x) + f(y), \quad \forall x, y \in X.$$

Thus f is additive. □

Now we prove the Hyers-Ulam stability of the functional inequality (1.1).

Theorem 2.2. *Let $f : X \to Y$ be an odd mapping for which there exists a function $\varphi : X^N \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x_1, \cdots, x_N) := \sum_{j=1}^{\infty} |N-1|^{j\beta} \varphi\left(\frac{x_1}{(N-1)^{j+1}}, \cdots, \frac{x_{N-1}}{(N-1)^{j+1}}, \frac{-x_N}{(N-1)^{j}}\right) < \infty,$$

$$\left\|\sum_{i=1}^{N} f(x_i)\right\| \leq \left\|Kf\left(\frac{\sum_{i=1}^{N} x_i}{K}\right)\right\| + \varphi(x_1, \cdots, x_N), \quad \forall x_i \in X. \tag{2.2}$$

Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{1}{(N-1)^\beta}\widetilde{\varphi}(x, \cdots, x) \tag{2.3}$$

for all $x \in X$.

Proof. Letting $x_1 = x_2 = \cdots = x_{N-1} = x$ and $x_N = -(N-1)x$ in (2.2), we get

$$\begin{aligned}\|(N-1)f(x) - f((N-1)x)\| &= \|(N-1)f(x) + f(-(N-1)x)\| \\ &\leq \varphi(x, \cdots, x, -(N-1)x)\end{aligned}$$

for all $x \in X$. So

$$\left\|f(x) - (N-1)f\left(\frac{x}{N-1}\right)\right\| \leq \varphi\left(\frac{x}{N-1}, \cdots, \frac{x}{N-1}, -x\right), \quad \forall x \in X.$$

Hence one may have the following formula for positive integers m, l with $m > l$,

$$\begin{aligned}&\left\|(N-1)^l f\left(\frac{x}{(N-1)^l}\right) - (N-1)^m f\left(\frac{x}{(N-1)^m}\right)\right\| \\ &\leq \sum_{i=l}^{m-1} |N-1|^{i\beta}\varphi\left(\frac{x}{(N-1)^{i+1}}, \cdots, \frac{x}{(N-1)^{i+1}}, \frac{-x}{(N-1)^i}\right)\end{aligned} \tag{2.4}$$

for all $x \in X$. It follows from (2.4) that the sequence $\left\{(N-1)^k f\left(\frac{x}{(N-1)^k}\right)\right\}$ is a Cauchy sequence for all $x \in X$. Since Y is an F-space, the sequence $\left\{(N-1)^k f\left(\frac{x}{(N-1)^k}\right)\right\}$ converges. So one may define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty} (N-1)^k f\left(\frac{x}{(N-1)^k}\right), \quad \forall x \in X.$$

Now, we show that the uniqueness of A. Let $T : X \to Y$ be another additive mapping satisfying (2.3). Then one has

$$\begin{aligned}\|A(x) - T(x)\| &= \left\|(N-1)^k A\left(\frac{x}{(N-1)^k}\right) - (N-1)^k T\left(\frac{x}{(N-1)^k}\right)\right\| \\ &\leq |N-1|^{k\beta}\left(\left\|A\left(\frac{x}{(N-1)^k}\right) - f\left(\frac{x}{(N-1)^k}\right)\right\|\right. \\ &\left.+ \left\|T\left(\frac{x}{(N-1)^k}\right) - f\left(\frac{x}{(N-1)^k}\right)\right\|\right) \\ &\leq 2|N-1|^{k\beta}\widetilde{\varphi}\left(\frac{x}{(N-1)^k}, \cdots, \frac{x}{(N-1)^k}\right)\end{aligned}$$

which tends to zero as $k \to \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$.

It follows from (2.2) that

$$\begin{aligned}\|A(x_1)+\cdots+A(x_N)\| = & \lim_{k\to\infty}\left\|(N-1)^k f\left(\frac{x_1}{(N-1)^k}\right)+(N-1)^k f\left(\frac{x_2}{(N-1)^k}\right)\right. \\ & \left. +\cdots+(N-1)^k f\left(\frac{x_N}{(N-1)^k}\right)\right\| \\ & \leq \lim_{k\to\infty}|N-1|^{k\beta}\left\|Kf\left(\frac{x_1+x_2+\cdots+x_N}{(N-1)^k K}\right)\right\| \\ & + \lim_{k\to\infty}|N-1|^{k\beta}\varphi\left(\frac{x_1}{(N-1)^k},\cdots,\frac{x_N}{(N-1)^k}\right) \\ & = \left\|KA\left(\frac{x_1+x_2+\cdots+x_N}{K}\right)\right\|, \quad \forall x_1,x_2,\cdots,x_N\in X.\end{aligned}$$

Thus

$$\|A(x_1)+\cdots+A(x_N)\| \leq \left\|KA\left(\frac{x_1+x_2+\cdots+x_N}{K}\right)\right\|, \quad \forall x_1,x_2,\cdots,x_N\in X.$$

By Proposition 2.1, the mapping $A : X \to Y$ is additive. This completes the proof. □

3. Hyers-Ulam stability of the functional inequality (1.2)

In this section, K, C, N are real numbers with $|K+C| < N$.

Proposition 3.1. *Let $f : X \to Y$ be a mapping such that*

$$\left\|\sum_{i=1}^N f(x_i)\right\| \leq \left\|Kf\left(\frac{\sum_{i=1}^M x_i}{K}\right)+Cf\left(\frac{\sum_{i=M+1}^N x_i}{C}\right)\right\|. \tag{3.1}$$

Then f is additive.

Proof. Letting $x_i = 0$ in (3.1) for all $i = 1,2,\cdots,N$, we obtain

$$|N|^\beta\|f(0)\| = \|Nf(0)\| \leq \|Kf(0)+Cf(0)\| \leq |K+C|^\beta\|f(0)\|.$$

Since $|K+C| < N, f(0) = 0$.

Next, we show that f is an additive mapping. Letting $x_2 = -x_1$ and $x_3 = x_4 = \cdots = x_N = 0$ in (3.1), we get

$$\|f(x_1)+f(-x_1)+f(0)+\cdots+f(0)\| \leq \|Kf(0)+Cf(0)\| = 0, \quad \forall x_1 \in X.$$

Thus $f(-x) = -f(x)$ for all $x \in X$.

Letting $x_3 = -x_1 - x_2$ and $x_4 = x_5 = \cdots = x_N = 0$, we get

$$\begin{aligned}\|f(x_1)+f(x_2)-f(x_1+x_2)+f(0)+\cdots+f(0)\| = & \|f(x_1)+f(x_2)+f(-x_1-x_2)\| \\ \leq & \|Kf(0)+Cf(0)\| = 0\end{aligned}$$

for all $x_1, x_2 \in X$. So

$$f(x+y) = f(x)+f(y), \quad \forall x,y \in X.$$

Thus f is additive. □

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Theorem 3.2. *Let $f : X \to Y$ be an odd mapping for which there exists a function $\varphi : X^N \to [0,\infty)$ such that*

$$\begin{aligned}\widetilde{\varphi}(x_1,\cdots,x_N) : &= \sum_{j=1}^{\infty} |(M-1)^j|^{\beta}\varphi\left(\frac{x_1}{(M-1)^j},\cdots,\frac{x_{M-1}}{(M-1)^j},\frac{x_M}{(M-1)^j},\cdots,\frac{x_N}{(M-1)^j}\right)\\ &< \infty,\end{aligned}$$

$$\left\|\sum_{i=1}^{N} f(x_i)\right\| \le \left\|Kf\left(\frac{\sum_{i=1}^{M} x_i}{K}\right) + Cf\left(\frac{\sum_{i=M+1}^{N} x_i}{C}\right)\right\| + \varphi(x_1,\cdots,x_N) \tag{3.2}$$

for all $x_1,\cdots,x_N \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{1}{|M-1|^{\beta}}\widetilde{\varphi}(x,\cdots,x,-(M-1)x,0,\cdots,0) \tag{3.3}$$

for all $x \in X$.

Proof. Letting $x_1 = x_2 = \cdots x_{M-1} = x$, $x_M = -(M-1)x$ and $x_{M+1} = \cdots = x_N = 0$ in (3.2), we get

$$\begin{aligned}\|(M-1)f(x) - f((M-1)x)\| &= \|(M-1)f(x) + f(-(M-1)x)\|\\ &\le \varphi(x,\cdots,x,-(M-1)x,0,\cdots,0),\end{aligned}$$

for all $x \in X$. So

$$\left\|f(x) - (M-1)f\left(\frac{x}{M-1}\right)\right\| \le \varphi\left(\frac{x}{M-1},\cdots,\frac{x}{M-1},-x,0,\cdots,0\right), \quad \forall x \in X.$$

Hence one may have the following formula for positive integers m,l with $m > l$,

$$\begin{aligned}&\left\|(M-1)^l f\left(\frac{x}{(M-1)^l}\right) - (M-1)^m f\left(\frac{x}{(M-1)^m}\right)\right\| \\ &\le \sum_{i=l}^{m-1} |(M-1)^i|^{\beta}\varphi\left(\frac{x}{(M-1)^{i+1}},\cdots,\frac{x}{(M-1)^{i+1}},\frac{-x}{(M-1)^i},0,\cdots,0\right).\end{aligned} \tag{3.4}$$

It follows from (3.4) that the sequence $\left\{(M-1)^k f\left(\frac{x}{(M-1)^k}\right)\right\}$ is a Cauchy sequence for all $x \in X$. Since Y is an F-space, the sequence $\left\{(M-1)^k f\left(\frac{x}{(M-1)^k}\right)\right\}$ converges. So one may define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty}(M-1)^k f\left(\frac{x}{(M-1)^k}\right), \quad \forall x \in X.$$

Now, we show that the uniqueness of A. Let $T : X \to Y$ be another additive mapping satisfying (3.3). Then one have

$$\begin{aligned}
\|A(x) - T(x)\| &= \left\|(M-1)^k A\left(\frac{x}{(M-1)^k}\right) - (M-1)^k T\left(\frac{x}{(M-1)^k}\right)\right\| \\
&\leq |(M-1)^k|^\beta \left(\left\|A\left(\frac{x}{(M-1)^k}\right) - f\left(\frac{x}{(M-1)^k}\right)\right\|\right. \\
&\left. + \left\|T\left(\frac{x}{(M-1)^k}\right) - f\left(\frac{x}{(M-1)^k}\right)\right\|\right) \\
&\leq 2|M-1|^{(k-1)\beta}\widetilde{\varphi}\left(\frac{x}{(M-1)^k}, \cdots, \frac{x}{(M-1)^k}, -\frac{x}{(M-1)^{k-1}}, 0, \cdots, 0\right)
\end{aligned}$$

which tends to zero as $k \to \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$.

It follows from (3.2) that

$$\begin{aligned}
\|A(x_1) + \cdots + A(x_N)\| &= \lim_{k\to\infty}\left\|(M-1)^k f\left(\frac{x_1}{(M-1)^k}\right) + (M-1)^k f\left(\frac{x_2}{(M-1)^k}\right)\right. \\
&\left. + \cdots + (M-1)^k f\left(\frac{x_N}{(M-1)^k}\right)\right\| \\
&\leq \lim_{k\to\infty}\left\|(M-1)^k K f\left(\frac{x_1 + \cdots + x_M}{(M-1)^k K}\right)\right. \\
&\left. + (M-1)^k C f\left(\frac{x_{M+1} + \cdots + x_N}{(M-1)^k C}\right)\right\| \\
&+ \lim_{k\to\infty} |M-1|^{k\beta}\varphi\left(\frac{x_1}{(M-1)^k}, \cdots, \frac{x_N}{(M-1)^k}\right) \\
&= \left\|KA\left(\frac{x_1 + \cdots + x_M}{K}\right) + CA\left(\frac{x_{M+1} + \cdots + x_N}{C}\right)\right\|
\end{aligned}$$

for all $x_1, x_2, \cdots, x_N \in X$.

Thus

$$\|A(x_1) + \cdots + A(x_N)\| \leq \left\|KA\left(\frac{x_1 + \cdots + x_M}{K}\right) + CA\left(\frac{x_{M+1} + \cdots + x_N}{C}\right)\right\|$$

for all $x_1, x_2, \cdots, x_N \in X$. By Proposition 3.1, the mapping $A : X \to Y$ is additive. This completes the proof. □

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Some New Solutions of the (3+1)-Dimensional Jimbo-Miwa Equation via the Improved (G'/G)-Expansion Method

Hasibun Naher[1], Farah Aini Abdullah[1], Abdur Rashid[2]

Abstract

The improved (G'/G)-expansion method is straightforward and effective mathematical tool for establishing exact traveling wave solutions of different nonlinear partial differential equations which arise in engineering sciences, applied mathematics and real time application fields. In this article, we have constructed some new traveling wave solutions of the nonlinear evolution equation, namely, the (3+1)-dimensional Jimbo-Miwa equation via the improved (G'/G)-expansion method. In this method, the general solution of the second order linear ordinary differential equation involving constants coefficients together with $A(\xi)=\sum_{j=-m}^{m} p_j\,(G'/G)^j$ is employed, where $p_j\,(j=0,\pm1,\pm2,...,\pm m)$ are constants. Further, it is worth stating that the obtained solutions become in special functional forms for the particular values of the arbitrary constants. In addition, it is noteworthy declaring that, some of our solutions are in good agreement with already published results. Moreover, some of the solutions are described in the figures with the aid of commercial software Maple.

Keywords: The improved (G'/G)-expansion method, the Jimbo-Miwa equation, traveling wave solutions, nonlinear evolution equations.

1. Introduction

The rigorous investigation of nonlinear evolution equations (NLEEs) for constructing exact solutions has become a hot topic for researchers. In the recent past, a wide range of methods have been established to obtain analytical solutions for nonlinear partial differential equations (PDEs). For example, the inverse scattering method [1], the generalized Riccati equation method [2], the Backlund transformation method [3], the Jacobi elliptic function expansion method [4], the Hirota's bilinear transformation method [5], the F-expansion method [6,7], the direct algebraic method [8], the Cole-Hopf transformation method [9], the Exp-function method [10-13] and others [14-17,39,40]. Wang *et al.* [18] presented a method to construct traveling wave solutions of some nonlinear evolution equations, called the (G'/G)-expansion method. In this method, they employed $u(\xi)=\sum_{i=0}^{m} a_i\,(G'/G)^i$ as traveling wave solutions, where $a_m \neq 0$. Later on, many researchers [19-24] investigated nonlinear partial differential equations to establish traveling wave solutions by using this method. More recently, Zhang *et al.* [25] extended this method and called the improved (G'/G)-expansion method. In this improved (G'/G)-expansion method, $u(\xi)=\sum_{i=0}^{m} a_i\,(G'/G)^i$ is implemented as traveling wave solutions, where either p_{-m} or p_m may be zero, but both p_{-m} and p_m cannot be zero at a time. Afterwards, many researchers executed this powerful method to establish abundant traveling wave solutions for different nonlinear partial differential equations. For instance, Zhao *et al.* [26] studied the variant Boussinesq equations by using this method to establish analytical solutions. In Ref. [27], Zayed and Al-Joudi investigated some nonlinear partial differential equations

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to obtain exact solutions via the same method. Naher *et al.* [28] constructed some new analytical solutions of the (3+1)-dimensional modified KdV-Zakharov-Kuznetsev equation by using this powerful method. Naher and Abdullah [29] investigated the nonlinear reaction diffusion equation to establish exact solutions by applying the same method whereas Naher and Abdullah [30] concerned about this method for obtaining traveling wave solutions of the combined KdV-MKdV equation. In Ref. [31] they constructed exact traveling wave solutions for the (2+1)-dimensional Modified Zakharov-Kuznetsov equation via this powerful method and so on.

Many researchers applied different methods to construct analytical solutions of the (3+1)-dimensional Jimbo-Miwa equation. For instance, Liu and Jiang [32] studied this equation by applying the extended homogeneous balance method to obtain exact solutions. In Ref. [33], Xu employed the truncated painleve expansion method to establish soliton solutions of the same equation. Wazwaz [34] applied the tanh-coth method to construct exact solutions of this equation. Zhaqilao and Li [35] investigated the same equation to construct traveling wave solutions via the Hirota bilinear method. Li and Dai [36] used the generalized Riccati equation mapping method to obtain exact solutions of this equation. Borhanifar and Kabir [37] implemented the Exp-function method to study the same equation for obtaining analytical solutions. In reference [38] Zayed investigated this equation by applying the basic (G'/G)-expansion method to construct traveling wave solutions. To the best of our knowledge, the (3+1)-dimensional Jimbo-Miwa equation is not investigated by applying the improved (G'/G)-expansion method to obtain traveling wave solutions. In the basic (G'/G)-expansion method, $u(\xi)=\sum_{i=0}^{m} a_i (G'/G)^i$ where $a_m \neq 0,$ is employed as traveling wave solutions, instead of $A(\xi)=\sum_{j=-m}^{m} p_j (G'/G)^j$ where either p_{-m} or p_m may be zero, but both p_{-m} and p_m cannot be zero at a time. The significant of this paper, we have studied the (3+1)-dimensional Jimbo-Miwa equation for obtaining some new traveling wave solutions including solitons and periodic wave solutions via the improved (G'/G)-expansion method.

2. Description of the improved (G'/G)-expansion method

Suppose the general nonlinear partial differential equation:

$$P(u,u_t,u_x,u_y,u_z,u_{tt},u_{xt},u_{yt},u_{zt},u_{xx},u_{xy},u_{zx},u_{yy},u_{zy},u_{zz},...)=0, \tag{1}$$

where $u=u(x,y,z,t)$ is an unknown function, P is a polynomial in $u(x,y,z,t)$ and the subscripts stand for the partial derivatives. The main steps of the improved (G'/G)-expansion method [25] are as follows:

Step 1. Consider the traveling wave variable:

$$u(x,y,z,t)=A(\xi), \quad \xi=x+y+z-Vt, \tag{2}$$

where V is the wave speed. Now using Eq. (2), Eq. (1) is converted into an ordinary differential equation for $A(\xi)$:

$$B(A,A',A'',A''',...)=0, \tag{3}$$

where the superscripts indicate the ordinary derivatives with respect to ξ.

Step 2. According to possibility, Eq. (3) can be integrated term by term one or more times, yields constant(s) of integration. The integral constant may be zero, for simplicity.

Step 3. Suppose that the traveling wave solution of Eq. (3) can be expressed in the form [25]:

$$A(\xi)=\sum_{j=-m}^{m} p_j (G'/G)^j \tag{4}$$

with $G=G(\xi)$ satisfies the second order linear ODE:

$$G''+\lambda G'+\mu G=0, \tag{5}$$

where $p_j\ (j=0,\pm1,\pm2,...,\pm m)$, λ and μ are constants.

Step 4. To determine the integer $m,$ substituting Eq. (4) along with Eq. (5) into Eq. (3) and then taking the homogeneous balance between the highest order nonlinear terms and the highest order derivatives appearing in Eq. (3).

Step 5. Substitute Eq. (4) and Eq. (5) into Eq. (3) with the value of w obtained in Step 4. Equating the coefficients of $(G'/G)^r$, $(r = 0, \pm 1, \pm 2, ...)$, then setting each coefficient to zero, we obtain a set of algebraic equations for p_j $(j = 0, \pm 1, \pm 2, ..., \pm m), V, \lambda$ and μ.

Step 6. Solve the system of algebraic equations which are obtained in step 5 with the aid of algebraic software Maple and we obtain values for p_j $(j = 0, \pm 1, \pm 2, ..., \pm m), V, \lambda$ and μ. Then, substitute obtained values in Eq. (4) along with Eq. (5) with the value of m, we can obtain the traveling wave solutions of Eq. (1).

3. Applications of the Method

In this section, we establish some new traveling wave solutions including three different families of the (3+1)-dimensional Jimbo-Miwa equation by applying the improved (G'/G)-expansion method.

3.1 The (3+1)-dimensional Jimbo-Miwa equation

We consider the (3+1)-dimensional Jimbo-Miwa equation followed by Zayed [38]:

$$u_{xxxy} + 3u_y u_{xx} + 3u_x u_{xy} + 2u_{yt} - 3u_{xz} = 0, \tag{6}$$

Now, we use the wave transformation Eq. (2) into the Eq. (6), which yields:

$$A^{(4)} + 3A'A'' + 3A'A'' - 2VA'' - 3A'' = 0, \tag{7}$$

Eq. (7) is integrable, therefore, integrating with respect ξ once yields:

$$C + A''' + 3A'^2 - (2V + 3)A' = 0, \tag{8}$$

where C is an integral constant which is to be determined later.

Taking the homogeneous balance between highest order derivative and nonlinear term in (8), we obtain $m = 1$. Therefore, the solution of (8) is of the form:

$$A(\xi) = p_{-1}(G'/G)^{-1} + p_0 + p_1(G'/G), \tag{9}$$

where p_{-1}, p_0 and p_1 are constants to be determined.

Substituting Eq. (9) together with Eq. (5) into the Eq. (8), the left-hand side of Eq. (8) is converted into a polynomial of $(G'/G)^r$, $(r = 0, \pm 1, \pm 2, ...)$. According to Step 5, collecting all terms with the same power of (G'/G). Then, setting each coefficient of the resulted polynomial to zero, yields a set of algebraic equations (for simplicity, which are not presented) for $p_{-1}, p_0, p_1, C, V, \lambda$ and μ.

Solving the system of obtained algebraic equations with the help of algebraic software Maple, we obtain two different values.

Case 1:

$$p_{-1} = 0,\ p_0 = p_0,\ p_1 = 2,\ V = \frac{1}{2}(\lambda^2 - 4\mu - 3),\ C = 0, \tag{10}$$

where p_0, λ and μ are free parameters.

Case 2:

$$p_{-1} = -2\mu,\ p_0 = p_0,\ p_1 = 0,\ V = \frac{1}{2}(\lambda^2 - 4\mu - 3),\ C = 0, \tag{11}$$

where p_0, λ and μ are free parameters.

Substituting the general solution Eq. (5) into Eq. (9), we obtain three different families of traveling wave solutions of Eq. (8):

Family 1: Hyperbolic function solutions:

When $\lambda^2 - 4\mu > 0$, we obtain

$$\begin{aligned} A(\xi) = p_{-1}\left(\frac{-\lambda}{2}+\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\frac{U\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi+V\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi}{U\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi+V\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi}\right)^{-1} + p_0 \\ + p_1\left(\frac{-\lambda}{2}+\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\frac{U\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi+V\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi}{U\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi+V\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi}\right), \end{aligned} \quad (12)$$

If U and V are taken particular values, various known solutions can be rediscovered.

Family 2: Trigonometric function solutions:

When $\lambda^2-4\mu<0$, we obtain

$$\begin{aligned} A(\xi) = p_{-1}\left(\frac{-\lambda}{2}+\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\frac{-U\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\xi+V\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\xi}{U\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\xi+V\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\xi}\right)^{-1} + p_0 \\ + p_1\left(\frac{-\lambda}{2}+\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\frac{-U\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\xi+V\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\xi}{U\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\xi+V\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\xi}\right) \end{aligned} \quad (13)$$

If U and V are taken particular values, various known solutions can be rediscovered.

Family 3: Rational function solution:

When $\lambda^2-4\mu=0$, we obtain

$$A(\xi) = p_{-1}\left(\frac{-\lambda}{2}+\frac{V}{U+V\xi}\right)^{-1} + p_0 + p_1\left(\frac{-\lambda}{2}+\frac{V}{U+V\xi}\right), \quad (14)$$

Family 1: Hyperbolic function solutions:

Substituting Eqs. (10) and (11) together with the general solution Eq. (5) into the Eq. (9), yields the hyperbolic function solution Eq. (12), the obtained solutions become respectively (if $U=0$ but $V\neq 0$):

$$A_1(\xi) = \sqrt{(\lambda^2-4\mu)}\coth\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\;\xi\right) - \lambda + p_0,$$

where $\xi = x+y+z-\frac{1}{2}(\lambda^2-4\mu-3)\,t.$

$$A_2(\xi) = -2\mu\left(\frac{-\lambda}{2}+\frac{\sqrt{\lambda^2-4\mu}}{2}\coth\frac{1}{2}\sqrt{\lambda^2-4\mu}\;\xi\right)^{-1} + p_0,$$

where $\xi = x+y+z-\frac{1}{2}(\lambda^2-4\mu-3)\,t.$

Moreover, substituting Eqs. (10) and (11) together with the general solution Eq. (5) into the Eq. (9), we obtain the hyperbolic function solution Eq. (12), our traveling wave solutions become respectively (if $V=0$ but $U\neq 0$):

$$A_3(\xi) = \sqrt{(\lambda^2-4\mu)}\tanh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\;\xi\right) - \lambda + p_0.$$

$$A_4(\xi) = -2\mu\left(\frac{-\lambda}{2}+\frac{\sqrt{\lambda^2-4\mu}}{2}\tanh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\;\xi\right)\right)^{-1} + p_0.$$

Further, substituting Eqs. (10) and (11) together with the general solution Eq. (5) into the Eq. (9), yields the hyperbolic function solution Eq. (12), our exact solutions become respectively (if $U\neq 0,\ U>V$):

$$A_5(\xi) = \sqrt{(\lambda^2-4\mu)}\tanh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\;\xi+\xi_0\right) - \lambda + p_0,$$

where $\xi_0 = \tanh^{-1}\frac{V}{U}.$

$$A_6(\xi) = -2\mu\left(\frac{-\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\tanh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\ \xi + \xi_0\right)\right)^{-1} + p_0\,,$$

where $\xi_0 = \tanh^{-1}\frac{V}{U}.$

Family 2: Trigonometric function solutions:

Substituting Eqs. (10) and (11) together with the general solution Eq. (5) into the Eq. (9), yields the trigonometric function solution Eq. (13), our obtained solutions become respectively (if $U=0$ but $V \neq 0$):

$$A_7(\xi) = \sqrt{\left(4\mu-\lambda^2\right)}\cot\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\ \xi\right) - \lambda + p_0,$$

where $\xi = x + y + z - \frac{1}{2}\left(\lambda^2 - 4\mu - 3\right) t.$

$$A_8(\xi) = -2\mu\left(\frac{-\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2}\cot\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\ \xi\right)\right)^{-1} + p_0\,,$$

where $\xi = x + y + z - \frac{1}{2}\left(\lambda^2 - 4\mu - 3\right) t.$

Again, substituting Eqs. (10) and (11) together with the general solution Eq. (5) into the Eq. (9), yields the trigonometric function solution Eq. (13), we obtain following solutions respectively (if $V=0$ but $U \neq 0$):

$$A_9(\xi) = -\sqrt{\left(4\mu-\lambda^2\right)}\tan\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\ \xi\right) - \lambda + p_0.$$

$$A_{10}(\xi) = -2\mu\left(\frac{-\lambda}{2} - \frac{\sqrt{4\mu-\lambda^2}}{2}\tan\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\ \xi\right)\right)^{-1} + p_0\,.$$

Furthermore, substituting Eqs. (10) and (11) together with the general solution Eq. (5) into the Eq. (9), yields the trigonometric function solution Eq. (13), our wave solutions become respectively (if $U \neq 0, U > V$):

$$A_{11}(\xi) = \sqrt{\left(4\mu-\lambda^2\right)}\tan\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\ \xi - \xi_0\right) - \lambda + p_0,$$

where $\xi_0 = \tan^{-1}\frac{V}{U}.$

$$A_{12}(\xi) = -2\mu\left(\frac{-\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2}\tan\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\ \xi - \xi_0\right)\right)^{-1} + p_0\,,$$

where $\xi_0 = \tan^{-1}\frac{V}{U}.$

Family 3: Rational function solutions:

Substituting Eqs. (10) and (11) together with the general solution Eq. (5) into the Eq. (9), we obtain the rational function solution Eq. (14), we construct following traveling wave solutions respectively (if $\lambda^2 - 4\mu = 0$):

$$A_{13}(\xi) = \frac{2V}{U + V\xi} - \lambda + p_0,$$

where $\xi = x + y + z - \frac{1}{2}\left(\lambda^2 - 4\mu - 3\right) t.\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$A_{14}(\xi) = -2\mu\left(\frac{-\lambda}{2} + \frac{V}{U + V\xi}\right)^{-1} + p_0,$$

where $\xi = x + y + z - \frac{1}{2}\left(\lambda^2 - 4\mu - 3\right) t.$

4. Results and Discussion

It is important to state that some of obtained solutions are in good harmony with already established results and are presented in the following table. Additionally, some of obtained traveling wave solutions are illustrated in figure 1 to figure 8 with Maple.

4.1 Table. Comparison between Zayed [38] solutions and Newly obtained solutions

Zayed [38] solutions	New solutions
i. If $B=0, A\neq 0, \mu=0, \lambda=1$ and $\alpha_0=1$ solution Eq. (3.46) (from section 3, Example 5) becomes: $u(\xi)=\coth\frac{1}{2}\xi.$	i. If $\mu=0, \lambda=1, p_0=1$ and $A_1(\xi)=u(\xi)$, solution $A_1(\xi)$ becomes: $u(\xi)=\coth\frac{1}{2}\xi.$
ii. If $A=0, B\neq 0, \mu=0, \lambda=2$ and $\alpha_0=2$ solution Eq. (3.46) (from section 3, Example 5) becomes: $u(\xi)=2\tanh\xi.$	ii If $\mu=0, \lambda=2, p_0=2$ and $A_3(\xi)=u(\xi)$, solution $A_3(\xi)$ becomes: $u(\xi)=2\tanh\xi.$
iii. If $A=0, B\neq 0, \mu=5, \lambda=2$ and $\alpha_0=2$ solution Eq. (3.47) (from section 3, Example 5) becomes: $u(\xi)=4\cot(2\xi).$	iii. If $\mu=5, \lambda=2, p_0=2$ and $A_7(\xi)=u(\xi)$, solution $A_7(\xi)$ becomes: $u(\xi)=4\cot(2\xi).$
iv. If $B=0, A\neq 0, \mu=\frac{5}{2}, \lambda=1$ and $\alpha_0=1$ solution Eq. (3.47) (from section 3, Example 5) becomes: $u(\xi)=\mp 3\tan\left(\pm\frac{3}{2}\xi\right).$	iv. If $\mu=\frac{5}{2}, \lambda=1, p_0=1$ and $A_9(\xi)=u(\xi)$, solution $A_9(\xi)$ becomes: $u(\xi)=\mp 3\tan\left(\pm\frac{3}{2}\xi\right).$
v. If $B=1, A=1, \lambda=1$ and $\alpha_0=1$ solution Eq. (3.48) (from section 3, Example 5) becomes: $u(\xi)=\frac{2}{1+\xi}.$	v. If $V=1, U=1, \lambda=1, p_0=1$ and $A_{13}(\xi)=u(\xi)$, solution $A_{13}(\xi)$ becomes: $u(\xi)=\frac{2}{1+\xi}.$

Beyond this table, we have constructed new traveling wave solutions $A_2, A_4, A_6, A_8, A_{10}, A_{12}$ and A_{14} which are not stated in the earlier literature.

4.2 Graphical presentations of some solutions

The descriptions of some traveling wave solutions are being depicted in the figures with the help of commercial software Maple:

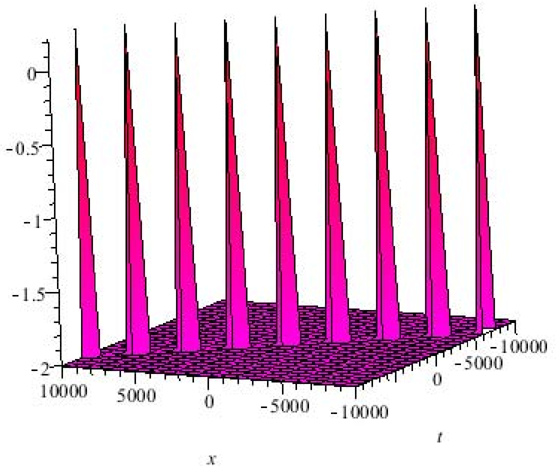

Fig. 1: Solutions solution for $\lambda=6, \mu=9, p_0=4, U=9, V=10$

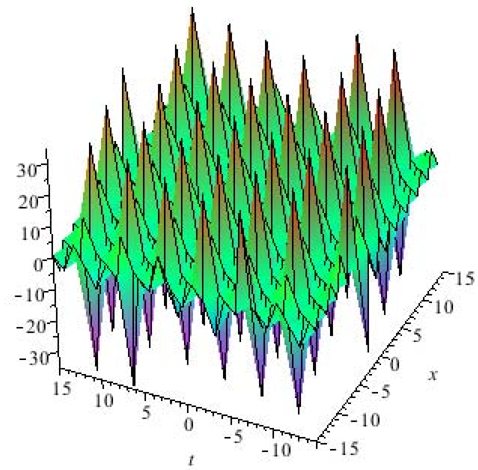

Fig. 2: Solitons solution for $\lambda=1, \mu=2, p_0=1$

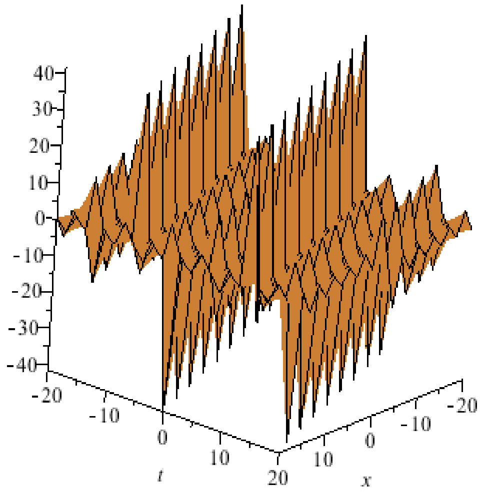

Fig. 3: Solitons solution for
$\lambda = 1, \mu = 1, p_0 = 1$

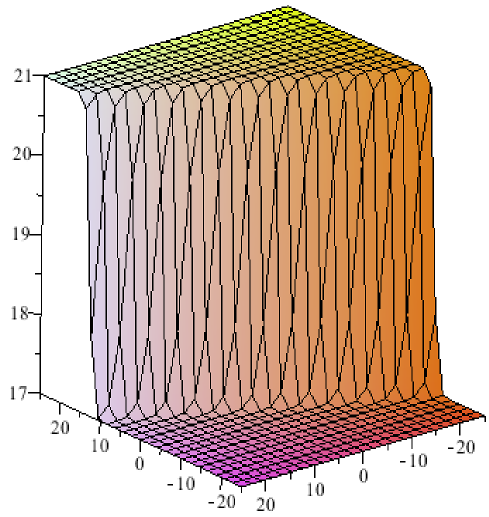

Fig. 4: Pweriodic solution for
$\lambda = 4, \mu = 3, p_0 = 15$

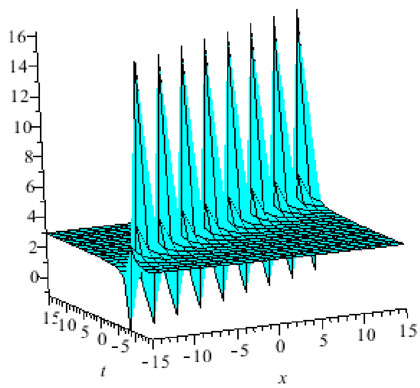

Fig. 5: Solitons solution for
$\lambda = 2, \mu = 1, p_0 = 5, U = 7, V = 5$

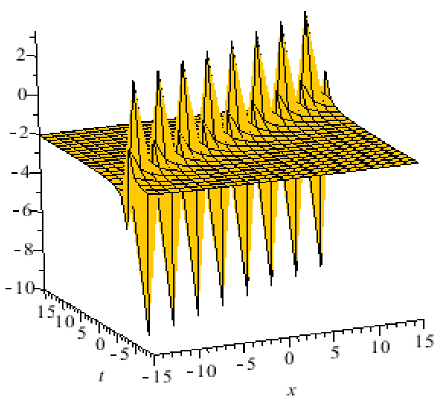

Fig. 6: Solitons solution for
$\lambda = 4, \mu = 4, p_0 = 2, U = 1, V = 1$

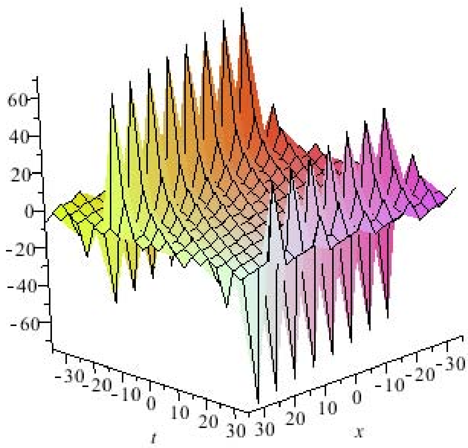

Fig. 7: Solitons solution for $\lambda = 3, \mu = 3, p_0 = 2$

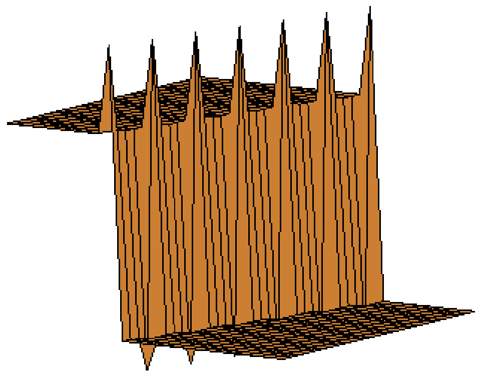

Fig. 8: Solitons solution for $\lambda = 2, \mu = 0.125, p_0 = 1$

5. Conclusions

In this article, the highly nonlinear partial differential equation, namely, the (3+1)-dimensional Jimbo-Miwa equation is investigated by applying the improved (G'/G)-expansion method. We have constructed some new traveling wave solutions including solitons and periodic wave solutions. The solutions are presented through three different families, like as, the hyperbolic, the trigonometric and the rational functions. Further, it is significant to reveal that some of our solutions are identical with the published results, for some special cases and others are new. The obtained solutions show that the improved (G'/G)-expansion method gives more general solutions and it has more advantages. As a result, we hope this straightforward and efficient method would be applied to construct abundant traveling wave solutions of different nonlinear partial differential equations which frequently arise in engineering sciences, mathematical physics and other technical arena.

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Statistical convergence in random paranormed space

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Abstract. In this paper we define the notion of statistical convergence and statistical Cauchy in a random paranormed space. We establish some relations between them and obtain a subsequential characterization for statistical convergence in random paranormed space.

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1. Introduction and Preliminaries

Probabilistic metric spaces were introduced by K. Menger in 1942 [8]. In this spaces instead of distance $d(p, q)$ between points p and q, we consider distribution function $F_{p,q}(x)$ and interpreted its values as probability that distance from p to q is less than x. Sherstnev [16] introduced the notion of random normed spaces and later on Hadžić [6] introduced the notion of random paranormed spaces as further generalization of random normed spaces.

A *paranorm* is a function $g : X \to \mathbb{R}$ defined on a linear space X such that for all $x, y, z \in X$

$(P1)$ $g(x) = 0$ if $x = \theta$

$(P2)$ $g(-x) = g(x)$

$(P3)$ $g(x + y) \leq g(x) + g(y)$

$(P4)$ If (α_n) is a sequence of scalars with $\alpha_n \to \alpha_0$ $(n \to \infty)$ and $x_n, a \in X$ with $x_n \to a$ $(n \to\to \infty)$ in the sense that $g(x_n - a) \to 0$ $(n \to \infty)$, then $\alpha_n x_n \to \alpha_0 a$ $(n \to \infty)$, in the sense that $g(\alpha_n x_n - \alpha_0 a) \to 0$ $(n \to \infty)$.

A paranorm g for which $g(x) = 0$ implies $x = \theta$ is called a *total paranorm* on X, and the pair (X, g) is called a *total paranormed space.*

A function $f : \mathbb{R} \to \mathbb{R}_0^+$ is called a *distribution function* if it is a non-decreasing and left continuous with $\inf_{t\in\mathbb{R}} f(t) = 0$ and $\sup_{t\in\mathbb{R}} f(t) = 1$. By D^+, we denote the set of all distribution functions such that $f(0) = 0$.

If $a \in \mathbb{R}_0^+$, then $H_a \in D^+$, where

$$H_a(t) := \begin{cases} 0 & \text{if } t \leq a, \\ 1 & \text{if } t > a. \end{cases}$$

It is obvious that $H_0 \geq f$ for all $f \in D^+$.

A *t-norm* is a continuous mapping $* : [0,1] \times [0,1] \to [0,1]$ such that $([0,1], *)$ is abelian monoid with unit one and $c * d \geq a * b$ if $c \geq a$ and $d \geq b$ for all $a, b, c \in [0,1]$. A *triangle function* τ is a binary operation on D^+ which is commutative, associative and $\tau(f, H_0) = f$ for every $f \in D^+$.

A *probabilistic metric* space (briefly a *PM-space*) ([8], [15]) is an ordered pair $(X, \mathcal{F})$ where X is an nonempty set and $\mathcal{F} : X \times X \to D^+$. The value of $\mathcal{F}$ at $(u, v) \in X \times X$ will be denoted by $F_{u,v}$. The functions $F_{u,v}$ $(u, v \in X)$ are assumed to satisfy the following conditions:

(a) $F_{u,v}(x) = 1$ for all $x > 0$ if and only if $u = v$,

(b) $F_{u,v}(0) = 0$,

(c) $F_{u,v} = F_{v,u}$,

(d) $F_{u,v}(x) = 1$ and $F_{v,w}(y) = 1$ imply $F_{u,w}(x + y) = 1$.

A *Menger space* is a triplet $(X, \mathcal{F}, *)$, where $(X, \mathcal{F})$ is a PM-space and t-norm $*$ is such that the Menger's triangle inequality

(e) $F_{u,w}(x + y) \geq F_{u,v}(x) * F_{v,w}(y)$

is satisfied for all $u, v, w \in X$ and for all $x \geq 0, y \geq 0$.

Let X be a linear space of dimension greater than one, τ a triangle, and $\mathcal{F} : X \times X \to D^+$ (the value of $\mathcal{F}$ at $u \in X$ will be denoted by F_u). Then $\mathcal{F}$ is called a *probabilistic norm* and triplet $(X, \mathcal{F}, \tau)$ a *probabilistic normed space* (briefly a *PN-space*) ([15], [16]) if the following conditions are satisfied:

(1) $F_x(t) = H_0(t)$ if and only if $x = 0$, where $F_u(t)$ denotes the value of F_x at $t \in \mathbb{R}$,

(2) $F_u(0) = 0$ for each $x \in X$,

(3) $F_{\alpha x}(t) = F_x(\frac{t}{|\alpha|})$ for every $t > 0, \alpha \neq 0$ and $x \in X$,

(4) $F_{x+y}(t) \geq \tau(F_x(t), F_x(t))$ whenever $x, y, z \in X$.

If (4) is replaced by

$(4)'$ $F_{x+y}(t_1 + t_2) \geq F_x(t_1) * F_y(t_2)$ for all $x, y, z \in X$ and $t_1, t_2 \in \mathbb{R}_0^+$;

then $(X, \mathcal{F}, *)$ is called a *random normed space.*

A *random paranormed space* (briefly a *RPN-space*) [6] is an ordered triple $(X, \mathcal{F}, *)$, where X is a linear space X, $\mathcal{F}$ is a mapping of X into D^+ such that:

$(RP1)$ $F_x = H$ if and only if $x = \theta$;

$(RP2)$ $F_x(0) = 0$ for each $x \in X$;

$(RP3)$ for any $x \in X$ and each $\alpha \in \mathbb{R}$, $F_{-x}(\alpha) = F_x(\alpha)$;

$(RP4)$ $F_{x+y}(\alpha+\beta) \geq F_x(\alpha) * F_y(\beta)$ for any $x, y \in X$ and $\alpha, \beta \in \mathbb{R}$;

$(RP5)$ If (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ $(n \to \infty)$ and $\lim F_{x_n - x}(\varepsilon) = 1$ for every $\varepsilon > 0$, then in the sense that $\lim F_{\lambda_n x_n - \lambda x} = 1$.

In this paper, we define and study the notion of convergence, statistical convergence and statistical Cauchy in random paranormed space.

2. Statistical convergence

The concept of statistical convergence for sequences of real numbers was introduced by Fast [4] and further studied by Fridy [5] and since then several generalizations and applications of this notion have been investigated by various authors, namely [2], [7], [6], [9], [11], [12], [14]. Recently, in [10] and [13], the concept of statistical convergence is studied in probabilistic normed space and in intuitionistic fuzzy normed spaces, respectively. Recently, Alotaibi and Alroqi [1] have studied the notion of statistical convergence in paranormed space. In this paper we shall study the concept of statistical convergence and statistical Cauchy in random paranormed space, hence generazing the results of [1].

Let K be a subset of the set of natural numbers $\mathbb{N}$. Then the *asymptotic density* of K denoted by $\delta(K)$, is defined as $\delta(K) = \lim_n \frac{1}{n} \mid \{k \leq n : k \in K\} \mid$, where the vertical bars denote the cardinality of the enclosed set.

A number sequence $x = (x_k)$ is said to be *statistically convergent* to the number L if for each $\epsilon > 0$, the set $K(\epsilon) = \{k \leq n : |x_k - L| > \epsilon\}$ has asymptotic density zero, i.e.

$$\lim_n \frac{1}{n} |\{k \leq n : |x_k - L\ |\}| = 0.$$

In this case we write st-$\lim x = L$.

A number sequence $x = (x_k)$ is said to be *statistically Cauchy* sequence if for every $\epsilon > 0$ there exists a number $N = N(\epsilon)$ such that

$$\lim_n \frac{1}{n} |\{k \leq n : |x_k - x_N| \geq \epsilon\}| = 0.$$

Let $(X, \mathcal{F}, *)$ be a random paranormed space.

Definition 2.1. A sequence $x = (x_k)$ is said to be *convergent* to the number ξ in $(X, \mathcal{F}, *)$ if for every $\varepsilon > 0$ and $\theta \in (0,1)$, there exists a positive integer k_0 such that $F_{x_k - \xi}(\varepsilon) > 1 - \theta$ whenever $k \geq k_0$. In this we write F-$\lim x = \xi$, and ξ is called the *F-limit* of x.

Definition 2.2. A sequence $x = (x_k)$ is said to be *statistically convergent to the number* ξ *in* $(X, \mathcal{F}, t)$ (or $F(st)$-*convergent*) if for each $\varepsilon > 0$ and $\theta \in (0,1)$

$$\delta(\{k \in \mathbb{N} : F_{x_k-\xi}(\varepsilon) \leq 1-\theta\}) = 0,$$

or equivalently

$$\delta(\{k \in \mathbb{N} : F_{x_k-\xi}(\varepsilon) > \theta\}) = 1;$$

i.e.

$$\lim_n \frac{1}{n}|\{k \leq n : F_{x_k-\xi}(\varepsilon) \leq 1-\theta\}| = 0.$$

In this case we write $F(st)$-$\lim x = \xi$. We denote the set of all $F(st)$-convergent sequences by S_F.

Definition 2.3. A number sequence $x = (x_k)$ is said to be *statistically Cauchy in* $(X, \mathcal{F}, *)$ (or $F(st)$-*Cauchy*) if for every $\epsilon > 0$ and $\theta \in (0,1)$ there exists a number $N = N(\epsilon)$ such that

$$\lim_n \frac{1}{n}|\{k \leq n : F_{x_k-x_N}(\varepsilon) \leq 1-\theta\}| = 0.$$

3. Main results

Theorem 3.1. If a sequence $x = (x_k)$ is statistically convergent in $(X, \mathcal{F}, *)$ then $F(st)$-limit is unique.

Proof. Suppose that $F(st)$-$\lim x = \xi_1$ and $F(st)$-$\lim x = \xi_2$. Given $\alpha > 0$ choose $\theta \in (0,1)$ such that $(1-\theta)*(1-\theta) > 1-\alpha$. Given $\varepsilon > 0$, define the following sets as:

$$K_1(\varepsilon) = \{k \in \mathbb{N} : F_{x_k-\xi_1}(\varepsilon/2) \leq 1-\theta\},$$

$$K_2(\varepsilon) = \{k \in \mathbb{N} : F_{x_k-\xi_2}(\varepsilon/2) \leq 1-\theta\}.$$

Since $F(st)$-$\lim x = \xi_1$, we have $\delta(K_1(\varepsilon)) = 0$. Similarly $F(st)$-$\lim x = \xi_2$ implies that $\delta(K_2(\varepsilon)) = 0$.Now let $K(\varepsilon) = K_1(\varepsilon) \cup K_2(\varepsilon)$. Then $\delta(K(\varepsilon)) = 0$ and hence the compliment $K^C(\varepsilon)$ is a nonempty set and $\delta(K^C(\varepsilon)) = 1$. Now if $k \in K^C(\varepsilon) = K_1^C(\varepsilon) \cap K_2^C(\varepsilon)$, then we have

$$F_{x_k-\xi_1}(\varepsilon/2) > 1-\theta \text{ and } F_{x_k-\xi_2}(\varepsilon/2) > 1-\theta.$$

Now if $k \in \mathbb{N}\backslash K(\varepsilon)$, then we have $F_{\xi_1-\xi_2}(\varepsilon) \geq F_{x_n-\xi_1}(\varepsilon/2) * F_{x_n-\xi_2}(\varepsilon/2) > (1-\theta) * (1-\theta) > 1-\alpha$.
Since $\alpha > 0$ was arbitrary, we get $F_{\xi_1-\xi_2}(\varepsilon) = 1$ for all $\varepsilon > 0$, and hence $\xi_1 = \xi_2$.

This completes the proof of the theorem.

Theorem 3.2. If F-$\lim x = \xi$ then $F(st)$-$\lim x = \xi$ but converse need not be true in general.

Proof. Let F-$\lim x = \xi$. Then for every $\varepsilon > 0$ and given $\theta \in (0,1)$, there is a positive integer N such that

$$F_{x_k - \xi}(\varepsilon) > 1 - \theta$$

for all $k \geq N$. Since the set $A(\epsilon) := \{k \in \mathbb{N} : F_{x_k-\xi}(\varepsilon) \leq 1 - \theta\} \subseteq \{1, 2, 3, ..., N-1\}$, $\delta(A(\epsilon)) = 0$. Hence $F(st)$-$\lim x = \xi$.
The following example shows that the converse need not be true.

Example. Let $X = \ell(1/k) := \{x = (x_k) : \sum_k \mid x_k \mid^{1/k} < \infty\}$. Then X is a paranormed space with paranorm $g(x) = \sum_k \mid x_k \mid^{1/k}$; and X is a random paranormed space with

$$F_{x_k}(\varepsilon) := \begin{cases} 1, & g(x) < \varepsilon \\ 0, & g(x) \geq \varepsilon. \end{cases}$$

Define a sequence $x = (x_k)$ by

$$x_k := \begin{cases} k, & \text{if } k = n^2, n \in \mathbb{N}; \\ 0, & \text{otherwise}; \end{cases}$$

and write for $\varepsilon > 0$,

$$K(\varepsilon) := \{k \leq n : F_{x_k}(\varepsilon) \leq 1 - \theta\}, 0 < \theta < 1.$$

We see that

$$F_{x_k}(\varepsilon) := \begin{cases} k^{1/k}, & \text{if } k = n^2, n \in \mathbb{N}; \\ 0, & \text{otherwise}; \end{cases}$$

and hence

$$\lim_k F_{x_k}(\varepsilon) := \begin{cases} 1, & \text{if } k = n^2, n \in \mathbb{N}; \\ 0, & \text{otherwise}; \end{cases}$$

Therefore F-$\lim x$ does not exist. On the other hand $\delta(K(\varepsilon)) = 0$, that is, $F(st)$-$\lim x = 0$.

This completes the proof of the theorem.

Theorem 3.3. Let $F(st)$-$\lim x = \xi_1$ and $F(st) - \lim y = \xi_2$. Then
(*i*) $F(st)$-$\lim(x \pm y) = \xi_1 \pm \xi_2$,
(*ii*) $F(st)$-$\lim \alpha x = \alpha \xi_1$, $\alpha \in \mathbb{R}$.

Proof. It is easy to prove.

Theorem 3.4. A sequence $x = (x_k)$ in $(X, \mathcal{F}, *)$ is statistically convergent to ξ if and only if there exists a set $K = \{k_1 < k_2 < ... < k_n < ..\} \subseteq \mathbb{N}$ with $\delta(K) = 1$ such that F-$\lim_{n\to\infty} x_{k_n} = \xi$. for every $\varepsilon > 0$.

Proof. Suppose that $F(st)$-$\lim x = \xi$. Now, write for every $\varepsilon > 0$ and for $r = 1, 2, ...$

$$K_r := \{n \in \mathbb{N} : F_{x_{k_n}-\xi}(\varepsilon) \geq 1 - \frac{1}{r}\},$$

and

$$M_r := \{n \in \mathbb{N} : F_{x_{k_n}-\xi}(\varepsilon) < \frac{1}{r}\}.$$

Then $\delta(K_r) = 0$,

$$M_1 \supset M_2 \supset ...... \supset M_i \supset M_{i+1} \supset ......, \tag{3.1}$$

and

$$\delta(M_r) = 1, r = 1, 2, ... \tag{3.2}$$

Now we have to show that for $n \in M_r$, (x_{k_n}) is F-convergent to ξ. On contrary suppose that (x_{k_n}) is not F-convergent to ξ. Therefore there is $\theta > 0$ such that $F_{x_{k_n}-\xi}(\varepsilon) \geq \theta$ for infinitely many terms. Let $M_\varepsilon := \{n \in \mathbb{N} : F_{x_{k_n}-\xi}(\varepsilon) < \theta\}$ and $\theta > \frac{1}{r}, r \in \mathbb{N}$. Then

$$\delta(M_\varepsilon) = 0\ , \tag{3.3}$$

and by (3.1), $M_r \subset M_\theta$. Hence $\delta(M_r) = 0$,which contradicts (3.2) and we get that (x_{k_n}) is F-convergent to ξ.

Conversely, suppose that there exists a set $K = \{k_1 < k_2 < k_3 < .... < k_n < ...\}$ with $\delta(K) = 1$ such that F-$\lim_{n\to\infty} x_{k_n} = \xi$. Then there is a positive integer N such that $F_{x_n-\xi}(\varepsilon) > 1 - \theta$ for $n > N$. Put $K_\varepsilon := \{n \in \mathbb{N} : F_{x_n-\xi}(\varepsilon) \leq 1 - \theta\}$ and $K' :=$

$\{k_{N+1}, k_{N+2}, ...\}$. Then $\delta(K') = 1$ and $K_\varepsilon \subseteq \mathbb{N} - K'$ which implies that $\delta(K_\varepsilon) = 0$. Hence $F(st)$-$\lim x = \xi$.

This completes the proof of the theorem.

Theorem 3.5. A sequence $x = (x_k)$ in a random paranormed space $(X, \mathcal{F}, *)$ is $F(st)$-convergent if and only if it is $F(st)$-Cauchy.

Proof. Suppose that $F(st)$-$\lim x = \xi$. Choose $r > 0$ such that $(1-r)*(1-r) > 1-\theta$. Then, for all $\varepsilon > 0$, we get

$$\delta(A(r)) = 0, \tag{2.5.1}$$

where $A(r) := \{n \in \mathbb{N} : F_{x_n-\xi}(\varepsilon/2) \leq 1-r\}$. This implies that

$$\delta(A^C(r)) = \delta(\{n \in \mathbb{N} : F_{x_n-\xi}(\varepsilon/2) > 1-r\}) = 1.$$

Let $m \in A^C(r)$. Then $F_{x_n-\xi}(\varepsilon/2) > 1-r$. Now, let $B(r) := \{n \in \mathbb{N} : F_{x_m-x_n}(\varepsilon) \leq 1-r\}$. We need to show that $B(r) \subset A(r)$. Let $n \in B(r)$. Then $F_{x_m-x_n}(\varepsilon) \leq 1-r$ and hence $F_{x_n-\xi}(\varepsilon/2) \leq 1-r$, i.e. $n \in A(r)$. Otherwise, if $F_{x_n-\xi}(\varepsilon/2) > 1-r$ then

$$1-r \geq F_{x_m-x_n}(\varepsilon) \geq F_{x_n-\xi}(\varepsilon/2) * F_{x_m-\xi}(\varepsilon/2) > (1-r)*(1-r) > 1-\theta,$$

which is not possible. Hence $B(r) \subset A(r)$, which implies that $x = (x_k)$ is $F(st)$-convergent.

Conversely, suppose that $x = (x_k)$ is $F(st)$-Cauchy but not $F(st)$-convergent. Then there exists $M \in \mathbb{N}$ such that $\delta(G(r) = 0$, where $G(r) := \{n \in \mathbb{N} : F_{x_n-x_M}(\varepsilon) \leq 1-r\}$, and $\delta(D(r)) = 0$,where $D(r) := \{n \in \mathbb{N} : F_{x_n-\xi}(\varepsilon) > 1-r\}$, i.e. $\delta(D^C(r)) = 1$. Since $F_{x_n-x_m}(\varepsilon) \geq 2F_{x_n-\xi}(\varepsilon/2) > 1-r$, if $F_{x_n-\xi}(\varepsilon/2) > (1-r)/2$. Therefore $\delta(G^C(r)) = 0$, i.e. $\delta(G(r) = 1$, which leads to a contradiction, since $x = (x_k)$ was $F(st)$-Cauchy. Hence $x = (x_k)$ must be $F(st)$-convergent.

This completes the proof of the theorem.

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Stability and Solutions for Rational Recursive Sequence of Order Three

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ABSTRACT

In this paper we deal with the behavior of the local stability, global attractor and boundedness of solutions of the following difference equation

$$x_{n+1} = ax_{n-1} + \frac{bx_{n-1}x_{n-2}}{cx_n + dx_{n-2}}, \quad n = 0, 1, ...,$$

where the initial conditions $x_{-2},\ x_{-1},\ x_0$ are arbitrary positive real numbers and a, b, c, d are constants. Also, we give a specific form of solution of four special cases of this equation.

Keywords: stability, global attractor, boundedness, periodicity, solution of difference equations.
Mathematics Subject Classification: 39A10

1 Introduction

In this paper we deal with properties of the local stability, global attractor and boundedness of the solutions of the following difference equation

$$x_{n+1} = ax_{n-1} + \frac{bx_{n-1}x_{n-2}}{cx_n + dx_{n-2}}, \quad n = 0, 1, ..., \tag{1}$$

where the initial conditions $x_{-2},\ x_{-1},\ x_0$ are arbitrary positive real numbers and a, b, c, d are constants. Also, we obtain the form of the solution of some special cases of Eq.(1).
Let I be some interval of real numbers and let $f : I^{k+1} \to I$, be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, ..., x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, ..., \tag{2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

The linearized equation of Eq.(2) about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f(\overline{x}, \overline{x}, ..., \overline{x})}{\partial x_{n-i}} y_{n-i}. \tag{3}$$

Theorem A [22] Assume that $p_i \in R,\ i = 1, 2, ..., k$ and $k \in \{0, 1, 2, ...\}$. Then

$$\sum_{i=1}^{k} |p_i| < 1,$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+k} + p_1 x_{n+k-1} + ... + p_k x_n = 0, \ \ n = 0, 1, ... \ . \tag{4}$$

Consider the following equation

$$x_{n+1} = g(x_n, x_{n-1}, x_{n-2}) \tag{5}$$

The following theorem will be useful for the proof of our results in this paper.
Theorem B [23]: Let $[a, b]$ be an interval of real numbers and assume that $g : [a, b]^3 \to [a, b]$ is a continuous function satisfying the following properties :
(a) $g(x, y, z)$ is non-decreasing in y and z in $[a, b]$ for each $x \in [a, b]$, and is non-increasing in $x \in [a, b]$ for each y and z in $[a, b]$;
(b) If $(m, M) \in [a, b] \times [a, b]$ is a solution of the system

$$M = g(m, M, M) \ \ and \ \ m = g(M, m, m)$$

then $m = M$. Then Eq.(5) has a unique equilibrium $\overline{x} \in [a, b]$ and every solution of Eq.(5) converges to $\overline{x}$.
The nature of many biological systems naturally leads to their study by means of a discrete variable. Particular examples include population dynamics and genetics. Some elementary models of biological phenomena, including a single species population model, harvesting of fish, the production of red blood cells, ventilation volume and blood CO_2 levels, a simple epidemics model and a model of waves of disease that can be analyzed by difference equations are shown in [26]. Recently, there has been interest in so-called dynamical diseases, which correspond to physiological disorders for which a generally stable control system becomes unstable. One of the first papers on this subject was that of Mackey [25]. In that paper they investigated a simple first order difference-delay equation that models the concentration of blood-level CO_2. They also discussed models of a second class of diseases associated with the production of red cells, white cells, and platelets in the bone marrow.
The study of asymptotic stability and oscillatory properties of solutions of difference equations is extremely useful in the behavior of mathematical models of various biological systems and other applications. This is due to the fact that difference equations are appropriate models for describing situations where the variable is assumed to take only a discrete set of values and they arise frequently in the study of biological models, in the formulation and analysis of discrete time systems, the numerical integration of differential equations by finite-difference schemes, the study of deterministic chaos, etc. For example, [24] the study of oscillation of positive solutions about the positive steady state N in the delay logistic difference equation

$$N_{n+1} = N_n \ \exp\left[r\left(1 - \sum_{j=0}^{m} p_j N_{n-j}\right)\right],$$

which describes situations where population growth is not continuous but seasonal with non-overlapping generations, leads to the study of oscillations about zero of a linear difference equation of the form

$$x_{n+1} - x_n + \sum_{i=0}^{m} p_i x_{n-k_i} = 0, \quad n = 0, 1, \ldots .$$

Also, difference equations are appropriate models for describing situations where population growth is not continuous but seasonal with overlapping generations. For example, the difference equation,

$$y_{n+1} = y_n \exp\left[r\left(1 - \tfrac{y_n}{K}\right)\right],$$

has been used to model various animal populations.
El-Metwally et al. [16] investigated the asymptotic behavior of the population model:

$$x_{n+1} = \alpha + \beta x_{n-1} e^{-x_n}.$$

The generalized Beverton-Holt stock recruitment model has investigated in [4,7]:

$$x_{n+1} = a x_n + \tfrac{b x_{n-1}}{1 + c x_{n-1} + d x_n}.$$

See also [8,14-15, 27]. The long term behavior of the solutions of nonlinear difference equations of order greater than one has been extensively studied during the last decade. For example: Cinar [6] has got the solutions of the following difference equation

$$x_{n+1} = \tfrac{a x_{n-1}}{1 + b x_n x_{n-1}}.$$

Elabbasy et al. [12] investigated the behavior and obtained the solution of some special cases of the difference equation

$$x_{n+1} = \tfrac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n-i}}.$$

Other related results on difference equations can be found in refs. [1-3,5,9-11,13,17-18,28-35].

2 Qualitative Behavior of Solutions of Eq.(1)

In this section we study some qualitative behavior of Eq.(1) such that local stability, global attractor of the equilibrium point and boundedness character of solutions of Eq.(1) when the constants a, b, c, d are positive real numbers.

2.1 Local Stability of Eq.(1)

In this section we investigate the local stability character of the solutions of Eq.(1). Eq.(1) has a unique equilibrium point which is given by

$$\overline{x} = a\overline{x} + \frac{b\overline{x}^2}{c\overline{x} + d\overline{x}}, \quad \Rightarrow \quad \overline{x}^2(1-a)(c+d) = b\overline{x}^2.$$

When $(c+d)(1-a) = b$, Eq.(1) has an infinite number of equilibrium points, and every non-negative real number $\overline{x}$ is its equilibrium point. If $(c+d)(1-a) \neq b$. Then, it follows the unique equilibrium point is $\overline{x} = 0$.

Let $f:(0,\infty)^3 \longrightarrow (0,\infty)$ be a function defined by

$$f(u,v,w)=av+\frac{bvw}{cu+dw}. \tag{6}$$

Therefore it follows that

$$f_u(u,v,w)=\frac{-bcvw}{(cu+dw)^2},\ f_v(u,v,w)=a+\frac{bw}{cu+dw},\ f_w(u,v,w)=\frac{bcuv}{(cu+dw)^2},$$

we see that

$$f_u(\overline{x},\overline{x},\overline{x})=\frac{-bc}{(c+d)^2},\ f_v(\overline{x},\overline{x},\overline{x})=a+\frac{b}{c+d},\ f_w(\overline{x},\overline{x},\overline{x})=\frac{bc}{(c+d)^2}.$$

The linearized equation of Eq.(1) about $\overline{x}$ is

$$y_{n+1}+\frac{bc}{(c+d)^2}y_n-\left(a+\frac{b}{c+d}\right)y_{n-1}-\frac{bc}{(c+d)^2}y_{n-2}=0. \tag{7}$$

Theorem 1. Assume that

$$a<1,\ \ b(3c+d)<(1-a)(c+d)^2.$$

Then the equilibrium point of Eq.(1) is locally asymptotically stable.
Proof: It follows by Theorem A that, Eq.(7) is asymptotically stable if

$$\left|\frac{bc}{(c+d)^2}\right|+\left|a+\frac{b}{c+d}\right|+\left|\frac{bc}{(c+d)^2}\right|<1 \ \ \Rightarrow \ \ a+\frac{b}{c+d}+\frac{2bc}{(c+d)^2}<1.$$

That is,

$$\frac{b(3c+d)}{(c+d)^2}<(1-a).$$

The proof is complete.

2.2 Global Attractor of the Equilibrium Point of Eq.(1)

In this section we investigate the global attractivity character of solutions of Eq.(1).
Theorem 2. If $d(1-a)\neq b,$ then the equilibrium point $\overline{x}$ of Eq.(1) is a global attractor.
Proof: Let p,q be real numbers and assume that $g:[p,q]^3\longrightarrow[p,q]$ is a function defined by $g(u,v,w)=av+\frac{bvw}{cu+dw}$, then we can easily see that the function $g(u,v,w)$ is increasing in v,w and is decreasing in $u.$
Suppose that (m,M) is a solution of the system $M=g(m,M,M)\ and\ m=g(M,m,m)$. Then from Eq.(1), we see that

$$M=aM+\frac{bM^2}{cm+dM},\quad m=am+\frac{bm^2}{cM+dm}.$$

Then

$$c(1-a)Mm+d(1-a)M^2=bM^2,\ c(1-a)Mm+d(1-a)m^2=bm^2.$$

Subtracting we obtain

$$d(1-a)(M^2-m^2)=b(M^2-m^2),\ d(1-a)\neq b.$$

Thus $M=m$. It follows by Theorem B that $\overline{x}$ is a global attractor of Eq.(1) and then the proof is complete.

2.3 Boundedness of Solutions of Eq.(1)

In this section we study the boundedness of solutions of Eq.(1).

Theorem 3. Every solution of Eq.(1) is bounded when $\left(a+\frac{b}{d}\right)<1.$

Proof: Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of Eq.(1). It follows from Eq.(1) that

$$x_{n+1}=ax_{n-1}+\frac{bx_{n-1}x_{n-2}}{cx_n+dx_{n-2}}\leq ax_{n-1}+\frac{bx_{n-1}x_{n-2}}{dx_{n-2}}=(a+\frac{b}{d})x_{n-1}.$$

Then

$$x_{n+1}\leq x_{n-1}\quad \text{for all}\quad n\geq 0.$$

Thus, the subsequences $\{x_{2n-1}\}_{n=0}^{\infty}$, $\{x_{2n}\}_{n=0}^{\infty}$ are decreasing and so are bounded from above by $M=\max\{x_{-2},x_{-1},x_0\}$.

3 Explicit Solutions of Some Special Cases of Eq.(1)

Our goal in this section is to find a specific form of the solutions of some special cases of Eq.(1) and give numerical examples in each case when the constants a,b,c,d are nonzero real numbers.

3.1 Case (1)

In this section we study the following special case of Eq.(1)

$$x_{n+1}=x_{n-1}+\frac{x_{n-1}x_{n-2}}{x_n+x_{n-2}}, \tag{8}$$

where the initial conditions $x_{-2},\ x_{-1},\ x_0$ are arbitrary nonzero real numbers.

Theorem 4. Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of Eq.(8). Then for $n=0,1,2,...$

$$x_{2n-1}=k\prod_{i=1}^{n}\left(\frac{A_{2i-1}k+2B_{2i-1}r}{B_{2i-1}k+A_{2i-1}r}\right),\ x_{2n}=h\prod_{i=1}^{n}\left(\frac{A_{2i}k+2B_{2i}r}{B_{2i}k+A_{2i}r}\right),$$

where $x_{-2}=r, x_{-1}=k,\ x_0=h,\ \{A_m\}_{m=1}^{\infty}=\{1,3,7,17,41,...\},\ \{B_m\}_{m=1}^{\infty}=\{1,2,5,12,29,...\}$ i. e. $A_m=2A_{m-1}+A_{m-2},\ B_m=2B_{m-1}+B_{m-2},\ m\geq 1,\ A_{-1}=-1,\ A_0=1,\ B_{-1}=1,\ B_0=0$ $($or $A_m=2B_{m-1}+A_{m-1},\ B_m=B_{m-1}+A_{m-1},\ m\geq 0,\ A_{-1}=-1,\ B_{-1}=1)$.

Proof: For $n=0$ the result holds. Now suppose that $n>0$ and that our assumption holds for $n-1,\ n-2$. That is;

$$x_{2n-3}=k\prod_{i=1}^{n-1}\left(\frac{A_{2i-1}k+2B_{2i-1}r}{B_{2i-1}k+A_{2i-1}r}\right),\ x_{2n-2}=h\prod_{i=1}^{n-1}\left(\frac{A_{2i}k+2B_{2i}r}{B_{2i}k+A_{2i}r}\right),\ x_{2n-4}=h\prod_{i=1}^{n-2}\left(\frac{A_{2i}k+2B_{2i}r}{B_{2i}k+A_{2i}r}\right).$$

Now, it follows from Eq.(8) that

$$\begin{aligned}
x_{2n-1} &= x_{2n-3}+\frac{x_{2n-3}x_{2n-4}}{x_{2n-2}+x_{2n-4}}\\
&= k\prod_{i=1}^{n-1}\left(\frac{A_{2i-1}k+2B_{2i-1}r}{B_{2i-1}k+A_{2i-1}r}\right)+\frac{k\prod_{i=1}^{n-1}\left(\frac{A_{2i-1}k+2B_{2i-1}r}{B_{2i-1}k+A_{2i-1}r}\right)h\prod_{i=1}^{n-2}\left(\frac{A_{2i}k+2B_{2i}r}{B_{2i}k+A_{2i}r}\right)}{h\prod_{i=1}^{n-1}\left(\frac{A_{2i}k+2B_{2i}r}{B_{2i}k+A_{2i}r}\right)+h\prod_{i=1}^{n-2}\left(\frac{A_{2i}k+2B_{2i}r}{B_{2i}k+A_{2i}r}\right)}\\
&= k\prod_{i=1}^{n-1}\left(\frac{A_{2i-1}k+2B_{2i-1}r}{B_{2i-1}k+A_{2i-1}r}\right)+\frac{k\prod_{i=1}^{n-1}\left(\frac{A_{2i-1}k+2B_{2i-1}r}{B_{2i-1}k+A_{2i-1}r}\right)}{\left(\frac{A_{2n-2}k+2B_{2n-2}r}{B_{2n-2}k+A_{2n-2}r}\right)+1}
\end{aligned}$$

$$
\begin{aligned}
&= k\prod_{i=1}^{n-1}\left(\tfrac{A_{2i-1}k+2B_{2i-1}r}{B_{2i-1}k+A_{2i-1}r}\right)+\frac{(B_{2n-2}k+A_{2n-2}r)k\prod\limits_{i=1}^{n-1}\left(\frac{A_{2i-1}k+2B_{2i-1}r}{B_{2i-1}k+A_{2i-1}r}\right)}{B_{2n-1}k+A_{2n-1}r}\\
&= k\prod_{i=1}^{n-1}\left(\tfrac{A_{2i-1}k+2B_{2i-1}r}{B_{2i-1}k+A_{2i-1}r}\right)\left(1+\tfrac{B_{2n-2}k+A_{2n-2}r}{B_{2n-1}k+A_{2n-1}r}\right)=k\prod_{i=1}^{n-1}\left(\tfrac{A_{2i-1}k+2B_{2i-1}r}{B_{2i-1}k+A_{2i-1}r}\right)\left(\tfrac{A_{2n-1}k+2B_{2n-1}r}{B_{2n-1}k+A_{2n-1}r}\right).
\end{aligned}
$$

Therefore

$$x_{2n-1}=k\prod_{i=1}^{n}\left(\tfrac{A_{2i-1}k+2B_{2i-1}r}{B_{2i-1}k+A_{2i-1}r}\right).$$

Also, we see from Eq.(8) that

$$
\begin{aligned}
x_{2n} &= x_{2n-2}+\tfrac{x_{2n-2}x_{2n-3}}{x_{2n-1}+x_{2n-3}}\\
&= h\prod_{i=1}^{n-1}\left(\tfrac{A_{2i}k+2B_{2i}r}{B_{2i}k+A_{2i}r}\right)+\frac{h\prod\limits_{i=1}^{n-1}\left(\frac{A_{2i}k+2B_{2i}r}{B_{2i}k+A_{2i}r}\right)k\prod\limits_{i=1}^{n-1}\left(\frac{A_{2i-1}k+2B_{2i-1}r}{B_{2i-1}k+A_{2i-1}r}\right)}{k\prod\limits_{i=1}^{n}\left(\frac{A_{2i-1}k+2B_{2i-1}r}{B_{2i-1}k+A_{2i-1}r}\right)+k\prod\limits_{i=1}^{n-1}\left(\frac{A_{2i-1}k+2B_{2i-1}r}{B_{2i-1}k+A_{2i-1}r}\right)}\\
&= h\prod_{i=1}^{n-1}\left(\tfrac{A_{2i}k+2B_{2i}r}{B_{2i}k+A_{2i}r}\right)+\frac{h\prod\limits_{i=1}^{n-1}\left(\frac{A_{2i}k+2B_{2i}r}{B_{2i}k+A_{2i}r}\right)}{\left(\frac{A_{2n-1}k+2B_{2n-1}r}{B_{2n-1}k+A_{2n-1}r}\right)+1}\\
&= h\prod_{i=1}^{n-1}\left(\tfrac{A_{2i}k+2B_{2i}r}{B_{2i}k+A_{2i}r}\right)\left(1+\tfrac{B_{2n-1}k+A_{2n-1}r}{B_{2n}k+A_{2n}r}\right).
\end{aligned}
$$

Thus

$$x_{2n}=h\prod_{i=1}^{n}\left(\tfrac{A_{2i}k+2B_{2i}r}{B_{2i}k+A_{2i}r}\right).$$

Hence, the proof is completed.

For supporting the results of this section, we consider numerical example for $x_{-2}=2,\ x_{-1}=6,\ x_0=11$. [See Fig. 1].

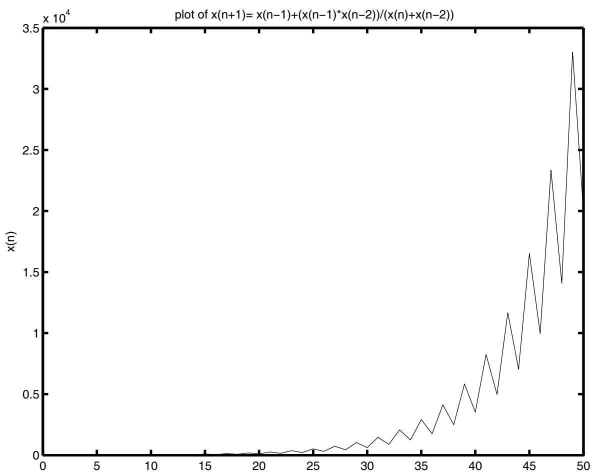

Figure 1.

3.2 Case (2)

In this section we give a specific form for the solutions of the difference equation

$$x_{n+1}=x_{n-1}+\frac{x_{n-1}x_{n-2}}{x_n-x_{n-2}}, \tag{9}$$

where the initial conditions $x_{-2},\ x_{-1},\ x_0$ are arbitrary non zero real numbers with $x_{-2} \neq x_0$.

Theorem 5. Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of Eq.(9). Then for $n = 0, 1, 2, ...$

$$x_{2n-2} = \frac{h^n}{r^{n-1}}, \qquad x_{2n-1} = \frac{kh^n}{(h-r)^n},$$

where $x_{-2} = r,\ x_{-1} = k,\ x_0 = h.$

Proof: For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n-1,\ n-2$. That is;

$$x_{2n-3} = \frac{kh^{n-1}}{(h-r)^{n-1}},\ x_{2n-5} = \frac{kh^{n-2}}{(h-r)^{n-2}},\ x_{2n-4} = \frac{h^{n-1}}{r^{n-2}}.$$

Now, it follows from Eq.(9) that

$$\begin{aligned} x_{2n-2} &= x_{2n-4} + \frac{x_{2n-4}x_{2n-5}}{x_{2n-3}+x_{2n-5}} = \frac{h^{n-1}}{r^{n-2}} + \frac{\left(\frac{h^{n-1}}{r^{n-2}}\right)\left(\frac{kh^{n-2}}{(h-r)^{n-2}}\right)}{\left(\frac{kh^{n-1}}{(h-r)^{n-1}}\right)-\left(\frac{kh^{n-2}}{(h-r)^{n-2}}\right)} \\ &= \frac{h^{n-1}}{r^{n-2}} + \frac{\frac{h^{n-1}}{r^{n-2}}}{\frac{h}{h-r}-1} = \frac{h^{n-1}}{r^{n-2}} + \frac{\frac{h^{n-1}}{r^{n-2}}(h-r)}{h-h+r} = \frac{h^{n-1}}{r^{n-2}}\left(1+\frac{h-r}{r}\right) = \frac{h^{n-1}}{r^{n-2}}\left(\frac{h}{r}\right). \end{aligned}$$

Therefore

$$x_{2n-2} = \frac{h^n}{r^{n-1}}.$$

Also, we see from Eq.(9) that

$$\begin{aligned} x_{2n-1} &= x_{2n-3} + \frac{x_{2n-3}x_{2n-4}}{x_{2n-2}+x_{2n-4}} = \frac{kh^{n-1}}{(h-r)^{n-1}} + \frac{\left(\frac{kh^{n-1}}{(h-r)^{n-1}}\right)\left(\frac{h^{n-1}}{r^{n-2}}\right)}{\left(\frac{h^n}{r^{n-1}}\right)-\left(\frac{h^{n-1}}{r^{n-2}}\right)} \\ &= \frac{kh^{n-1}}{(h-r)^{n-1}} + \frac{\left(\frac{kh^{n-1}}{(h-r)^{n-1}}\right)}{\frac{h}{r}-1} = \frac{kh^{n-1}}{(h-r)^{n-1}}\left(1+\frac{1}{\frac{h}{r}-1}\right) = \frac{kh^{n-1}}{(h-r)^{n-1}}\left(\frac{h}{h-r}\right). \end{aligned}$$

Thus

$$x_{2n-1} = \frac{kh^n}{(h-r)^n}.$$

Hence, the proof is completed.

Assume that $x_{-2} = 6, x_{-1} = 1.2,\ x_0 = 3$. [See Fig. 2], for $x_{-2} = 9, x_{-1} = 5,\ x_0 = 21$. [See Fig. 3], and for $x_{-2} = 21, x_{-1} = 11,\ x_0 = 9$. [See Fig. 4].

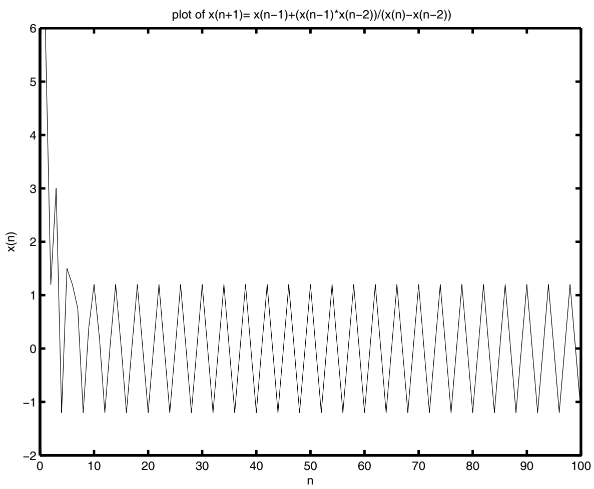

Figure 2.

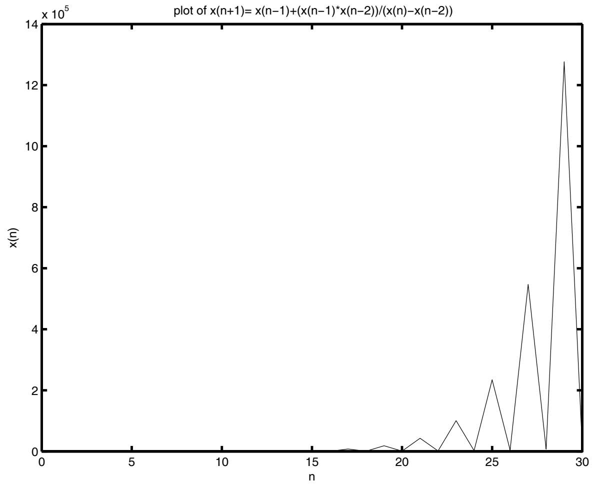

Figure 3.

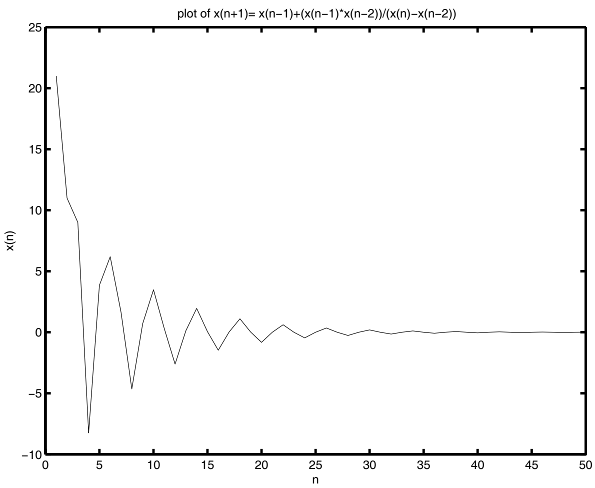

Figure 4.

The following cases can be treated similarly.

3.3 Case (3)

In this section we obtain the solution of the following special case of Eq.(1)

$$x_{n+1}=x_{n-1}-\frac{x_{n-1}x_{n-2}}{x_n+x_{n-2}}, \tag{10}$$

where the initial conditions $x_{-2},\ x_{-1},\ x_0$ are arbitrary positive real numbers.
Theorem 6. Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of Eq.(10). Then for $n=0,1,2,...$

$$x_{2n-1}=\frac{kh^n}{\prod\limits_{i=1}^{n}((2i-1)h+r)},\quad x_{2n}=\frac{h^{n+1}}{\prod\limits_{i=1}^{n}(2ih+r)}.$$

Fig. 5 shows the solution when $x_{-2}=7,\ x_{-1}=2,\ x_0=9.$

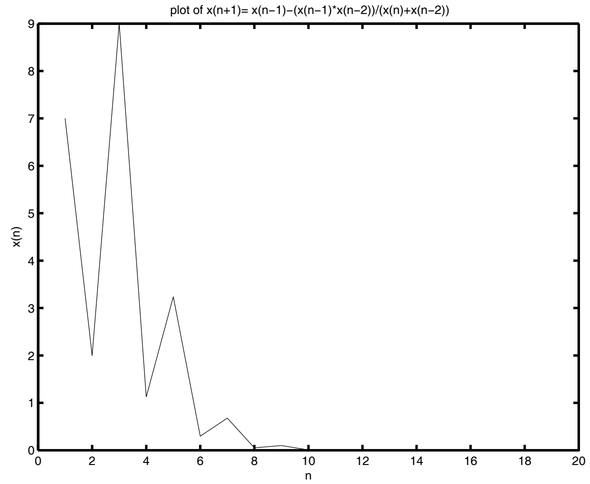

Figure 5.

3.4 Case (4)

In this section we study the following special case of Eq.(1)

$$x_{n+1} = x_{n-1} - \frac{x_{n-1}x_{n-2}}{x_n - x_{n-2}}, \tag{11}$$

where the initial conditions $x_{-2},\ x_{-1},\ x_0$ are arbitrary non zero real numbers.with $x_{-2} \neq x_0$.
Theorem 7. Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of Eq.(11). Then for $n = 0, 1, 2, ...$

$$x_{2n-2} = h\left(\frac{h}{r}\right)^{n-1},\ \ x_{2n-1} = k\left(\frac{h-2r}{h-r}\right)^{n}.$$

Fig. 6 shows the solution when $x_{-2} = 3, x_{-1} = 7,\ x_0 = 8.$

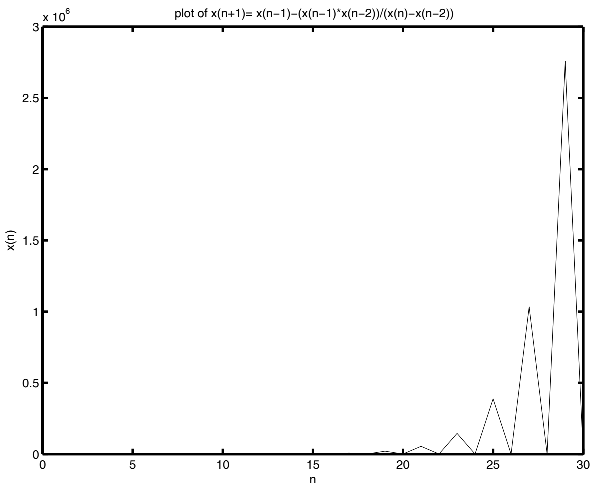

Figure 6.

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Convergence of a generalized MHSS iterative method for augmented systems *

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Abstract

Bai et al. [Modified HSS iteration methods for a class of complex symmetric linear systems, Computing 87 (2010) 93-111.], have presented the modified HSS (MHSS) algorithm to solve a class of complex symmetric linear systems. In this paper, we establish the generalized MHSS(GMHSS) method for solving augmented systems. We prove the convergence of the proposed method under suitable restrictions on the iteration parameters. Lastly, numerical experiments are carried out and experimental results show that the new iterative methods are feasible and effective.

Key words: Complex symmetric matrix, Hermitian and skew-Hermitian splitting, positive definite matrix, iterative methods, convergence.

AMSC: 65F10; 65F15; 65F50

1 Introduction

For solving the linear system

$$Ax = b, \qquad x, b \in \mathbb{C}^n, \tag{1}$$

where $A = (a_{i,j}) \in \mathbb{C}^{n\times n}$ is a complex symmetric matrix of the form

$$A = W + iT,$$

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and $W, T \in \mathbb{R}^{n\times n}$ are real symmetric matrices with W be positive definite and T positive semidefinite. If $T \neq 0$, we obtain that A is non-Hermitian. In this paper, we use $i = \sqrt{-1}$ to denote the imaginary unit.

Since the matrix A naturally possesses a Hermitian/skew-Hermitian splitting (HSS) [1-6]

$$A = H + S,$$

where

$$H = \frac{1}{2}(A + A^*), \qquad S = \frac{1}{2}(A - A^*),$$

with A^* being the conjugate transpose of A, based on this particular matrix splitting for solving the system of linear equations (1), the HSS splitting method was considered and was further discussed in [7-11]. Recently, Huang et al. [12] studied the spectral properties of HSS preconditioner for saddle point problems. To make the HSS method more attractive, an asymmetric Hermitian/skew-Hermitian (AHSS) iteration in [11] was considered. Bai et al. [13] presented the modified HSS to the complex symmetric linear systems, and the modified HSS(MHSS) iteration method is as following:

The MHSS iteration method: Given an initial guess $x^{(0)}$, for $k = 0, 1, 2\ldots$ until $x^{(k)}$ converges, compute

$$\begin{cases} (W + \alpha I)x_{k+\frac{1}{2}} = (\alpha I - iT)x_k + b, \\ (T + \alpha I)x_{k+1} = (\alpha I + iW)x_{k+\frac{1}{2}} - ib, \end{cases} \tag{2}$$

$(k = 0, 1, \cdots)$, where x_0 is an arbitrary initial guess and α is a given positive constant. They have also proved that for any positive α the MHSS method converges unconditionally to the unique solution of the complex symmetric system of linear equations. Guo and Wang in [14] studied the convergence of MHSS when W and T are not real nonsymmetric matrices.

Based on the MHSS splitting method, in this paper, we present a different approach to solve Eq.(1), called the generalized modified Hermitian/ skew-Hermitian splitting iteration method, shortened to the *GMHSS* iteration. Let us describe it as follows.

The GMHSS iteration method: Given an initial guess $x^{(0)}$, for $k = 0, 1, 2\ldots$ until $x^{(k)}$ converges, compute

$$\begin{cases} (W + \alpha P)x_{k+\frac{1}{2}} = (\alpha P - iT)x_k + b, \\ (T + \beta P)x_{k+1} = (\beta P + iW)x_{k+\frac{1}{2}} - ib, \end{cases} \tag{3}$$

where α and β are given positive constants, W is positive definite and T is positkive semidefinite. We asume that the matrix P is symmetric positive defi-

nite and P is exchangeable with the matrices W and T, respectively.

Remark 1.1. The above matrix P is obtained easily, such as P is positive definite diagonal matrix. Obviously, P is exchangeable with matrices W and T.

In this paper, we will analyze the GMHSS iteration method and we find that if the matrix W is positive definite and the matrix T is positive semidefinite, the GMHSS iteration can converge to the unique solution of linear system (1) with any given positive α and β, which is restricted in an appropriate region. And the upper bound of the contraction factor of the GMHSS iteration is only related to paramaters of α and β, the spectrum of the $P^{-1}W$ and $P^{-1}T$.

The outline of this paper is as follows. First in section 2, we will discuss the convergence properties of the GMHSS iterative method. In Section 3 the IGMHSS iteration is described. In Section 4, we provide numerical experiments to illustrate the theoretical results obtained in Section 2, and the effectiveness of proposed methods. Lastly, we obtain some conclusions in section 5.

2 Convergence analysis of the GMHSS method

In this section, we will study the convergence properties of the generalized MHSS iteration and derive the upper bound of the contraction factor. First, we notice that the GMHSS iteration method is a kind of two-splitting iteration s-plitting iteration, acturally, so we quote the convergence criterion for a two-step splitting iteration.

Lemma 2.1.[2] *let $A \in C^{n\times n}$, $A = M_i - N_i$ $(i = 1, 2)$ be two splitting of the matrix A, and $x_0 \in C^n$ be a given initial vector. If x_k is a two-step iteration sequence defined by*

$$\begin{cases} M_1 x_{k+\frac{1}{2}} = N_1 x_k + b, \\ M_2 x_{k+1} = N_2 x_k + b, \end{cases}$$

$k = 0, 1, 2, \cdots$, then

$$x_{k+1} = M_2^{-1} N_2 M_1^{-1} N_1 x_k + M_2^{-1}(I + N_2 M_1^{-1})b, \quad k = 0, 1, 2, \cdots.$$

Moreover, if the spectral radius $\rho(M_2^{-1} N_2 M_1^{-1} N_1) < 1$, then the iterative sequence x_k converges to the unique solution $x_k \in C^n$ of the linear equations (1) for all initial vectors $x_0 \in C^n$.

For the convergence property of the GMHSS iteration, we obtain the following convergence property based on above lemma 2.1.

3

Theorem 2.2. Let $A \in \mathbb{C}^{n\times n}$, with $W \in R^{n\times n}$ and $T \in R^{n\times n}$ symmetric positive definite and symmetric positive semidefinite, respectively. Let α and β be positive constants, the matrix P be symmetric positive definite being exchangeable with the matrices W and T, respectively. Then the iteration matrix $M(\alpha, \beta)$ of the GMHSS method is

$$M(\alpha, \beta) = (\beta P + T)^{-1}(\beta P + iW)(\alpha P + W)^{-1}(\alpha P - iT), \tag{4}$$

and its spectral radius $\rho(M(\alpha, \beta))$ is bounded by

$$\delta(\alpha, \beta) \equiv \max_{\lambda_i \in \lambda(P^{-1}W)} \frac{\sqrt{\beta^2 + {\lambda_i}^2}}{\alpha + \lambda_i} \max_{u_i \in \lambda(P^{-1}T)} \frac{\sqrt{\alpha^2 + {u_i}^2}}{\beta + u_i}, \tag{5}$$

where $\lambda(M)$ is the spectral set of matrix M. And, for any given positive parameter α, if paramater β satisfies

$$\alpha\sqrt{\frac{\lambda_{max}}{2\alpha + \lambda_{max}}} < \beta \leq \sqrt{\alpha(\alpha + 2\lambda_{min})}, \tag{6}$$

then $\delta(\alpha, \beta) < 1$, i.e. the GMHSS iteration converges, where $\lambda_{\max}$ and $\lambda_{\min}$ are the maximum and minimum eigenvalues of matrix $P^{-1}W$.

Proof. We set

$$M_1 = \alpha P + W, \quad N_1 = \alpha P - iT, \quad M_2 = \beta P + T \quad and \quad N_2 = \beta P + iW,$$

in Lemma 2.1. Since α and β are positive constants, $\alpha P + W$ and $\beta P + T$ are nonsingular, we obtain (4).

By direct computations, we have

$$\begin{aligned}
\rho(M(\alpha, \beta)) &= \rho((\beta P + T)^{-1}(\beta P + iW)(\alpha P + W)^{-1}(\alpha P - iT)) \\
&\leq \| (\beta P + T)^{-1}(\beta P + iW)(\alpha P + W)^{-1}(\alpha P - iT) \|_2 \\
&\leq \| (\beta P + iW)(\alpha P + W)^{-1} \|_2 \| (\alpha P - iT)(\beta P + T)^{-1} \|_2 \\
&= \| (\beta I + iP^{-1}W)(\alpha I + P^{-1}W)^{-1} \|_2 \| (\alpha I - iP^{-1}T)(\beta I + P^{-1}T)^{-1} \|_2 \\
&= \max_{\lambda_i \in \lambda(P^{-1}W)} \tfrac{\sqrt{\beta^2 + {\lambda_i}^2}}{\alpha + \lambda_i} \max_{u_i \in \lambda(P^{-1}T)} \tfrac{\sqrt{\alpha^2 + {u_i}^2}}{\beta + u_i},
\end{aligned}$$

then we obtain the upper bound of $\rho(M(\alpha, \beta))$ by (5).

Since W, P are symmetric positive definite and T is symmetric positive semidefinite, $\lambda_i > 0$, $u_i \geq 0$, and $\alpha > 0$ and $\beta > 0$, we obtain the following equality:

$$\begin{aligned}\max_{\lambda_i} \frac{\sqrt{\beta^2+{\lambda_i}^2}}{\alpha+\lambda_i} &= \max\left\{\frac{\sqrt{\beta^2+{\lambda_{min}}^2}}{\alpha+\lambda_{min}}, \frac{\sqrt{\beta^2+{\lambda_{max}}^2}}{\alpha+\lambda_{max}}\right\} \\ &= \begin{cases} \frac{\sqrt{\beta^2+{\lambda_{max}}^2}}{\alpha+\lambda_{max}}, & \beta \le \beta^*, \\ \frac{\sqrt{\beta^2+{\lambda_{min}}^2}}{\alpha+\lambda_{min}}, & \beta \ge \beta^*, \end{cases}\end{aligned}$$

where β^* is a function of λ_{max}, λ_{min} and α.

Case1: If $\beta > \alpha$, then $\max\limits_{u_i} \frac{\sqrt{\alpha^2+{u_i}^2}}{\beta+u_i} < 1$, thus,

$$\delta(\alpha,\beta) < \max_{\lambda_i} \frac{\sqrt{\beta^2+{\lambda_i}^2}}{\alpha+\lambda_i} = \begin{cases} \frac{\sqrt{\beta^2+{\lambda_{max}}^2}}{\alpha+\lambda_{max}}, & \beta \le \beta^*, \\ \frac{\sqrt{\beta^2+{\lambda_{min}}^2}}{\alpha+\lambda_{min}}, & \beta \ge \beta^*, \end{cases}$$

When $\beta \le \beta^*$, if $\frac{\sqrt{\beta^2+{\lambda_{max}}^2}}{\alpha+\lambda_{max}} \le 1$, then

$$\delta(\alpha,\beta) < 1$$

we can obtain the following inequality easily:

$$\alpha < \beta \le \beta^*. \tag{7}$$

And when $\beta \ge \beta^*$, by simple computation, we obtain

$$\beta^* < \beta \le \sqrt{\alpha(\alpha+2\lambda_{\min})}. \tag{8}$$

Combining (7) and (8), if

$$\alpha < \beta \le \sqrt{\alpha(\alpha+2\lambda_{\min})}, \tag{9}$$

we have $\delta(\alpha,\beta) < 1$.

Case2: If $\beta < \alpha$, then $\max\limits_{u_i} \frac{\sqrt{\alpha^2+{u_i}^2}}{\beta+u_i} \le \max\limits_{u_i} \frac{\sqrt{\alpha^2+{u_i}^2}}{\sqrt{\beta^2+{u_i}^2}} \le \frac{\alpha}{\beta}$, thus,

$$\delta(\alpha,\beta) \;\le\; \frac{\alpha}{\beta} \max_{\lambda_i} \frac{\sqrt{\beta^2+{\lambda_i}^2}}{\alpha+\lambda_i}.$$

In order to make the bound $\delta(\alpha,\beta) < 1$, the following inequality must hold

$$\max_{\lambda_i} \frac{\sqrt{\beta^2+{\lambda_i}^2}}{\alpha+\lambda_i} < \frac{\beta}{\alpha}.$$

Similarly, when $\beta \leq \beta^*$, we get

$$\alpha\sqrt{\frac{\lambda_{max}}{2\alpha+\lambda_{max}}} < \beta \leq \beta^*, \tag{10}$$

and when $\beta \geq \beta^*$, we obtain

$$\beta^* \leq \beta < \alpha. \tag{11}$$

Combining (10) and (11), if

$$\alpha\sqrt{\frac{\lambda_{max}}{2\alpha+\lambda_{max}}} < \beta < \alpha, \tag{12}$$

we have $\delta(\alpha, \beta) < 1$.

Case 3: If $\beta = \alpha$, and $P = I$, this is just the case in [13], then it is unconditionally convergent.

With the combination of Case 1, 2 and 3, the proof is completed. □

For any given positive α, Theorem 2.2 mainly discuss the available β for convergence of the GMHSS iteration. We can also notice that the choice of β is only dependent on the spectrum of the matrix $P^{-1}W$ and the choice of α. Since

$$\alpha(\alpha+2\lambda_{min}) - \frac{\alpha^2\lambda_{max}}{2\alpha+\lambda_{max}} = \alpha\frac{2\alpha^2+4\alpha\lambda_{min}+2\lambda_{min}\lambda_{max}}{2\alpha+\lambda_{max}} > 0,$$

that is to say the available β always exists for any given positive α. And if λ_{min} and α are large, η_{max} is small, then the area of available β is larger. An upper bound of the contraction factor of the GMHSS iteration is given by $\delta(\alpha, \beta)$, which is dependent on the spectrum of $P^{-1}W$ and $P^{-1}T$ and the parameters choice of α and β.

Corollary 2.3. Let A, W, T and P be the matrices defined in Theorem 2.2, and λ_{max} and λ_{min} are the upper and the lower bounds for the matrix $P^{-1}W$, respectively. Then for any given positive parameter α, the optimal β should be

$$\overline{\beta} = \sqrt{\frac{\alpha^2(\lambda_{\min}+\lambda_{\max})+2\alpha\lambda_{\min}\lambda_{\max}}{2\alpha+\lambda_{\min}+\lambda_{\max}}}. \tag{13}$$

Proof. In order to minimize the bound in (5), we can obtain the following equality:

$$\frac{\sqrt{\beta^2+{\lambda_{min}}^2}}{\alpha+\lambda_{min}} = \frac{\sqrt{\beta^2+{\lambda_{max}}^2}}{\alpha+\lambda_{max}}.$$

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By simple calculation, we get (13) and the proof is completed. □

Remark 2.4. Let $\overline{\delta}(\alpha,\beta)=\max\limits_{\lambda_i}\frac{\sqrt{\beta^2+\lambda_i{}^2}}{\alpha+\lambda_i}\max\left\{1,\frac{\alpha}{\beta}\right\}$, we can obtain its optimal parameter α when the first derivative of $\overline{\delta}(\alpha,\overline{\beta})$ is zero, here $\overline{\beta}$ is a function of α. It is difficult to find the explicit expression of α, but we can estimate the better values of α by numerical experiments.

3 The IGMHSS iteration method

In the GMHSS iteration, we need to solve two linear sub-systems whose coefficient matrices are $\alpha P+W$ and $\beta P+T$, respectively. This is very costly and impractical to inverse them exactly in actual implementations . In order to improve the GMHSS iteration more efficiently, we can employ inexact GMHSS(IGMHSS) iteration to solve the two subproblems. As supposed, $\alpha P+W$ and $\beta P+T$ are symmetric positive definite, we can employ conjugate gradient (CG) method to solve these two linear systems. The IGMHSS iteration scheme is as following.

1. k:=0;
2. $r^{(k)}=b-Ax^{(k)}$;
3. employing CG method approximately to solve $(\alpha P+W)y^{(k)}=r^{(k)}$;
4. $x^{(k+\frac{1}{2})}=x^{(k)}+y^{(k)}$;
5. $r^{(k+\frac{1}{2})}=-ib+iAx^{(k+\frac{1}{2})}$;
6. employing CG method approximately to solve $(\beta P+T)z^{(k)}=r^{(k+\frac{1}{2})}$;
7. $x^{(k+1)}=x^{(k+\frac{1}{2})}+z^{(k)}$;
8. Set $k=k+1$ and goto 2;
9. Set $x=x^{(k)}$ and output x.

In our implementations of IGMHSS iteration scheme, $b=Ae$, (e is $(1,1,\cdots 1)^T\in C^m$). All tests are started from the zero vector. The iteration is terminated once the current iterate x^k satisfies

$$\frac{\|b-Ax^k\|_2}{\|b\|_2}\leq 10^{-6},$$

and the stopping criteria for the inner CG is

$$\frac{\|r^{(k,l_k)}\|_2}{\|b-Ax^k\|_2}\leq 10^{-2},$$

where $r^{(k,l_k)}$ represents the residual of the l_kth inner iterate in the kth outer iterate, and parameter β is chosen to be $\overline{\beta}$ in (13).

4 Numerical experiments

In this section, we provide numerical experiments to illustrate the theoretical results obtained in Section 2 and the effectiveness of GMHSS iterations. All numerical experiments are carried out on a PC equipped with Intel Core i3 2.3 GHz CPU and 2.00 GB RAM memory Using MATLAB R2010a.

Example 4.1.[13] We consider the linear system of the form

$$[(K+\frac{3-\sqrt{3}}{\tau}I)+i(K-\frac{3+\sqrt{3}}{\tau}I)]x=b, \tag{14}$$

on the unit square $[0,1]\times[0,1]$ with the mesh-size $h=\frac{1}{m+1}$. τ is the time step-size and K is the five-point centered difference matrix approximating the negative Laplacian operator $L=-\Delta$ with homogeneous Dirichlet boundary conditions. The matrix $K\in R^{n\times n}$ possesses the tensor-product form $K=I_m\otimes V_m+V_m\otimes I_m$, where $\otimes$ denotes the Kronecker product, and V_m is tridiagonal matrix given by with $V_m=h^{-2}tridiag(-1,2,-1)\in R^{m\times m}$. Hence, K is an $n\times n$ blocktridiagonal matrix, with $n=m^2$. We take

$$W=K+\frac{3-\sqrt{3}}{\tau}I \qquad and \qquad T=K-\frac{3+\sqrt{3}}{\tau}I.$$

Different τ result in different A. In our tests, we take $\tau=h$. Furthermore, we normalize the linear system by multiplying both sides through by h^2. For more details, we refer to [15].

Example 4.2.[13] consider the linear system of the form

$$[(-\omega^2M+K)+i(\omega C_V+C_H)]x=b, \tag{15}$$

on the unit square $[0,1]\times[0,1]$ with the mesh-size $h=\frac{1}{m+1}$. M and K are the inertia and the stiffness matrices, C_V and C_H are the viscous and the hysteretic damping matrices, respectively, and ω is the driving circular frequency. We take $C_H=\mu K$ with a damping coefficient, $M=I$, $C_V=10I$, and K the five point centered difference matrix approximating the negative Laplacian operator with homogeneous Dirichlet boundary conditions. The matrix $K\in R^{n\times n}$ possesses the tensor-product form $K=I_m\otimes V_m+V_m\otimes I_m$, where $\otimes$ denotes the Kronecker product, and V_m is tridiagonal matrix given by with $V_m=h^{-2}tridiag(-1,2,-1)\in R^{m\times m}$. Hence, K is an $n\times n$ blocktridiagonal matrix, with $n=m^2$. We also normalize the system by multiplying both sides through by h^2 as before. In our tests, we set $\omega=\pi$, and $\mu=0.02$. We take

$$W=K-\omega^2I \qquad and \qquad T=\mu K+10\omega I.$$

In Fig.1 and 2, we plot the spectral radius of the iteration matrix of MHSS and GMHSS methods with different α and different matrix P for example 4.1 and 4.2 when $m = 16, 24$, respectively, *MHSS* represents the spectral radius of the iteration matrix of the MHSS method and *GMHSS* represents that of the iteration matrix of the GMSS method with corresponding matrix P.

From Fig.1 and 2, it is clear that if β is chosen to be the optimal one, the spectral radius of the iteration matrix of the GMHSS method is always smaller than that of the MHSS method. If $P = 0.1 \times I$, spectral radius of the GMHSS is much smaller than that of MHSS and GMHSS. It is obviously that different choices of matrix P can decrease spectral radius of the iteration matrix greatly. However, the matrix P of GMHSS method is not optimal one and only satisfies the condition of theorem 2.2.

In Fig.3 and 4, we dipict the number of iterations with different α and different matrix P for example 4.1 and 4.2 when $m = 16, 24$, respectively. From Fig.3 and 4, we find that the choices of matrix P have less impact on number of iterations. However, the number of GMHSS iterations is still much less than that of MHSS as α is increasing.

In [13], they showed that the upper bound $\delta(\alpha, \beta)$ is minimized when $\alpha = \sqrt{\lambda_{\min}\lambda_{\max}}$. In order to compare MHSS with GMHSS method conveniently, we just use this α and report the number of iterations (IT) and the elapsed CPU time (CPU) by MHSS and GMHSS methods for Example 4.1 and Example 4.2, which is listed in table 1 and table 2. Parameter β is still chosen to be $\overline{\beta}$ in (13). From table 1 and 2, we find that IT of GMHSS is just half of IT by MHSS when $P = I$. If $P \neq I$, IT of GMHSS is still less than that of MHSS. However, the CPU for different methods are almost the same.

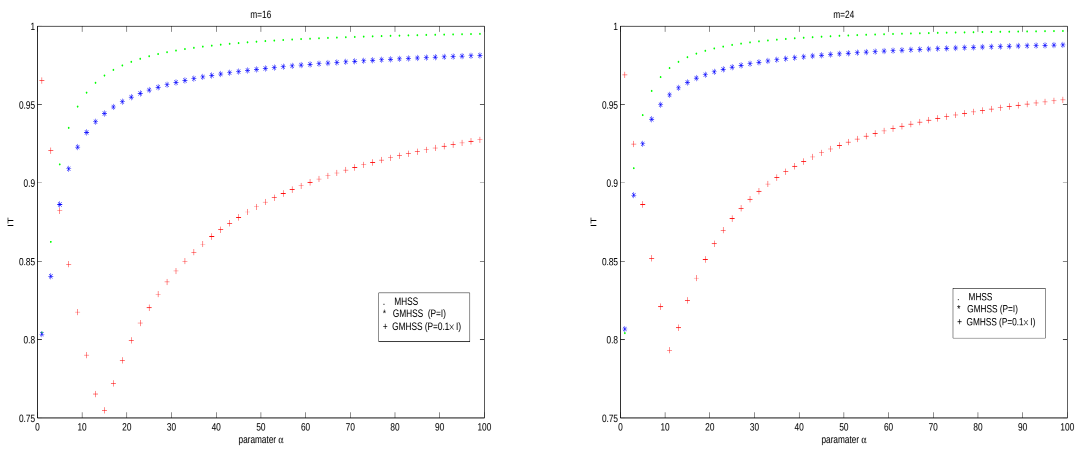

Figure 1: Spectral radius of iteration matrices of MHSS and GMHSS methods with different α for example 4.1.

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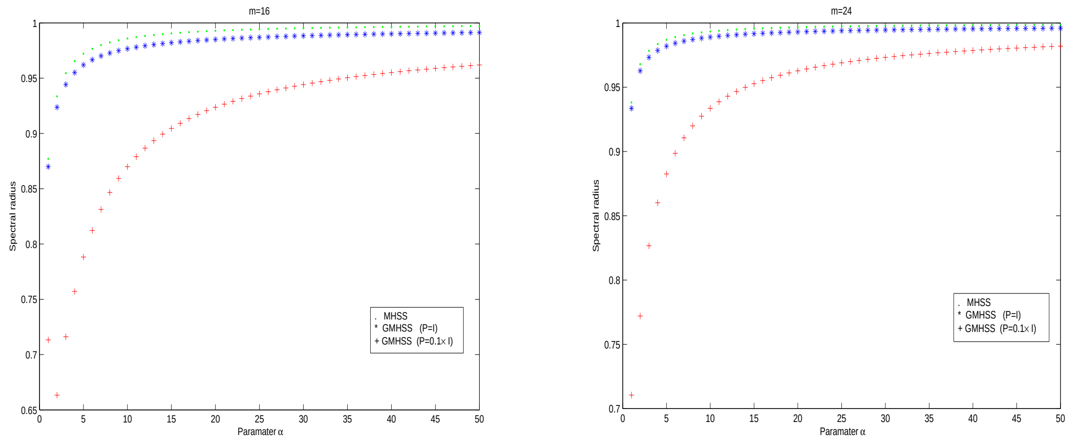

Figure 2: Spectral radius of iteration matrices of MHSS and GMHSS methods with different α for example 4.2.

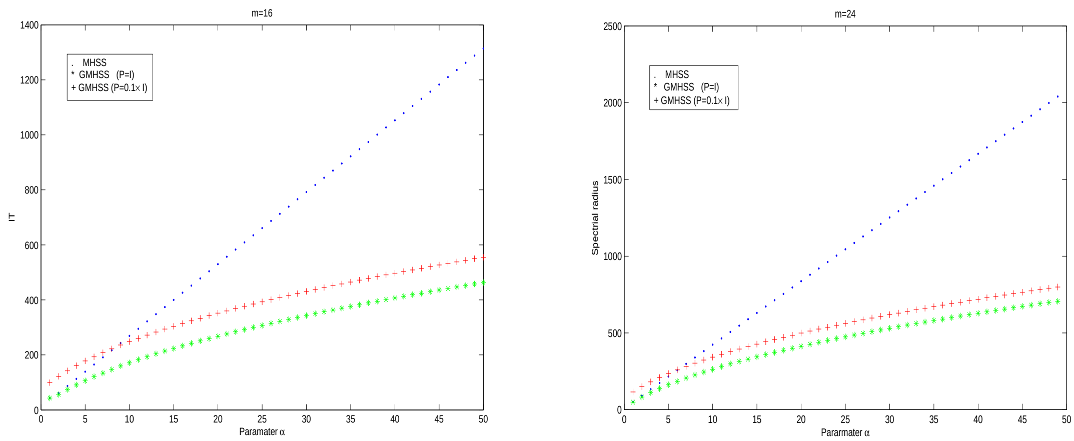

Figure 3: Number of iterations with different α for example 4.1.

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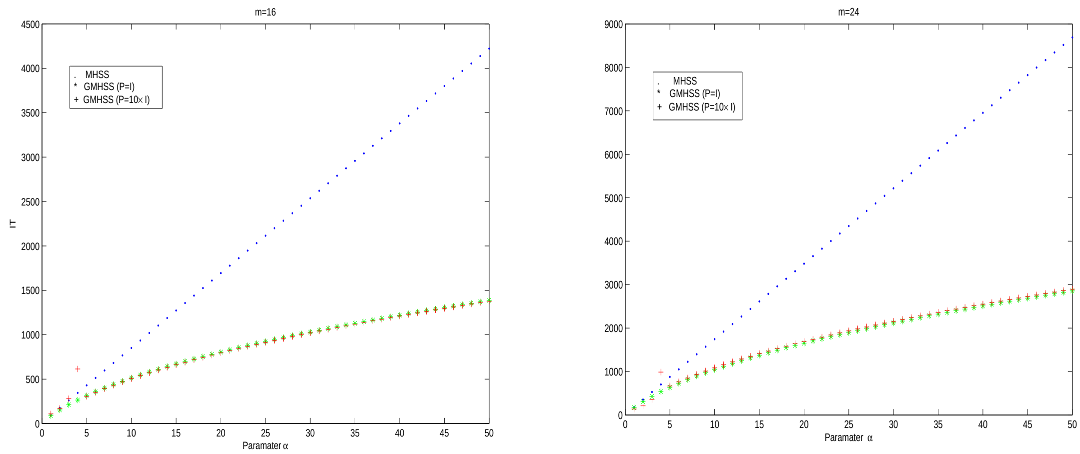

Figure 4: Number of iterations with different α for example 4.2.

Table 1
IT and CPU with MHSS and GMHSS methods for Example 4.1

		m=8	m=16	m=24	m=32
MHSS	IT	60	82	102	118
	CPU	0.0226	0.2405	2.1704	8.8142
GMHSS (P=I)	IT	30	41	51	59
	CPU	0.0210	0.2369	2.1905	8.6661
GMHSS ($P = 0.5 \times I$)	IT	40	48	54	59
	CPU	0.0210	0.2315	2.0479	8.7551

Table 2
IT and CPU with MHSS and GMHSS methods for Example 4.2

		m=8	m=16	m=24	m=32
MHSS	IT	70	106	140	172
	CPU	0.0213	0.2923	2.6583	10.5413
GMHSS (P=I)	IT	35	53	70	86
	CPU	0.0213	0.2838	2.6104	10.4714
GMHSS ($P = 10 \times I$)	IT	99	79	79	68
	CPU	0.0353	0.2889	1.8923	5.6304

5 Conclusions and remarks

In this paper, we present a generalized MHSS iterative method for augmented systems and demonstrate that this method converges to the unique solution of the linear system. When chosen the various parameters and matrix P, the spectral radii of the iteration matrices, the number of iterations (IT) and the elapsed CPU time with the proposed method are smaller than those in [13], which is shown through numerical experiments. Particularly, one may discuss how to shink the upper bound $\delta(\alpha, \beta)$ furtherly, and obtain a set of new optimal parameters in order to accelerate the convergence of the considered method quickly.

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AN ADDITIVE FUNCTIONAL INEQUALITY IN MATRIX NORMED MODULES OVER A C^*-ALGEBRA

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ABSTRACT. Using the direct method, we prove the Hyers-Ulam stability of an additive functional inequality in matrix normed modules over a C^*-algebra.

1. INTRODUCTION AND PRELIMINARIES

The abstract characterization given for linear spaces of bounded Hilbert space operators in terms of *matricially normed spaces* [32] implies that quotients, mapping spaces and various tensor products of operator spaces may again be regarded as operator spaces. Owing in part to this result, the theory of operator spaces is having an increasingly significant effect on operator algebra theory (see [6]).

The proof given in [32] appealed to the theory of ordered operator spaces [3]. Effros and Ruan [7] showed that one can give a purely metric proof of this important theorem by using a technique of Pisier [26] and Haagerup [17] (as modified in [5]).

The stability problem of functional equations originated from a question of Ulam [34] concerning the stability of group homomorphisms.

The functional equation

$$f(x+y) = f(x) + f(y)$$

is called the *Cauchy additive functional equation*. In particular, every solution of the Cauchy additive functional equation is said to be an *additive mapping*. Hyers [18] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [28] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [14] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

In 1990, Rassias [29] during the 27^{th} International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. In 1991, Gajda [13] following the same approach as in Rassias [28], gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [13], as well as by Rassias and Šemrl [30] that one cannot prove a Rassias' type theorem when $p = 1$ (cf. the books of Czerwik [4], Hyers, Isac and Rassias [19]). The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1, 8, 9, 10, 11, 21, 22, 24, 27]).

In [15], Gilányi showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(xy^{-1})\| \leq \|f(xy)\| \tag{1.1}$$

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then f satisfies the Jordan-von Neumann functional equation

$$2f(x)+2f(y)=f(xy)+f(xy^{-1}).$$

See also [31]. Gilányi [16] and Fechner [12] proved the Hyers-Ulam stability of the functional inequality (1.1).

Park, Cho and Han [25] proved the Hyers-Ulam stability of the following functional inequalities

$$\begin{aligned}\|f(x)+f(y)+f(z)\| &\leq \left\|2f\left(\frac{x+y+z}{2}\right)\right\|,\\ \|f(x)+f(y)+f(z)\| &\leq \|f(x+y+z)\|, \\ \|f(x)+f(y)+2f(z)\| &\leq \left\|2f\left(\frac{x+y}{2}+z\right)\right\|.\end{aligned} \tag{1.2}$$

We will use the following notations:
$M_n(X)$ is the set of all $n\times n$-matrices in X;
$e_j\in M_{1,n}(\mathbb{C})$ is that j-th component is 1 and the other components are zero;
$E_{ij}\in M_n(\mathbb{C})$ is that (i,j)-component is 1 and the other components are zero;
$E_{ij}\otimes x\in M_n(X)$ is that (i,j)-component is x and the other components are zero;
For $x\in M_n(X), y\in M_k(X)$,

$$x\oplus y=\begin{pmatrix} x & 0\\ 0 & y\end{pmatrix}.$$

Note that $(X,\{\|\cdot\|_n\})$ is a matrix normed space if and only if $(M_n(X),\|\cdot\|_n)$ is a normed space for each positive integer n and $\|AxB\|_k\leq\|A\|\|B\|\|x\|_n$ holds for $A\in M_{k,n}(\mathbb{C})$, $x=(x_{ij})\in M_n(X)$ and $B\in M_{n,k}(\mathbb{C})$, and that $(X,\{\|\cdot\|_n\})$ is a matrix Banach space if and only if X is a Banach space and $(X,\{\|\cdot\|_n\})$ is a matrix normed space.

Let E,F be vector spaces. For a given mapping $h:E\to F$ and a given positive integer n, define $h_n:M_n(E)\to M_n(F)$ by

$$h_n([x_{ij}])=[h(x_{ij})]$$

for all $[x_{ij}]\in M_n(E)$.

A matrix normed space $(X,\{\|\cdot\|_n\})$ is called an L^∞-*matrix normed space* if $\|x\oplus y\|_{n+k}=\max\{\|x\|_n,\|y\|_k\}$ holds for all $x\in M_n(X)$ and all $y\in M_k(X)$.

Throughout this paper, assume that A is a unital C^*-algebra with unitary group $U(A)$. Let $(X,\{\|\cdot\|_n\})$ be a matrix normed module over A and $(Y,\{\|\cdot\|_n\})$ a matrix Banach module over A.

2. Hyers-Ulam stability of the additive functional inequality (1.2) in matrix normed modules over a C^*-algebra

In this section, we prove the Hyers-Ulam stability of the additive functional inequality (1.2) in matrix normed modules over a C^*-algebra by using the direct method.

Lemma 2.1. *Let $(X,\{\|\cdot\|_n\})$ be a matrix normed space.*
(1) $\|E_{kl}\otimes x\|_n=\|x\|$ *for* $x\in X$.
(2) $\|x_{kl}\|\leq\|[x_{ij}]\|_n\leq\sum_{i,j=1}^n\|x_{ij}\|$ *for* $[x_{ij}]\in M_n(X)$.
(3) $\lim_{n\to\infty}x_n=x$ *if and only if* $\lim_{n\to\infty}x_{nij}=x_{ij}$ *for* $x_n=[x_{nij}],x=[x_{ij}]\in M_k(X)$.

Proof. (1) Since $E_{kl}\otimes x=e_k^*xe_l$ and $\|e_k^*\|=\|e_l\|=1$, $\|E_{kl}\otimes x\|_n\leq\|x\|$. Since $e_k(E_{kl}\otimes x)e_l^*=x$, $\|x\|\leq\|E_{kl}\otimes x\|_n$. So $\|E_{kl}\otimes x\|_n=\|x\|$.
(2) Since $e_kxe_l^*=x_{kl}$ and $\|e_k\|=\|e_l^*\|=1$, $\|x_{kl}\|\leq\|[x_{ij}]\|_n$.

Since $[x_{ij}] = \sum_{i,j=1}^{n} E_{ij} \otimes x_{ij}$,

$$\|[x_{ij}]\|_n = \left\| \sum_{i,j=1}^{n} E_{ij} \otimes x_{ij} \right\|_n \leq \sum_{i,j=1}^{n} \|E_{ij} \otimes x_{ij}\|_n = \sum_{i,j=1}^{n} \|x_{ij}\|.$$

(3) By (2), we have

$$\|x_{nkl} - x_{kl}\| \leq \|[x_{nij} - x_{ij}]\|_n = \|[x_{nij}] - [x_{ij}]\|_n \leq \sum_{i,j=1}^{n} \|x_{nij} - x_{ij}\|.$$

So we get the result. □

We need the following result.

Lemma 2.2. ([25, Proposition 2.2]) *Let $f : X \to Y$ be a mapping such that*

$$\|f(a) + f(b) + f(c)\| \leq \|f(a + b + c)\|$$

for all $a, b, c \in X$. Then $f : X \to Y$ is additive.

Theorem 2.3. *Let $f : X \to Y$ be a mapping and let $\phi : X^3 \to [0, \infty)$ be a function such that*

$$\Phi(a, b, c) := \frac{1}{2} \sum_{l=0}^{\infty} \frac{1}{2^l} \phi(2^l a, 2^l b, 2^l c) < +\infty, \tag{2.1}$$

$$\begin{aligned} \|uf_n([x_{ij}]) + uf_n([y_{ij}]) + f_n(u[z_{ij}])\|_n &\leq \|f_n([x_{ij}] + [y_{ij}] + [z_{ij}])\|_n \\ &+ \sum_{i,j=1}^{n} \phi(x_{ij}, y_{ij}, z_{ij}) \end{aligned} \tag{2.2}$$

for all $a, b, c \in X$, $u \in U(A)$ and all $x = [x_{ij}], y = [y_{ij}], z = [z_{ij}] \in M_n(X)$. Then there exists a unique A-linear mapping $L : X \to Y$ such that

$$\|f_n([x_{ij}]) - L_n([x_{ij}])\|_n \leq \sum_{i,j=1}^{n} \Phi(x_{ij}, x_{ij}, -2x_{ij}) \tag{2.3}$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. When $n = 1$, by putting $u = 1 \in U(A)$ in (2.2), we have

$$\|f(x) + f(y) + f(z)\| \leq \|f(x + y + z)\| + \phi(x, y, z)$$

for all $x, y, z \in X$. By the same reasoning as in the proof of [25, Theorem 3.2], one can show that there is a unique additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\| \leq \Phi(x, x, -2x)$$

for all $x \in X$. The mapping $L : X \to Y$ is given by

$$L(x) = \lim_{l \to \infty} \frac{1}{2^l} f(2^l x)$$

for all $x \in X$.

By the assumption $\lim_{l\to\infty} \frac{1}{2^l}\Phi(2^l x, 2^l y, 2^l z) = 0$ for all $x, y, z \in X$. So for each $u \in U(A)$, we get

$$\begin{aligned} \|uL(x) + uL(y) + L(uz)\| &= \lim_{l\to\infty} \frac{1}{2^l}\|uf(2^l x) + uf(2^l y) + f(u(2^l z))\| \\ &\le \lim_{l\to\infty} \frac{1}{2^l}\|f(2^l x + 2^l y + 2^l z)\| + \lim_{l\to\infty} \frac{1}{2^l}\Phi(2^l x, 2^l y, 2^l z) \\ &= \|L(x + y + z)\| \end{aligned} \tag{2.4}$$

for all $x, y, z \in X$.

Putting $y = 0, z = -x$ in (2.4), we get $\|uL(x) + L(-ux)\| \le 0$. Since L is additive,

$$L(ux) = uL(x) \tag{2.5}$$

for all $u \in U(A)$ and all $x \in X$.

Now let $a \in A (a \neq 0)$ and M an integer greater than $4|a|$. Then

$$|\frac{a}{M}| < \frac{1}{4} < \frac{1}{3} = 1 - \frac{2}{3}$$

By [23], there exist three elements $u_1, u_2, u_3 \in U(A)$ such that $3a/M = u_1 + u_2 + u_3$. So by (2.5)

$$\begin{aligned} L(ax) &= L(\frac{M}{3} \cdot 3\frac{a}{M}x) = M \cdot L(\frac{1}{3} \cdot 3\frac{a}{M}x) = \frac{M}{3}L(3\frac{a}{M}x) \\ &= \frac{M}{3}L(u_1x + u_2x + u_3x) = \frac{M}{3}(L(u_1x) + L(u_2x) + L(u_3x)) \\ &= \frac{M}{3}(u_1 + u_2 + u_3)L(x) = \frac{M}{3} \cdot 3\frac{a}{M}L(x) = aL(x) \end{aligned}$$

for all $a \in A$ and all $x \in X$. Hence

$$L(ax + by) = L(ax) + L(by) = aL(x) + bL(y)$$

for all $a, b \in A\,(a, b \neq 0)$ and all $x, y \in X$. And $L(0x) = 0 = 0L(x)$ for all $x \in X$. So the unique additive mapping $L : X \to Y$ is an A-linear mapping.

By Lemma 2.1,

$$\|f_n([x_{ij}]) - L_n([x_{ij}])\|_n \le \sum_{i,j=1}^{n} \|f(x_{ij}) - L(x_{ij})\| \le \sum_{i,j=1}^{n} \Phi(x_{ij}, x_{ij}, -2x_{ij})$$

for all $x = [x_{ij}] \in M_n(X)$. Thus $L : X \to Y$ is a unique A-linear mapping satisfying (2.3), as desired. □

Corollary 2.4. *Let r, θ be positive real numbers with $r < 1$. Let $f : X \to Y$ be a mapping such that*

$$\begin{aligned} \|uf_n([x_{ij}]) + uf_n([y_{ij}]) + f_n(u[z_{ij}])\|_n \quad &\le \quad \|f_n\left([x_{ij}] + [y_{ij}] + [z_{ij}]\right)\|_n \\ &+ \quad \sum_{i,j=1}^{n} \theta(\|x_{ij}\|^r + \|y_{ij}\|^r + \|z_{ij}\|^r) \end{aligned} \tag{2.6}$$

for all $x = [x_{ij}], y = [y_{ij}], z = [z_{ij}] \in M_n(X)$. Then there exists a unique A-linear mapping $L : X \to Y$ such that

$$\|f_n([x_{ij}]) - L_n([x_{ij}])\|_n \le \sum_{i,j=1}^{n} \frac{2 + 2^r}{2 - 2^r}\theta\|x_{ij}\|^r$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. Letting $\phi(a,b,c) = \theta(\|a\|^r + \|b\|^r + \|c\|^r)$ in Theorem 2.3, we obtain the result. □

Theorem 2.5. *Let $f : X \to Y$ be a mapping and let $\phi : X^3 \to [0,\infty)$ be a function satisfying* (2.2) *and*

$$\Phi(x,y,z) := \frac{1}{2}\sum_{l=1}^{\infty} 2^l \phi\left(\frac{x}{2^l}, \frac{y}{2^l}, \frac{z}{2^l}\right) < +\infty, \tag{2.7}$$

for all $x,y,z \in X$. Then there exists a unique A-linear mapping $L : X \to Y$ such that

$$\|f_n([x_{ij}]) - L_n([x_{ij}])\|_n \le \sum_{i,j=1}^{n} \Phi(x_{ij}, x_{ij}, -2x_{ij})$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. The proof is similar to the proof of Theorem 2.3. □

Corollary 2.6. *Let r,θ be positive real numbers with $r > 1$. Let $f : X \to Y$ be a mapping satisfying* (2.6)*. Then there exists a unique A-linear mapping $L : X \to Y$ such that*

$$\|f_n([x_{ij}]) - L_n([x_{ij}])\|_n \le \sum_{i,j=1}^{n} \frac{2^r+2}{2^r-2}\theta\|x_{ij}\|^r$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. Letting $\phi(x,y,z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ in Theorem 2.5, we obtain the result. □

We need the following result.

Lemma 2.7. ([33]) *If E is an L^∞-matrix normed space, then $\|[x_{ij}]\|_n \le \|[\|x_{ij}\|]\|_n$ for all $[x_{ij}] \in M_n(E)$.*

Theorem 2.8. *Let Y be an L^∞-normed Banach space. Let $f : X \to Y$ be a mapping and let $\phi : X^3 \to [0,\infty)$ be a function satisfying* (2.1) *and*

$$\begin{aligned} \|uf_n([x_{ij}]) + uf_n([y_{ij}]) + f_n(u[z_{ij}])\|_n &\le \|f_n([x_{ij}] + [y_{ij}] + [z_{ij}])\|_n \\ &+ \|[\phi(x_{ij}, y_{ij}, z_{ij})]\|_n \end{aligned} \tag{2.8}$$

for all $x = [x_{ij}], y = [y_{ij}], z = [z_{ij}] \in M_n(X)$. Then there exists a unique A-linear mapping $L : X \to Y$ such that

$$\|[f(x_{ij}) - L(x_{ij})]\|_n \le \|[\Phi(x_{ij}, x_{ij}, -2x_{ij})]\|_n \tag{2.9}$$

for all $x = [x_{ij}] \in M_n(X)$. Here Φ is given in Theorem 2.3.

Proof. By the same reasoning as in the proof of Theorem 2.3, there exists a unique A-linear mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\| \le \Phi(x, x, -2x)$$

for all $x \in X$. The mapping $L : X \to Y$ is given by

$$L(x) = \lim_{l\to\infty} \frac{1}{2^l} f(2^l x)$$

for all $x \in X$.

It is easy to show that if $0 \le a_{ij} \le b_{ij}$ for all i,j, then

$$\|[a_{ij}]\|_n \le \|[b_{ij}]\|_n. \tag{2.10}$$

By Lemma 2.7 and (2.10),

$$\|[f(x_{ij}) - L(x_{ij})]\|_n \le \|[\|f(x_{ij}) - L(x_{ij})\|]\|_n \le \|[\Phi(x_{ij}, x_{ij}, -2x_{ij})]\|_n$$

for all $x = [x_{ij}] \in M_n(X)$. So we obtain the inequality (2.9). ☐

Corollary 2.9. *Let Y be an L^∞-normed Banach space. Let r, θ be positive real numbers with $r < 1$. Let $f : X \to Y$ be a mapping such that*

$$\begin{aligned} \|uf_n([x_{ij}]) + uf_n([y_{ij}]) + f_n(u[z_{ij}])\|_n &\leq \|f_n([x_{ij}] + [y_{ij}] + [z_{ij}])\|_n \\ &+ \|[\theta(\|x_{ij}\|^r + \|y_{ij}\|^r + \|z_{ij}\|^r)]\|_n \end{aligned} \tag{2.11}$$

for all $x = [x_{ij}], y = [y_{ij}], z = [z_{ij}] \in M_n(X)$. Then there exists a unique A-linear mapping $L : X \to Y$ such that

$$\|f_n([x_{ij}]) - L_n([x_{ij}])\|_n \leq \left\| \left[\frac{2-2^r}{2-2^r} \theta \|x_{ij}\|^r \right] \right\|_n$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. Letting $\phi(x, y, z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ in Theorem 2.8, we obtain the result. ☐

Theorem 2.10. *Let Y be an L^∞-normed Banach space. Let $f : X \to Y$ be a mapping and let $\phi : X^3 \to [0, \infty)$ be a function satisfying (2.7) and (2.8). Then there exists a unique A-linear mapping $L : X \to Y$ such that*

$$\|[f(x_{ij}) - L(x_{ij})]\|_n \leq \|[\Phi(x_{ij}, x_{ij}, -2x_{ij})]\|_n$$

for all $x = [x_{ij}] \in M_n(X)$. Here Φ is given in Theorem 2.5.

Proof. The proof is similar to the proof of Theorem 2.8. ☐

Corollary 2.11. *Let Y be an L^∞-normed Banach space. Let r, θ be positive real numbers with $r > 1$. Let $f : X \to Y$ be a mapping satisfying (2.11). Then there exists a unique A-linear mapping $L : X \to Y$ such that*

$$\|f_n([x_{ij}]) - L_n([x_{ij}])\|_n \leq \left\| \left[\frac{2^r+2}{2^r-2} \theta \|x_{ij}\|^r \right] \right\|_n$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. Letting $\phi(x, y, z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ in Theorem 2.10, we obtain the result. ☐

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FUNCTIONAL EQUATIONS IN MATRIX NORMED MODULES

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Abstract. In this paper, we prove the Hyers-Ulam stability of the Cauchy additive functional equation in matrix normed modules over a C^*-algebra.

1. Introduction and preliminaries

The abstract characterization given for linear spaces of bounded Hilbert space operators in terms of *matricially normed spaces* [25] implies that quotients, mapping spaces and various tensor products of operator spaces may again be regarded as operator spaces. Owing in part to this result, the theory of operator spaces is having an increasingly significant effect on operator algebra theory (see [6]).

The proof given in [25] appealed to the theory of ordered operator spaces [3]. Effros and Ruan [7] showed that one can give a purely metric proof of this important theorem by using a technique of Pisier [21] and Haagerup [13] (as modified in [5]).

The stability problem of functional equations originated from a question of Ulam [29] concerning the stability of group homomorphisms.

The functional equation

$$f(x+y) = f(x) + f(y)$$

is called the *Cauchy additive functional equation*. In particular, every solution of the Cauchy additive functional equation is said to be an *additive mapping*. Hyers [14] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [22] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [12] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

In 1990, Rassias [23] during the 27^{th} International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. In 1991, Gajda [11] following the same approach as in Rassias [22], gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [11], as well as by Rassias and Šemrl [24] that one cannot prove a Rassias' type theorem when $p = 1$ (cf. the books of Czerwik [4], Hyers, Isac and Rassias [15]).

The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1, 8, 9, 10, 16, 17, 18, 20, 26, 27]).

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We will use the following notations:
$M_n(X)$ is the set of all $n \times n$-matrices in X;
$e_j \in M_{1,n}(\mathbb{C})$ is that j-th component is 1 and the other components are zero;
$E_{ij} \in M_n(\mathbb{C})$ is that (i,j)-component is 1 and the other components are zero;
$E_{ij} \otimes x \in M_n(X)$ is that (i,j)-component is x and the other components are zero;
For $x \in M_n(X), y \in M_k(X)$,

$$x \oplus y = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}.$$

Note that $(X, \{\|\cdot\|_n\})$ is a matrix normed space if and only if $(M_n(X), \|\cdot\|_n)$ is a normed space for each positive integer n and $\|AxB\|_k \leq \|A\|\|B\|\|x\|_n$ holds for $A \in M_{k,n}(\mathbb{C})$, $x = (x_{ij}) \in M_n(X)$ and $B \in M_{n,k}(\mathbb{C})$, and that $(X, \{\|\cdot\|_n\})$ is a matrix Banach space if and only if X is a Banach space and $(X, \{\|\cdot\|_n\})$ is a matrix normed space.

A matrix normed space $(X, \{\|\cdot\|_n\})$ is called an L^∞-*matrix normed space* if $\|x \oplus y\|_{n+k} = \max\{\|x\|_n, \|y\|_k\}$ holds for all $x \in M_n(X)$ and all $y \in M_k(X)$.

Let E, F be vector spaces. For a given mapping $h : E \to F$ and a given positive integer n, define $h_n : M_n(E) \to M_n(F)$ by

$$h_n([x_{ij}]) = [h(x_{ij})]$$

for all $[x_{ij}] \in M_n(E)$.

Throughout this paper, assume that A is a unital C^*-algebra with unitary group $U(A)$. Let $(X, \{\|\cdot\|_n\})$ be a matrix normed module over A and $(Y, \{\|\cdot\|_n\})$ a matrix Banach module over A.

2. Hyers-Ulam stability of the Cauchy additive functional equation in matrix normed modules over a C^*-algebra

In this section, we prove the Hyers-Ulam stability of the Cauchy additive functional equation in matrix normed modules over a C^*-algebra.

Lemma 2.1. *Let $(X, \{\|\cdot\|_n\})$ be a matrix normed space.*
(1) $\|E_{kl} \otimes x\|_n = \|x\|$ *for* $x \in X$.
(2) $\|x_{kl}\| \leq \|[x_{ij}]\|_n \leq \sum_{i,j=1}^n \|x_{ij}\|$ *for* $[x_{ij}] \in M_n(X)$.
(3) $\lim_{n\to\infty} x_n = x$ *if and only if* $\lim_{n\to\infty} x_{nij} = x_{ij}$ *for* $x_n = [x_{nij}], x = [x_{ij}] \in M_k(X)$.

Proof. (1) Since $E_{kl} \otimes x = e_k^* x e_l$ and $\|e_k^*\| = \|e_l\| = 1$, $\|E_{kl} \otimes x\|_n \leq \|x\|$. Since $e_k(E_{kl} \otimes x)e_l^* = x$, $\|x\| \leq \|E_{kl} \otimes x\|_n$. So $\|E_{kl} \otimes x\|_n = \|x\|$.

(2) Since $e_k x e_l^* = x_{kl}$ and $\|e_k\| = \|e_l^*\| = 1$, $\|x_{kl}\| \leq \|[x_{ij}]\|_n$.

Since $[x_{ij}] = \sum_{i,j=1}^n E_{ij} \otimes x_{ij}$,

$$\|[x_{ij}]\|_n = \left\| \sum_{i,j=1}^n E_{ij} \otimes x_{ij} \right\|_n \leq \sum_{i,j=1}^n \|E_{ij} \otimes x_{ij}\|_n = \sum_{i,j=1}^n \|x_{ij}\|.$$

(3) By (2), we have

$$\|x_{nkl} - x_{kl}\| \leq \|[x_{nij} - x_{ij}]\|_n = \|[x_{nij}] - [x_{ij}]\|_n \leq \sum_{i,j=1}^n \|x_{nij} - x_{ij}\|.$$

So we get the result. □

For a mapping $f : X \to Y$, define $D_u f : X^2 \to Y$ and $D_u f_n : M_n(X^2) \to M_n(Y)$ by

$$D_u f(a, b) = f(u(a + b)) - uf(a) - uf(b),$$

$$D_u f_n([x_{ij}], [y_{ij}]) := f_n(u[x_{ij} + y_{ij}]) - uf_n([x_{ij}]) - uf_n([y_{ij}])$$

for all $a, b \in X, u \in U(A)$ and all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$.

Theorem 2.2. *Let $f : X \to Y$ be a mapping and let $\phi : X^2 \to [0, \infty)$ be a function such that*

$$\Phi(a, b) := \frac{1}{2} \sum_{l=0}^{\infty} \frac{1}{2^l} \phi(2^l a, 2^l b) < +\infty, \tag{2.1}$$

$$\|D_u f_n([x_{ij}], [y_{ij}])\|_n \leq \sum_{i,j=1}^{n} \phi(x_{ij}, y_{ij}) \tag{2.2}$$

for all $a, b \in X$, $u \in U(A)$ and all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique A-linear mapping $L : X \to Y$ such that

$$\|f_n([x_{ij}]) - L_n([x_{ij}])\|_n \leq \sum_{i,j=1}^{n} \Phi(x_{ij}, x_{ij}) \tag{2.3}$$

for all $x = [x_{ij}] \in M_n(n)$.

Proof. Let $n = 1, u = 1 \in U(A)$ in (2.2). Then (2.2) is equivalent to

$$\|f(a + b) - f(a) - f(b)\| \leq \phi(a, b)$$

for all $a, b \in X$. By the same reasoning as in [12], there exists a unique additive mapping $L : X \to Y$ such that

$$\|f(a) - L(a)\| \leq \Phi(a, a)$$

for all $a \in X$. The mapping $L : X \to Y$ is given by

$$L(a) = \lim_{l\to\infty} \frac{1}{2^l} f(2^l a)$$

for all $a \in X$.

By the assumption,

$$\lim_{x\to\infty} \frac{1}{2^l} \phi(2^l x, 2^l y) = 0$$

holds for all $x, y \in X$. So for each $u \in U(A)$, we get

$$\begin{aligned}\|L(u(x + y)) - uL(x) - uL(y)\| &= \lim_{l\to\infty} \frac{1}{2^l} \|f(u(2^l(x + y))) - uf(2^l x) - uf(2^l y)\| \\ &= \lim_{l\to\infty} \frac{1}{2^l} \|D_u f(2^l x, 2^l y)\| \leq \lim_{l\to\infty} \frac{1}{2^l} \phi(2^l x, 2^l y) = 0.\end{aligned}$$

Hence

$$L(u(x + y)) - uL(x) - uL(y) = 0 \tag{2.4}$$

for all $u \in U(A)$ and all $x, y \in X$.

Putting $y = 0$ in (2.4), we get

$$L(ux) = uL(x) \tag{2.5}$$

for all $u \in U(A)$ and all $x \in X$.

Now let $a \in A(a \neq 0)$ and M an integer greater than $4|a|$. Then

$$|\frac{a}{M}| < \frac{1}{4} < \frac{1}{3} = 1 - \frac{2}{3}$$

By [19], there exist three elements $u_1, u_2, u_3 \in U(A)$ such that $3a/M = u_1 + u_2 + u_3$. So by (2.5)

$$\begin{aligned} L(ax) &= L(\frac{M}{3} \cdot 3\frac{a}{M}x) = M \cdot L(\frac{1}{3} \cdot 3\frac{a}{M}x) = \frac{M}{3}L(3\frac{a}{M}x) \\ &= \frac{M}{3}L(u_1x + u_2x + u_3x) = \frac{M}{3}(L(u_1x) + L(u_2x) + L(u_3x)) \\ &= \frac{M}{3}(u_1 + u_2 + u_3)L(x) = \frac{M}{3} \cdot 3\frac{a}{M}L(x) = aL(x) \end{aligned}$$

for all $a \in A$ and all $x \in X$. Hence

$$L(ax + by) = L(ax) + L(by) = aL(x) + bL(y)$$

for all $a, b \in A\,(a, b \neq 0)$ and all $x, y \in X$. And $L(0x) = 0 = 0L(x)$ for all $x \in X$. So the unique additive mapping $L : X \to Y$ is an A-linear mapping.

By Lemma 2.1,

$$\|f_n([x_{ij}]) - L_n([x_{ij}])\|_n \leq \sum_{i,j=1}^{n} \|f(x_{ij}) - L(x_{ij})\| \leq \sum_{i,j=1}^{n} \Phi(x_{ij}, x_{ij})$$

for all $x = [x_{ij}] \in M_n(X)$. Thus $L : X \to Y$ is a unique A-linear mapping satisfying (2.3), as desired. □

Corollary 2.3. *Let r, θ be positive real numbers with $r < 1$. Let $f : X \to Y$ be a mapping such that*

$$\|D_u f_n([x_{ij}], [y_{ij}])\|_n \leq \sum_{i,j=1}^{n} \theta(\|x_{ij}\|^r + \|y_{ij}\|^r) \tag{2.6}$$

for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique A-linear mapping $L : X \to Y$ such that

$$\|f_n([x_{ij}]) - L_n([x_{ij}])\|_n \leq \sum_{i,j=1}^{n} \frac{2\theta}{2 - 2^r}\|x_{ij}\|^r$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. Letting $\phi(a, b) = \theta(\|a\|^r + \|b\|^r)$ in Theorem 2.2, we obtain the result. □

Theorem 2.4. *Let $f : X \to Y$ be a mapping and let $\phi : X^2 \to [0, \infty)$ be a function satisfying* (2.2) *and*

$$\Phi(a, b) := \frac{1}{2}\sum_{l=1}^{\infty} 2^l \phi\left(\frac{a}{2^l}, \frac{b}{2^l}\right) < +\infty, \tag{2.7}$$

for all $a, b \in X$. Then there exists a unique A-linear mapping $L : X \to Y$ such that

$$\|f_n([x_{ij}]) - L_n([x_{ij}])\|_n \le \sum_{i,j=1}^{n} \Phi(x_{ij}, x_{ij})$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. The proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let r, θ be positive real numbers with $r > 1$. Let $f : X \to Y$ be a mapping satisfying* (2.6)*. Then there exists a unique A-linear mapping $L : X \to Y$ such that*

$$\|f_n([x_{ij}]) - L_n([x_{ij}])\|_n \le \sum_{i,j=1}^{n} \frac{2\theta}{2^r - 2}\|x_{ij}\|^r$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. Letting $\phi(a, b) = \theta(\|a\|^r + \|b\|^r)$ in Theorem 2.4, we obtain the result. □

We need the following result.

Lemma 2.6. ([28]) *If E is an L^∞-matrix normed space, then $\|[x_{ij}]\|_n \le \|[\|x_{ij}\|]\|_n$ for all $[x_{ij}] \in M_n(E)$.*

Theorem 2.7. *Let Y be an L^∞-normed Banach space. Let $f : X \to Y$ be a mapping and let $\phi : X^2 \to [0, \infty)$ be a function satisfying* (2.1) *and*

$$\|Df_n([x_{ij}], [y_{ij}])\|_n \le \|[\phi(x_{ij}, y_{ij})]\|_n \tag{2.8}$$

for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique A-linear mapping $L : X \to Y$ such that

$$\|[f(x_{ij}) - L(x_{ij})]\|_n \le \|[\Phi(x_{ij}, x_{ij})]\|_n \tag{2.9}$$

for all $x = [x_{ij}] \in M_n(X)$. Here Φ is given in Theorem 2.2.

Proof. By the same reasoning as in the proof of Theorem 2.2, there exists a unique additive mapping $L : X \to Y$ such that

$$\|f(a) - L(a)\| \le \Phi(a, a)$$

for all $a \in X$. The mapping $L : X \to Y$ is given by

$$L(a) = \lim_{l\to\infty} \frac{1}{2^l} f(2^l a)$$

for all $a \in X$.

It is easy to show that if $0 \le a_{ij} \le b_{ij}$ for all i, j, then

$$\|[a_{ij}]\|_n \le \|[b_{ij}]\|_n. \tag{2.10}$$

By Lemma 2.6 and (2.10),

$$\|[f(x_{ij}) - L(x_{ij})]\|_n \le \|[\|f(x_{ij}) - L(x_{ij})\|]\|_n \le \|[\Phi(x_{ij}, x_{ij})]\|_n$$

for all $x = [x_{ij}] \in M_n(X)$. So we obtain the inequality (2.9). □

Corollary 2.8. *Let Y be an L^∞-normed Banach space. Let r, θ be positive real numbers with $r < 1$. Let $f : X \to Y$ be a mapping such that*

$$\|Df_n([x_{ij}], [y_{ij}])\|_n \leq \|[\theta(\|x_{ij}\|^r + \|y_{ij}\|^r)]\|_n \tag{2.11}$$

for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists a unique A-linear mapping $L : X \to Y$ such that

$$\|f_n([x_{ij}]) - L_n([x_{ij}])\|_n \leq \left\| \left[\frac{2\theta}{2 - 2^r} \|x_{ij}\|^r \right] \right\|_n$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. Letting $\phi(a, b) = \theta(\|a\|^r + \|b\|^r)$ in Theorem 2.7, we obtain the result. □

Theorem 2.9. *Let Y be an L^∞-normed Banach space. Let $f : X \to Y$ be a mapping and let $\phi : X^2 \to [0, \infty)$ be a function satisfying (2.7) and (2.8). Then there exists a unique A-linear mapping $L : X \to Y$ such that*

$$\|[f(x_{ij}) - L(x_{ij})]\|_n \leq \|[\Phi(x_{ij}, x_{ij})]\|_n$$

for all $x = [x_{ij}] \in M_n(X)$. Here Φ is given in Theorem 2.4.

Proof. The proof is similar to the proof of Theorem 2.7. □

Corollary 2.10. *Let Y be an L^∞-normed Banach space. Let r, θ be positive real numbers with $r > 1$. Let $f : X \to Y$ be a mapping satisfying (2.11). Then there exists a unique A-linear mapping $L : X \to Y$ such that*

$$\|f_n([x_{ij}]) - L_n([x_{ij}])\|_n \leq \left\| \left[\frac{2\theta}{2^r - 2} \|x_{ij}\|^r \right] \right\|_n$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. Letting $\phi(a, b) = \theta(\|a\|^r + \|b\|^r)$ in Theorem 2.9, we obtain the result. □

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Existence results for fractional differential inclusions with multi–point and fractional integral boundary conditions

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Abstract

In this paper, some new existence results are obtained for fractional differential inclusions with a new class of multi-point and fractional integral boundary conditions by applying standard fixed point theorems for multivalued maps. The cases when the right-hand side has convex as well non-convex values are considered. Some illustrative examples are also presented.

Keywords: Fractional differential inclusions; nonlocal boundary conditions; fixed point theorems
2010 Mathematics Subject Classifications: 34A60, 34A08.

1 Introduction

The theory of fractional differential equations and inclusions has received much attention over the past years and become an important field of investigation due to its extensive applications in numerous branches of physics, economics and engineering sciences [1]-[4]. Fractional differential equations and inclusions are appropriate models for describing real world problems, which cannot be described using classical integer order differential equations. Some recent contributions to the subject can be seen in [5]-[18] and references cited therein.

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In this paper we discuss the existence of solutions for a boundary value problem of fractional differential inclusions of order $q \in (1, 2]$ with $m-$point and fractional Riemann-Liouville integral boundary conditions

$$D^q x(t) \in F(t, x(t)), \quad \text{a.e. } 0 < t < T, \ 1 < q \leq 2, \tag{1.1}$$

$$\sum_{i=1}^{m-2} \beta_i x(\eta_i) = 0, \quad I^p x(T) \equiv \int_0^T \frac{(T-s)^{p-1}}{\Gamma(p)} x(s)\, ds = 0, \tag{1.2}$$

where D^q denotes the Riemann-Liouville fractional derivative of order q, I^p is the Riemann-Liouville fractional integral of order $p > 0$, $F : [0, T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$, $\mathcal{P}(\mathbb{R})$ is the family of all subsets of $\mathbb{R}$, $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < T$ and $\beta_i, i = 1, 2, \ldots, m-2$ are real constants such that $\displaystyle\sum_{i=1}^{m-2} \beta_i \eta_i^{q-1} \neq \frac{(q-1)T}{(p+q-1)} \sum_{i=1}^{m-2} \beta_i \eta_i^{q-2}$.

A special case of this problem when $F = \{f\}$, $\beta_1 = 1, \beta_i = 0, i = 2, \ldots, m-2$ and $\eta_j = \eta$ for some $j \in \{1, \ldots, m-2\}$ was studied recently in [19], in which existence and uniqueness results are obtained by using Banach's and Krasnoselskii fixed point theorems and Leray-Schauder nonlinear alternative.

Integral boundary conditions have found useful applications in applied fields such as blood flow problems, chemical engineering, thermo-elasticity, underground water flow, population dynamics.

We establish new existence results for the problem (1.1)-(1.2), when the right hand side is convex as well as non-convex valued. The first result relies on the nonlinear alternative of Leray-Schauder type. In the second result, we shall combine the nonlinear alternative of Leray-Schauder type for single-valued maps with a selection theorem due to Bressan and Colombo for lower semicontinuous multivalued maps with nonempty closed and decomposable values, while in the third result, we shall use the fixed point theorem for contraction multivalued maps due to Covitz and Nadler. The methods used are well known, however their exposition in the framework of problem (1.1)-(1.2) is new.

The paper is organized as follows. In Section 2, we recall some preliminary facts that we need in the sequel and Section 3 deals with the main results. Some illustrative examples are presented in Section 4.

2 Preliminaries

Let us recall some basic definitions of fractional calculus [1, 3, 4].

Definition 2.1 *The Riemann-Liouville fractional integral of order $q > 0$ of a function $g \in L^1((0, T), \mathbb{R})$ is defined by*

$$I^q g(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} g(s) ds,$$

whre Γ *is the Gamma function.*

Definition 2.2 *The Riemann-Liouville fractional derivative of order* $q > 0$ *of a continuous function* $g : (0, \infty) \to \mathbb{R}$ *is defined by*

$$D^q g(t) = \frac{1}{\Gamma(n-q)} \Big(\frac{d}{dt}\Big)^n \int_0^t \frac{g(s)}{(t-s)^{q-n+1}} ds,$$

where $n = [q] + 1$, $[q]$ *denotes the integral part of real number* q.

Lemma 2.3 ([1]) *Let* $q > 0$ *and* $y \in C(0,T) \cap L(0,T)$. *Then fractional differential equation* $D^q y(t) = 0$ *has a unique solution*

$$y(t) = c_1 t^{q-1} + c_2 t^{q-2} + \cdots + c_n t^{q-n},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$ *and* $n = [q] + 1$.

Lemma 2.4 *Suppose that* $\sum_{i=1}^{m-2} \beta_i \eta_i^{q-1} \neq \frac{(q-1)T}{(p+q-1)} \sum_{i=1}^{m-2} \beta_i \eta_i^{q-2}$, $1 < q \le 2$, $p > 0$, $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < T$ *and* $h \in C[0,T]$. *Then the problem*

$$D^q x(t) = h(t), \quad 0 < t < T, \tag{2.1}$$

$$\sum_{i=1}^{m-2} \beta_i x(\eta_i) = 0, \quad I^p x(T) = 0, \tag{2.2}$$

has a unique solution

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} h(s) ds \\ &\quad - \frac{\tau_1 t^{q-2} - \tau_2 t^{q-1}}{(p+q-1)\Gamma(q-1) T^{p+q-2} \Phi} \int_0^T (T-s)^{p+q-1} h(s) ds \\ &\quad + \frac{(q-1)T t^{q-2} - (p+q-1) t^{q-1}}{(p+q-1)\Gamma(q)\Phi} \sum_{i=1}^{m-2} \beta_i \int_0^{\eta_i} (\eta_i - s)^{q-1} h(s) ds, \end{aligned} \tag{2.3}$$

where

$$\tau_1 = \sum_{i=1}^{m-2} \beta_i \eta_i^{q-1}, \quad \tau_2 = \sum_{i=1}^{m-2} \beta_i \eta_i^{q-2} \quad \text{and} \quad \Phi = \tau_1 - \frac{\tau_2 (q-1) T}{(p+q-1)}. \tag{2.4}$$

Proof. From (2.1) and Lemma 2.3, we have

$$x(t) = c_1 t^{q-1} + c_2 t^{q-2} + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} h(s) ds. \tag{2.5}$$

Substituting $t = \eta_i$, $i = 1, \dots, m-2$ in (2.5) with the first condition of (2.2), we get that

$$c_1\tau_1 + c_2\tau_2 = -\sum_{i=1}^{m-2}\beta_i\int_0^{\eta_i}\frac{(\eta_i - s)^{q-1}}{\Gamma(q)}h(s)ds, \tag{2.6}$$

where τ_1, τ_2 are defined by (2.4). Using the Riemann-Liouville fractional integral of order $p > 0$ for (2.5) and applying Dirichlet's formula [1, p.56], we obtain

$$\begin{aligned}
I^p x(t) &= \frac{c_1}{\Gamma(p)}\int_0^t (t-s)^{p-1}s^{q-1}ds + \frac{c_2}{\Gamma(p)}\int_0^t (t-s)^{p-1}s^{q-2}ds \\
&\quad + \frac{1}{\Gamma(p)\Gamma(q)}\int_0^t\int_0^s (t-s)^{p-1}(s-\rho)^{q-1}h(\rho)d\rho ds \\
&= c_1\frac{\Gamma(q)}{\Gamma(p+q)}t^{p+q-1} + c_2\frac{\Gamma(q-1)}{\Gamma(p+q-1)}t^{p+q-2} \\
&\quad + \frac{1}{\Gamma(p+q)}\int_0^t (t-s)^{p+q-1}h(s)ds.
\end{aligned}$$

The second condition of (2.2) implies

$$c_1\frac{\Gamma(q)}{\Gamma(p+q)}T^{p+q-1} + c_2\frac{\Gamma(q-1)}{\Gamma(p+q-1)}T^{q+p-2} = -\int_0^T\frac{(T-s)^{p+q-1}}{\Gamma(p+q)}h(s)ds. \tag{2.7}$$

Solving the linear equations (2.6)-(2.7) for unknown constants c_1 and c_2, we have that

$$\begin{aligned}
c_1 &= -\frac{1}{\Phi}\sum_{i=1}^{m-2}\beta_i\int_0^{\eta_i}\frac{(\eta_i - s)^{q-1}}{\Gamma(q)}h(s)ds \\
&\quad + \frac{\Gamma(p+q-1)\tau_2}{\Gamma(q-1)T^{p+q-2}\Phi}\int_0^T\frac{(T-s)^{p+q-1}}{\Gamma(p+q)}h(s)ds,
\end{aligned}$$

and

$$\begin{aligned}
c_2 &= \frac{(q-1)T}{(p+q-1)\Phi}\sum_{i=1}^{m-2}\beta_i\int_0^{\eta_i}\frac{(\eta_i - s)^{q-1}}{\Gamma(q)}h(s)ds \\
&\quad - \frac{\Gamma(p+q-1)\tau_1}{\Gamma(q-1)T^{p+q-2}\Phi}\int_0^T\frac{(T-s)^{p+q-1}}{\Gamma(p+q)}h(s)ds,
\end{aligned}$$

where Φ is defined by (2.4). Substituting constants c_1 and c_2 in (2.5), we obtain (2.3). □

Now we recall some basic concepts of multi-valued analysis ([20], [21]).

Let $C([0,T])$ denote a Banach space of continuous functions from $[0,T]$ into $\mathbb{R}$ with the norm $\|x\| = \sup_{t\in[0,T]}|x(t)|$. Let $L^1([0,T],\mathbb{R})$ be the Banach space of measurable functions $x : [0,T] \to \mathbb{R}$ which are Lebesgue integrable and normed by $\|x\|_{L^1} = \int_0^T |x(t)|dt$.

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For a normed space $(X, \|\cdot\|)$, let $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$, and $\mathcal{P}_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$.

A multi-valued map $G : X \to \mathcal{P}(X)$ is *convex (closed) valued* if $G(x)$ is convex (closed) for all $x \in X$; is *bounded* on bounded sets if $G(\mathbb{B}) = \cup_{x\in\mathbb{B}} G(x)$ is bounded in X for all $\mathbb{B} \in \mathcal{P}_b(X)$ (i.e. $\sup_{x\in\mathbb{B}}\{\sup\{|y| : y \in G(x)\}\} < \infty$); is called *upper semi-continuous (u.s.c.)* on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0) \subseteq N$; is said to be *completely continuous* if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in \mathcal{P}_b(X)$.

For each $y \in C([0,T],\mathbb{R})$, define the set of selections of F by

$$S_{F,y} := \{v \in L^1([0,T],\mathbb{R}) : v(t) \in F(t, y(t)) \text{ for a.e. } t \in [0,T]\}.$$

3 Existence results

Before studying the boundary value problem (1.1)-(1.2) let us begin by defining its solution.

Definition 3.1 *A function $x \in AC^1([0,T],\mathbb{R})$ is called a solution of problem (1.1)-(1.2) if there exists a function $v \in L^1([0,T],\mathbb{R})$ with $v(t) \in F(t,x(t))$, a.e. $[0,T]$ such that $D^q x(t) = v(t)$, a.e. $[0,T]$ and $\sum_{i=1}^{m-2} \beta_i x(\eta_i) = 0,\ I^p x(T) = 0$*

3.1 The Carathéodory case

Definition 3.2 *A multivalued map $F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if*

(i) $t \longmapsto F(t,x)$ is measurable for each $x \in \mathbb{R}$;

(ii) $x \longmapsto F(t,x)$ is upper semicontinuous for almost all $t \in [0,T]$;

Further a Carathéodory function F is called L^1-Carathéodory if

(iii) for each $\alpha > 0$, there exists $\varphi_\alpha \in L^1([0,T],\mathbb{R}^+)$ such that

$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \le \varphi_\alpha(t)$$

for all $\|x\| \le \alpha$ and for a. e. $t \in [0,T]$.

We recall the well-known nonlinear alternative of Leray-Schauder for multivalued maps and a useful result regarding closed graphs.

Lemma 3.3 *(Nonlinear alternative for Kakutani maps)[25]. Let E be a Banach space, C a closed convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to \mathcal{P}_{cp,c}(C)$ is a upper semicontinuous compact map.*

Then either

(i) F *has a fixed point in* $\overline{U}$, *or*

(ii) there is a $u \in \partial U$ *and* $\lambda \in (0,1)$ *with* $u \in \lambda F(u)$.

Lemma 3.4 *([22], [23]) Let* X *be a Banach space. Let* $F : [0,T] \times \mathbb{R} \to \mathcal{P}_{cp,c}(X)$ *be an* L^1- *Carathéodory multivalued map and let* Θ *be a linear continuous mapping from* $L^1([0,T],\mathbb{R})$ *to* $C([0,T],\mathbb{R})$. *Then the operator*

$$\Theta \circ S_F : C([0,T],\mathbb{R}) \to \mathcal{P}_{cp,c}(C([0,T],\mathbb{R})), \quad x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x})$$

is a closed graph operator in $C([0,T],\mathbb{R}) \times C([0,T],\mathbb{R})$.

Theorem 3.5 *Assume that:*

(H_1) $F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *is Carathéodory and has nonempty compact and convex values;*

(H_2) *there exists a continuous nondecreasing function* $\psi : [0,\infty) \to (0,\infty)$ *and a function* $p \in L^1([0,T],\mathbb{R}^+)$ *such that*

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \le p(t)\psi(\|x\|) \quad \text{for each} \quad (t,x) \in [0,T] \times \mathbb{R};$$

(H_3) *there exists a constant* $M > 0$ *such that*

$$\frac{M}{T^q \psi(M) \|p\|_{L^1}[\Lambda_1 + \Lambda_2 + \Lambda_3]} > 1,$$

where

$$\begin{aligned} \Lambda_1 &= \frac{1}{\Gamma(q+1)}, \\ \Lambda_2 &= \frac{|\tau_1| + T|\tau_2|}{(p+q-1)(p+q)\Gamma(q-1)|\Phi|}, \\ \Lambda_3 &= \frac{(p+2(q-1)) \sum_{i=1}^{m-2} |\beta_i| \eta_i^q}{(p+q-1)\Gamma(q+1)T|\Phi|}. \end{aligned} \tag{3.1}$$

Then the boundary value problem (1.1)-(1.2) has at least one solution on $[0,T]$.

Proof. Define the operator $\mathcal{H} : C([0,T],\mathbb{R}) \to \mathcal{P}(C([0,T],\mathbb{R}))$ by

$$\mathcal{H}(x) = \left\{ \begin{array}{l} h \in C([0,T],\mathbb{R}) : \\ h(t) = \left\{ \begin{array}{l} \displaystyle\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} h(s)ds \\ \displaystyle -\frac{\tau_1 t^{q-2} - \tau_2 t^{q-1}}{(p+q-1)\Gamma(q-1)T^{p+q-2}\Phi} \int_0^T (T-s)^{p+q-1} v(s)ds \\ \displaystyle +\frac{(q-1)Tt^{q-2} - (p+q-1)t^{q-1}}{(p+q-1)\Gamma(q)\Phi} \sum_{i=1}^{m-2} \beta_i \int_0^{\eta_i} (\eta_i - s)^{q-1} v(s)ds, \end{array} \right. \end{array} \right\}$$

for $v \in S_{F,x}$.

We will show that $\mathcal{H}$ satisfies the assumptions of the nonlinear alternative of Leray-Schauder type. The proof consists of several steps. As a first step, we show that $\mathcal{H}$ *is convex for each* $x \in C([0,T],\mathbb{R})$. This step is obvious since $S_{F,x}$ is convex (F has convex values), and therefore we omit the proof.
In the second step, we show that $\mathcal{H}$ *maps bounded sets (balls) into bounded sets in* $C([0,T],\mathbb{R})$. For a positive number ρ, let $B_\rho = \{x \in C([0,T],\mathbb{R}) : \|x\| \leq \rho\}$ be a bounded ball in $C([0,T],\mathbb{R})$. Then, for each $h \in \mathcal{H}(x), x \in B_\rho$, there exists $v \in S_{F,x}$ such that

$$\begin{aligned} h(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v(s)ds - \frac{\tau_1 t^{q-2} - \tau_2 t^{q-1}}{(p+q-1)\Gamma(q-1)T^{p+q-2}\Phi} \int_0^T (T-s)^{p+q-1} v(s)ds \\ &\quad + \frac{(q-1)Tt^{q-2} - (p+q-1)t^{q-1}}{(p+q-1)\Gamma(q)\Phi} \sum_{i=1}^{m-2} \beta_i \int_0^{\eta_i} (\eta_i - s)^{q-1} v(s)ds. \end{aligned}$$

Then for $t \in [0,T]$ we have

$$\begin{aligned} |h(t)| &\leq \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |v(s)|ds \\ &\quad + \frac{|\tau_1| + T|\tau_2|}{(p+q-1)\Gamma(q-1)T^p|\Phi|} \int_0^T (T-s)^{p+q-1} |v(s)|ds \\ &\quad + \frac{T^{q-1}(p+2(q-1))}{(p+q-1)\Gamma(q)|\Phi|} \sum_{i=1}^{m-2} |\beta_i| \int_0^{\eta_i} (\eta_i - s)^{q-1} |v(s)|ds \\ &\leq \frac{\psi(\|x\|)}{\Gamma(q)} \int_0^t (t-s)^{q-1} p(s)ds \\ &\quad + \frac{\psi(\|x\|)(|\tau_1| + T|\tau_2|)}{(p+q-1)\Gamma(q-1)T^p|\Phi|} \int_0^T (T-s)^{p+q-1} p(s)ds \\ &\quad + \frac{\psi(\|x\|)T^{q-1}(p+2(q-1))}{(p+q-1)\Gamma(q)|\Phi|} \sum_{i=1}^{m-2} |\beta_i| \int_0^{\eta_i} (\eta_i - s)^{q-1} p(s)ds \\ &\leq \frac{\psi(\|x\|)\|p\|_{L^1}}{\Gamma(q)} \int_0^T (T-s)^{q-1} ds \\ &\quad + \frac{\psi(\|x\|)\|p\|_{L^1}(|\tau_1| + T|\tau_2|)}{(p+q-1)\Gamma(q-1)T^p|\Phi|} \int_0^T (T-s)^{p+q-1} ds \\ &\quad + \frac{\psi(\|x\|)\|p\|_{L^1}T^{q-1}(p+2(q-1))}{(p+q-1)\Gamma(q)|\Phi|} \sum_{i=1}^{m-2} |\beta_i| \int_0^{\eta_i} (\eta_i - s)^{q-1} ds \\ &\leq \frac{T^q\psi(\|x\|)\|p\|_{L^1}}{\Gamma(q+1)} + \frac{\psi(\|x\|)\|p\|_{L^1}T^q(|\tau_1| + T|\tau_2|)}{(p+q-1)(p+q)\Gamma(q-1)|\Phi|} \\ &\quad + \frac{\psi(\|x\|)\|p\|_{L^1}T^{q-1}(p+2(q-1))\sum_{i=1}^{m-2} |\beta_i|\eta_i^q}{(p+q-1)\Gamma(q+1)|\Phi|} \end{aligned}$$

$$
\begin{aligned}
&= T^q\psi(\|x\|)\|p\|_{L^1}\left\{\frac{1}{\Gamma(q+1)}+\frac{|\tau_1|+T|\tau_2|}{(p+q-1)(p+q)\Gamma(q-1)|\Phi|}\right.\\
&\quad\left.+\frac{(p+2(q-1))\sum_{i=1}^{m-2}|\beta_i|\eta_i^q}{(p+q-1)\Gamma(q+1)T|\Phi|}\right\}.
\end{aligned}
$$

Consequently,

$$\|h\|\leq T^q\psi(\|\rho\|)\|p\|_{L^1}[\Lambda_1+\Lambda_2+\Lambda_3],$$

where $\Lambda_i, i=1,2,3$ are defined in (3.1).

Now we show that $\mathcal{H}$ *maps bounded sets into equicontinuous sets of* $C([0,T],\mathbb{R})$. Let $t_1,t_2\in[0,T]$ with $t_1<t_2$ and $x\in B_r$. For each $h\in\mathcal{H}(x)$, we obtain

$$
\begin{aligned}
&|h(t_2)-h(t_1)|\\
\leq&\ \left|\frac{1}{\Gamma(q)}\int_0^{t_2}(t_2-s)^{q-1}|v(s)|ds-\frac{1}{\Gamma(q)}\int_0^{t_1}(t_1-s)^{q-1}|v(s)|ds\right|\\
&+\frac{|\tau_1||t_2^{q-1}-t_1^{q-1}|+|\tau_2||t_2^{q-1}-t_1^{q-1}|}{(p+q-1)T^{p+q-2}\Gamma(q-1)|\Phi|}\int_0^T(T-s)^{p+q-1}|v(s)|ds\\
&+\frac{T(q-1)|t_2^{q-2}-t_1^{q-2}|+(p+q-1)|t_2^{q-1}-t_1^{q-1}|}{(p+q-1)\Gamma(q)|\Phi|}\int_0^{\eta_i}(\eta_i-s)^{q-1}|v(s)|ds\\
\leq&\ \left|\frac{1}{\Gamma(q)}\int_0^{t_2}(t_2-s)^{q-1}p(s)\psi(\rho)ds-\frac{1}{\Gamma(q)}\int_0^{t_1}(t_1-s)^{q-1}p(s)\psi(\rho)ds\right|\\
&+\frac{|\tau_1||t_2^{q-1}-t_1^{q-1}|+|\tau_2||t_2^{q-1}-t_1^{q-1}|}{(p+q-1)T^{p+q-2}\Gamma(q-1)|\Phi|}\int_0^T(T-s)^{p+q-1}p(s)\psi(\rho)ds\\
&+\frac{T(q-1)|t_2^{q-2}-t_1^{q-2}|+(p+q-1)|t_2^{q-1}-t_1^{q-1}|}{(p+q-1)\Gamma(q)|\Phi|}\int_0^{\eta_i}(\eta_i-s)^{q-1}p(s)\psi(\rho)ds.
\end{aligned}
$$

Obviously the right hand side of the above inequality tends to zero independently of $x\in B_\rho$ as $t_2-t_1\to 0$. As $\mathcal{H}$ satisfies the above three assumptions, therefore it follows by the Arzelá-Ascoli theorem that $\mathcal{H}: C([0,T],\mathbb{R})\to\mathcal{P}(C([0,T],\mathbb{R}))$ is completely continuous.

By [20, Proposition 1.2], since $\mathcal{H}$ is completely continuous, in order to prove that it is upper semi-continuous it is enough to prove that it has a closed graph. Thus, in our next step, we show that $\mathcal{H}$ *has a closed graph.* Let $x_n\to x_*, h_n\in\mathcal{H}(x_n)$ and $h_n\to h_*$. Then we need to show that $h_*\in\mathcal{H}(x_*)$. Associated with $h_n\in\mathcal{H}(x_n)$, there exists $v_n\in S_{F,x_n}$ such that for each $t\in[0,T]$,

$$
\begin{aligned}
h_n(t) &= \int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}v_n(s)ds-\frac{\tau_1t^{q-2}-\tau_2t^{q-1}}{(p+q-1)\Gamma(q-1)T^{p+q-2}\Phi}\int_0^T(T-s)^{p+q-1}v_n(s)ds\\
&\quad+\frac{(q-1)Tt^{q-2}-(p+q-1)t^{q-1}}{(p+q-1)\Gamma(q)\Phi}\sum_{i=1}^{m-2}\beta_i\int_0^{\eta_i}(\eta_i-s)^{q-1}v_n(s)ds.
\end{aligned}
$$

Thus it suffices to show that there exists $v_* \in S_{F,x_*}$ such that for each $t \in [0,T]$,

$$\begin{aligned} h_*(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v_*(s)ds - \frac{\tau_1 t^{q-2} - \tau_2 t^{q-1}}{(p+q-1)\Gamma(q-1)T^{p+q-2}\Phi} \int_0^T (T-s)^{p+q-1} v_*(s)ds \\ &+ \frac{(q-1)Tt^{q-2} - (p+q-1)t^{q-1}}{(p+q-1)\Gamma(q)\Phi} \sum_{i=1}^{m-2} \beta_i \int_0^{\eta_i} (\eta_i - s)^{q-1} v_*(s)ds. \end{aligned}$$

Let us consider the linear operator $\Theta : L^1([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$ given by

$$\begin{aligned} v \mapsto \Theta(v)(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v(s)ds \\ &- \frac{\tau_1 t^{q-2} - \tau_2 t^{q-1}}{(p+q-1)\Gamma(q-1)T^{p+q-2}\Phi} \int_0^T (T-s)^{p+q-1} h(s)ds \\ &+ \frac{(q-1)Tt^{q-2} - (p+q-1)t^{q-1}}{(p+q-1)\Gamma(q)\Phi} \sum_{i=1}^{m-2} \beta_i \int_0^{\eta_i} (\eta_i - s)^{q-1} h(s)ds. \end{aligned}$$

Observe that

$$\begin{aligned} &\|h_n(t) - h_*(t)\| \\ = &\left\| \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} (v_n(u) - v_*(u))ds \right. \\ &- \frac{\tau_1 t^{q-2} - \tau_2 t^{q-1}}{(p+q-1)\Gamma(q-1)T^{p+q-2}\Phi} \int_0^T (T-s)^{p+q-1} (v_n(u) - v_*(u))ds \\ &\left. + \frac{(q-1)Tt^{q-2} - (p+q-1)t^{q-1}}{(p+q-1)\Gamma(q)\Phi} \sum_{i=1}^{m-2} \beta_i \int_0^{\eta_i} (\eta_i - s)^{q-1} (v_n(u) - v_*(u))ds \right\| \to 0, \end{aligned}$$

as $n \to \infty$.

Thus, it follows by Lemma 3.4 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, therefore, we have

$$\begin{aligned} h_*(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v_*(s)ds - \frac{\tau_1 t^{q-2} - \tau_2 t^{q-1}}{(p+q-1)\Gamma(q-1)T^{p+q-2}\Phi} \int_0^T (T-s)^{p+q-1} v_*(s)ds \\ &+ \frac{(q-1)Tt^{q-2} - (p+q-1)t^{q-1}}{(p+q-1)\Gamma(q)\Phi} \sum_{i=1}^{m-2} \beta_i \int_0^{\eta_i} (\eta_i - s)^{q-1} v_*(s)ds, \end{aligned}$$

for some $v_* \in S_{F,x_*}$.

Finally, we show there exists an open set $U \subseteq C([0,T],\mathbb{R})$ with $x \notin \mathcal{H}(x)$ for any $\lambda \in (0,1)$ and all $x \in \partial U$. Let $\lambda \in (0,1)$ and $x \in \lambda\mathcal{H}(x)$. Then there exists $v \in L^1([0,T],\mathbb{R})$ with $v \in S_{F,x}$ such that, for $t \in [0,T]$, we have

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v(s)ds - \frac{\tau_1 t^{q-2} - \tau_2 t^{q-1}}{(p+q-1)\Gamma(q-1)T^{p+q-2}\Phi} \int_0^T (T-s)^{p+q-1} v(s)ds$$

$$+\frac{(q-1)Tt^{q-2}-(p+q-1)t^{q-1}}{(p+q-1)\Gamma(q)\Phi}\sum_{i=1}^{m-2}\beta_i\int_0^{\eta_i}(\eta_i-s)^{q-1}v(s)ds.$$

Repeating the computations of the second step, we have

$$\begin{aligned}|x(t)| \quad &\leq \quad T^q\psi(\|x\|)\|p\|_{L^1}\Bigg\{\frac{1}{\Gamma(q+1)}+\frac{|\tau_1|+T|\tau_2|}{(p+q-1)(p+q)\Gamma(q-1)|\Phi|}\\ &\quad +\frac{(p+2(q-1))\sum_{i=1}^{m-2}|\beta_i|\eta_i^q}{(p+q-1)\Gamma(q+1)T|\Phi|}\Bigg\}.\end{aligned}$$

Consequently, we have

$$\frac{\|x\|}{T^q\psi(\|x\|)\|p\|_{L^1}[\Lambda_1+\Lambda_2+\Lambda_3]}\leq 1.$$

In view of (H_3), there exists M such that $\|x\|\neq M$. Let us set

$$U=\{x\in C([0,T],\mathbb{R}):\|x\|<M\}.$$

Note that the operator $\mathcal{H}:\overline{U}\to\mathcal{P}(C([0,T],\mathbb{R}))$ is upper semicontinuous and completely continuous. From the choice of U, there is no $x\in\partial U$ such that $x\in\lambda\mathcal{H}(x)$ for some $\lambda\in(0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 3.3), we deduce that $\mathcal{H}$ has a fixed point $x\in\overline{U}$ which is a solution of the problem (1.1)-(1.2). This completes the proof. □

3.2 The lower semicontinuous case

As a next result, we study the case when F is not necessarily convex valued. Our strategy to deal with this problem is based on the nonlinear alternative of Leray Schauder type together with the selection theorem of Bressan and Colombo [27] for lower semicontinuous maps with decomposable values. We recall that a subset A of $[0,T]\times\mathbb{R}$ is $\mathcal{L}\otimes\mathcal{B}$ measurable if A belongs to the $\sigma-$algebra generated by all sets of the form $\mathcal{J}\times\mathcal{D}$, where $\mathcal{J}$ is Lebesgue measurable in $[0,T]$ and $\mathcal{D}$ is Borel measurable in $\mathbb{R}$. Also, a subset $\mathcal{A}$ of $L^1([0,T],\mathbb{R})$ is decomposable if for all $u,v\in\mathcal{A}$ and measurable $\mathcal{J}\subset[0,T]=J$, the function $u\chi_{\mathcal{J}}+v\chi_{J-\mathcal{J}}\in\mathcal{A}$, where $\chi_{\mathcal{J}}$ stands for the characteristic function of $\mathcal{J}$.

Lemma 3.6 *([23]) Let Y be a separable metric space and let $N:Y\to\mathcal{P}(L^1([0,T],\mathbb{R}))$ be a lower semi-continuous (l.s.c.) multivalued operator with nonempty closed and decomposable values. Then N has a continuous selection, that is, there exists a continuous function (single-valued) $g:Y\to L^1([0,T],\mathbb{R})$ such that $g(x)\in N(x)$ for every $x\in Y$.*

Theorem 3.7 *Assume that (H_2), (H_3) and the following condition holds:*

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(H_4) $F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *is a nonempty compact-valued multivalued map such that*

(a) $(t,x) \longmapsto F(t,x)$ *is* $\mathcal{L} \otimes \mathcal{B}$ *measurable,*

(b) $x \longmapsto F(t,x)$ *is lower semicontinuous for each* $t \in [0,T]$*;*

Then the boundary value problem (1.1)-(1.2) has at least one solution on $[0,T]$.

Proof. It follows from (H_2) and (H_4) that F is of l.s.c. type [23]. Then from Lemma 3.6, there exists a continuous function $f : AC^1([0,T],\mathbb{R}) \to L^1([0,T],\mathbb{R})$ such that $f(x) \in \mathcal{F}(x)$ for all $x \in C([0,T],\mathbb{R})$, where $\mathcal{F} : C([0,T] \times \mathbb{R}) \to \mathcal{P}(L^1([0,T],\mathbb{R}))$ is the Nemytskii operator associated with F, defined as

$$\mathcal{F}(x) = \{w \in L^1([0,T],\mathbb{R}) : w(t) \in F(t,x(t)) \text{ for a.e. } t \in [0,T]\}.$$

Consider the problem

$$\begin{cases} D^q x(t) = f(x(t)), \quad 0 < t < T, \\ \displaystyle\sum_{i=1}^{m-2} \beta_i x_i(\eta_i) = 0, \quad I^p x(T) = 0. \end{cases} \tag{3.2}$$

Observe that if $x \in AC^1([0,T],\mathbb{R})$ is a solution of (3.2), then x is a solution to the problem (1.1)-(1.2). In order to transform the problem (3.2) into a fixed point problem, we define the operator $\overline{\mathcal{H}}$ as

$$\begin{aligned} \overline{\mathcal{H}}x(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(x(s))ds \\ &\quad - \frac{\tau_1 t^{q-2} - \tau_2 t^{q-1}}{(p+q-1)\Gamma(q-1)T^{p+q-2}\Phi} \int_0^T (T-s)^{p+q-1} f(x(s))ds \\ &\quad + \frac{(q-1)Tt^{q-2} - (p+q-1)t^{q-1}}{(p+q-1)\Gamma(q)\Phi} \sum_{i=1}^{m-2} \beta_i \int_0^{\eta_i} (\eta_i - s)^{q-1} f(x(s))ds. \end{aligned}$$

It can easily be shown that $\overline{\mathcal{H}}$ is continuous and completely continuous. The remaining part of the proof is similar to that of Theorem 3.5. So we omit it. This completes the proof. □

3.3 The Lipschitz case

Now we prove the existence of solutions for the problem (1.1)-(1.2) with a nonconvex valued right hand side by applying a fixed point theorem for multivalued map due to Covitz and Nadler [26].

Definition 3.8 *A multivalued operator* $N : X \to \mathcal{P}_{cl}(X)$ *is called:*

(a) *$\gamma-$Lipschitz if and only if there exists $\gamma > 0$ such that*

$$H_d(N(x), N(y)) \leq \gamma d(x, y) \text{ for each } x, y \in X;$$

(b) *a contraction if and only if it is $\gamma-$Lipschitz with $\gamma < 1$.*

Lemma 3.9 *(Covitz and Nadler fixed point theorem) ([26]) Let (X, d) be a complete metric space. If $N : X \to \mathcal{P}_{cl}(X)$ is a contraction, then $FixN \neq \emptyset$.*

Theorem 3.10 *Assume that the following conditions hold:*

(H_5) $F : [0, T] \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ is such that $F(\cdot, x) : [0, T] \to \mathcal{P}_{cp}(\mathbb{R})$ is measurable for each $x \in \mathbb{R}$.

(H_6) $H_d(F(t, x), F(t, \bar{x})) \leq m(t)|x - \bar{x}|$ for almost all $t \in [0, T]$ and $x, \bar{x} \in \mathbb{R}$ with $m \in L^1([0, T], \mathbb{R}^+)$ and $d(0, F(t, 0)) \leq m(t)$ for almost all $t \in [0, T]$.

Then the boundary value problem (1.1)-(1.2) has at least one solution on $[0, T]$ if

$$T^q \|m\|_{L^1}[\Lambda_1 + \Lambda_2 + \Lambda_3] < 1,$$

($\Lambda_i, i = 1, 2, 3$ are defined in (3.1).)

Proof. Observe that the set $S_{F,x}$ is nonempty for each $x \in C([0, T], \mathbb{R})$ by the assumption (H_5), so F has a measurable selection (see Theorem III.6 [28]). Now we show that the operator $\mathcal{H} : C([0, T], \mathbb{R}) \to \mathcal{P}(C([0, T], \mathbb{R}))$ defined in the begining of proof of Theorem 3.5 satisfies the assumptions of Lemma 3.9. To show that $\mathcal{H}(x) \in \mathcal{P}_{cl}((C[0, T], \mathbb{R}))$ for each $x \in C([0, T], \mathbb{R})$, let $\{u_n\}_{n \geq 0} \in \mathcal{H}(x)$ be such that $u_n \to u$ $(n \to \infty)$ in $C([0, T], \mathbb{R})$. Then $u \in C([0, T], \mathbb{R})$ and there exists $v_n \in S_{F,x_n}$ such that, for each $t \in [0, T]$,

$$\begin{aligned} u_n(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v_n(s) ds - \frac{\tau_1 t^{q-2} - \tau_2 t^{q-1}}{(p+q-1)\Gamma(q-1)T^{p+q-2}\Phi} \int_0^T (T-s)^{p+q-1} v_n(s) ds \\ &\quad + \frac{(q-1)Tt^{q-2} - (p+q-1)t^{q-1}}{(p+q-1)\Gamma(q)\Phi} \sum_{i=1}^{m-2} \beta_i \int_0^{\eta_i} (\eta_i - s)^{q-1} v_n(s) ds. \end{aligned}$$

As F has compact values, we pass onto a subsequence (if necessary) to obtain that v_n converges to v in $L^1([0, T], \mathbb{R})$. Thus, $v \in S_{F,x}$ and for each $t \in [0, T]$, we have

$$\begin{aligned} v_n(t) \to v(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} h(s) ds \\ &\quad - \frac{\tau_1 t^{q-2} - \tau_2 t^{q-1}}{(p+q-1)\Gamma(q-1)T^{p+q-2}\Phi} \int_0^T (T-s)^{p+q-1} h(s) ds \end{aligned}$$

$$+\frac{(q-1)Tt^{q-2}-(p+q-1)t^{q-1}}{(p+q-1)\Gamma(q)\Phi}\sum_{i=1}^{m-2}\beta_i\int_0^{\eta_i}(\eta_i-s)^{q-1}h(s)ds.$$

Hence, $u\in\Omega(x)$.

Next we show that there exists $\delta<1$ such that

$$H_d(\mathcal{H}(x),\mathcal{H}(\bar{x}))\le\delta\|x-\bar{x}\| \ \text{ for each } \ x,\bar{x}\in AC^1([0,T],\mathbb{R}).$$

Let $x,\bar{x}\in AC^1([0,T],\mathbb{R})$ and $h_1\in\mathcal{H}(x)$. Then there exists $v_1(t)\in F(t,x(t))$ such that, for each $t\in[0,T]$,

$$\begin{aligned} h_1(t) &= \int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}v_1(s)ds-\frac{\tau_1t^{q-2}-\tau_2t^{q-1}}{(p+q-1)\Gamma(q-1)T^{p+q-2}\Phi}\int_0^T(T-s)^{p+q-1}v_1(s)ds \\ &\quad +\frac{(q-1)Tt^{q-2}-(p+q-1)t^{q-1}}{(p+q-1)\Gamma(q)\Phi}\sum_{i=1}^{m-2}\beta_i\int_0^{\eta_i}(\eta_i-s)^{q-1}v_1(s)ds.\end{aligned}$$

By (H_6), we have

$$H_d(F(t,x),F(t,\bar{x}))\le m(t)|x(t)-\bar{x}(t)|.$$

So, there exists $w(t)\in F(t,\bar{x}(t))$ such that

$$|v_1(t)-w(t)|\le m(t)|x(t)-\bar{x}(t)|,\quad t\in[0,T].$$

Define $U:[0,T]\to\mathcal{P}(\mathbb{R})$ by

$$U(t)=\{w\in\mathbb{R}:|v_1(t)-w(t)|\le m(t)|x(t)-\bar{x}(t)|\}.$$

Since the multivalued operator $U(t)\cap F(t,\bar{x}(t))$ is measurable (Proposition III.4 [28]), there exists a function $v_2(t)$ which is a measurable selection for U. So $v_2(t)\in F(t,\bar{x}(t))$ and for each $t\in[0,T]$, we have $|v_1(t)-v_2(t)|\le m(t)|x(t)-\bar{x}(t)|$.

For each $t\in[0,T]$, let us define

$$\begin{aligned} h_2(t) &= \int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}v_2(s)ds-\frac{\tau_1t^{q-2}-\tau_2t^{q-1}}{(p+q-1)\Gamma(q-1)T^{p+q-2}\Phi}\int_0^T(T-s)^{p+q-1}v_2(s)ds \\ &\quad +\frac{(q-1)Tt^{q-2}-(p+q-1)t^{q-1}}{(p+q-1)\Gamma(q)\Phi}\sum_{i=1}^{m-2}\beta_i\int_0^{\eta_i}(\eta_i-s)^{q-1}v_2(s)ds.\end{aligned}$$

Thus,

$$|h_1(t)-h_2(t)| \le \int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}|v_1(s)-v_2(s)|ds$$

$$
\begin{aligned}
&+\frac{|\tau_1|+T|\tau_2|}{(p+q-1)T^p\Gamma(q-1)|\Phi|}\int_0^T (T-s)^{p+q-1}|v_1(s)-v_2(s)|ds\\
&+\frac{T^{q-1}(p+2(q-1))}{(p+q-1)\Gamma(q)|\Phi|}\sum_{i=1}^{m-2}|\beta_i|\int_0^{\eta_i}(\eta_i-s)^{q-1}|v_1(s)-v_2(s)|ds\\
\leq\ & \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}m(s)\|x-\overline{x}\|ds\\
&+\frac{|\tau_1|+T|\tau_2|}{(p+q-1)T^p\Gamma(q-1)|\Phi|}\int_0^T (T-s)^{p+q-1}m(s)\|x-\overline{x}\|ds\\
&+\frac{T^{q-1}(p+2(q-1))}{(p+q-1)\Gamma(q)|\Phi|}\sum_{i=1}^{m-2}|\beta_i|\int_0^{\eta_i}(\eta_i-s)^{q-1}m(s)\|x-\overline{x}\|ds\\
\leq\ & T^q\|m\|_{L^1}\left\{\frac{1}{\Gamma(q+1)}+\frac{|\tau_1|+T|\tau_2|}{(p+q-1)(p+q)\Gamma(q-1)|\Phi|}\right.\\
&\left.+\frac{(p+2(q-1))\sum_{i=1}^{m-2}|\beta_i|\eta_i^q}{(p+q-1)\Gamma(q+1)T|\Phi|}\right\}\|x-\overline{x}\|.
\end{aligned}
$$

Hence,

$$\|h_1-h_2\|\leq T^q\|m\|_{L^1}[\Lambda_1+\Lambda_2+\Lambda_3].$$

Analogously, interchanging the roles of x and $\overline{x}$, we obtain

$$
\begin{aligned}
H_d(\mathcal{H}(x),\mathcal{H}(\bar{x})) &\leq \delta\|x-\bar{x}\|\\
&\leq T^q\|m\|_{L^1}[\Lambda_1+\Lambda_2+\Lambda_3]\|x-\overline{x}\|.
\end{aligned}
$$

Since $\mathcal{H}$ is a contraction, it follows by Lemma 3.9 that $\mathcal{H}$ has a fixed point x which is a solution of (1.1)-(1.2). This completes the proof. □

4 Examples

In this section, we illustrate our main results with the help of some examples. Let us consider the following boundary value problem of fractional differential inclusions with nonlocal and integral boundary conditions

$$
\begin{cases}
D^{3/2}x(t)\in F(t,x(t)), \qquad t\in\left(0,\dfrac{5}{2}\right),\\
\dfrac{5}{2}x\left(\dfrac{1}{2}\right)-\dfrac{4}{3}x(1)+\dfrac{3}{4}x\left(\dfrac{3}{2}\right)-\dfrac{2}{5}x(2)=0, \quad I^{3/4}x\left(\dfrac{5}{2}\right)=0.
\end{cases}
\tag{4.1}
$$

Note that (4.1) is a six-point fractional boundary value problem. Here we have $q=3/2$, $p=3/4$, $T=5/2$, $\eta_1=1/2$, $\eta_2=1$, $\eta_3=3/2$, $\eta_4=2$, $\beta_1=5/2$, $\beta_2=-4/3$, $\beta_3=3/4$,

$\beta_4 = -2/5$. We find that

$$
\begin{aligned}
\tau_1 &= \sum_{i=1}^{4} \beta_i \eta_i^{\frac{1}{2}} = \frac{102 - 80\sqrt{2} + 45\sqrt{3}}{60\sqrt{2}} \approx 0.7873068, \\
\tau_2 &= \sum_{i=1}^{4} \beta_i \eta_i^{-\frac{1}{2}} = \frac{276\sqrt{3} - 80\sqrt{6} + 90}{60\sqrt{6}} \approx 2.5317303, \\
|\Phi| &= \left| \tau_1 - \frac{\tau_2 (q-1) T}{(p+q-1)} \right| = \left| \frac{45 - 174\sqrt{3}}{60\sqrt{6}} \right| \approx 1.7444234 \neq 0, \\
\Lambda_1 &= \frac{1}{\Gamma(q+1)} = \frac{4}{3\sqrt{\pi}} \approx 0.7522528, \\
\Lambda_2 &= \frac{|\tau_1| + T|\tau_2|}{(p+q-1)(p+q)\Gamma(q-1)|\Phi|} = \frac{8(1584\sqrt{3} - 560\sqrt{6} + 720)}{45\sqrt{\pi}(174\sqrt{3} - 45)} \approx 0.8183803, \\
\Lambda_3 &= \frac{(p + 2(q-1)) \sum_{i=1}^{m-2} |\beta_i| \eta_i^q}{(p+q-1)\Gamma(q+1) T |\Phi|} = \frac{28(342\sqrt{3} + 160\sqrt{6} + 405)}{75\sqrt{\pi}(174\sqrt{3} - 45)} \approx 1.1413865.
\end{aligned}
$$

(a) Let $F : [0, 5/2] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued map given by

$$
x \to F(t,x) = \left[\frac{|x|}{|x| + \sin^2 x + 1} + t + 1,\ e^{-x^2} + \frac{4}{5} t^2 + 3 \right]. \tag{4.2}
$$

For $f \in F$, we have

$$
|f| \le \max \left(\frac{|x|}{|x| + \sin^2 x + 1} + t + 1,\ e^{-x^2} + \frac{4}{5} t^2 + 3 \right) \le 9, \quad x \in \mathbb{R}.
$$

Thus,

$$
\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| \,:\, y \in F(t,x)\} \le 9 = p(t)\psi(\|x\|), \quad x \in \mathbb{R},
$$

with $p(t) = 1$, $\psi(\|x\|) = 9$. Further, using the condition (H_3) we find that $M > 241.2044713$. Therefore, all the conditions of Theorem 3.5 are satisfied. So, problem (4.1) with $F(t,x)$ given by (4.2) has at least one solution on [0,5/2].

(b) If $F : [0, 5/2] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map given by

$$
x \to F(t,x) = \left[\frac{(t+1)x^2}{x^2+1},\ \frac{t|x|(\cos^2 x + 1)}{2(|x|+1)} \right]. \tag{4.3}
$$

For $f \in F$, we have

$$
|f| \le \max \left(\frac{(t+1)x^2}{x^2+1},\ \frac{t|x|(\cos^2 x + 1)}{2(|x|+1)} \right) \le t + 1, \quad x \in \mathbb{R}.
$$

Here $\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \leq (t+1) = p(t)\psi(\|x\|)$, $x \in \mathbb{R}$, with $p(t) = t+1$, $\psi(\|x\|) = 1$. It is easy to verify that $M > 60.3011178$. Then, by Theorem 3.5, the problem (4.1) with $F(t,x)$ given by (4.3) has at least one solution on $[0, 5/2]$.

(c) Consider the multivalued map $F : [0, 5/2] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ given by

$$x \to F(t,x) = \left[0,\ \frac{\sin x}{15(1+t)^2} + \frac{1}{200}\right]. \tag{4.4}$$

Then we have

$$\sup\{|u| : u \in F(t,x)\} \leq \frac{1}{15(1+t)^2} + \frac{1}{200},$$

and

$$H_d(F(t,x),\ F(t,\overline{x})) \leq \frac{1}{15(1+t)^2}|x - \overline{x}|.$$

Let $m(t) = \dfrac{1}{15(1+t)^2}$. Then $H_d(F(t,x),\ F(t,\overline{x})) \leq m(t)|x-\overline{x}|$, and $\|m\|_{L^1} = \dfrac{1}{21}$. We can show that $T^q\|m\|_{L^1}[\Lambda_1 + \Lambda_2 + \Lambda_3] \approx 0.5104857 < 1$.

By Theorem 3.10, the problem (4.1) with the $F(t,x)$ given by (4.4) has at least one solution on $[0, 5/2]$.

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On the stability of an additive functional inequality for the fixed point alternative

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Abstract

In this paper, we prove the generalized Hyers-Ulam stability of the additive functional inequality

$$\left\|f(3x+2y+z)+f(x+2y+2z)+f(2x+2y+3z)\right\| \le \left\|6f(x+y+z)\right\|$$

in Banach spaces.

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1. Introduction and preliminaries

In 1940, S.M. Ulam [8] suggested the stability problem of functional equations concerning the stability of group homomorphisms as follows: *Let* $(\mathcal{G}, \circ)$ *be a group and let* $(\mathcal{H}, \star, d)$ *be a metric group with the metric* $d(\cdot, \cdot)$. *Given* $\varepsilon > 0$, *does there*

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exist a $\delta = \delta(\varepsilon) > 0$ such that if a mapping $f : \mathcal{G} \to \mathcal{H}$ satisfies the inequality

$$d\big(f(x \circ y), f(x) \star f(y)\big) < \delta$$

for all $x, y \in \mathcal{G}$, then a homomorphism $F : \mathcal{G} \to \mathcal{H}$ exits with

$$d\big(f(x), F(x)\big) < \varepsilon$$

for all $x \in \mathcal{G}$?

In the next year, D.H. Hyers [4] gave a first (partial) affirmative answer to the question of Ulam for Banach spaces as follows: *If $\delta > 0$ and if $f : \mathcal{E} \to \mathcal{F}$ is a mapping between Banach spaces $\mathcal{E}$ and $\mathcal{F}$ satisfying*

$$\big\|f(x+y) - f(x) - f(y)\big\| \le \delta$$

for all $x, y \in \mathcal{E}$, then there is a unique additive mapping $A : \mathcal{E} \to \mathcal{F}$ such that

$$\big\|f(x) - A(x)\big\| \le \delta$$

for all $x \in \mathcal{E}$.

Thereafter, we call that type the Hyers-Ulam stability.

We will recall a fundamental result in fixed point theory for explicit later use.

Theorem 1.1. (The alternative of fixed point) [1, 7]
Let (S, d) be a complete generalized metric space and $\Lambda : S \to S$ a strictly contractive mapping with a Lipschitz constant L. Then, for each element $s \in S$, either

$$d(\Lambda^n s, \Lambda^{n+1} s) = \infty \text{ for all nonnegative integers } n$$

or there exists a positive integer n_0 such that

(a) $d(\Lambda^n s, \Lambda^{n+1} s) < \infty$ *for all* $n \ge n_0$;

(b) *the sequence*$(\Lambda^n s)$ *is convergent to a fixed point* t^* *of* Λ;

(c) t^* *is the unique fixed point of* Λ *in the set* $S^* = \{t \in S | d(\Lambda^{n_0} s, t) < \infty\}$;

(d) $d(t, t^*) \le \dfrac{1}{1-L} d(t, \Lambda t)$ *for all* $t \in S^*$.

In recent, several results [2, 3] on the Hyers-Ulam stability using fixed point theory was obtained.

2. Hyers-Ulam stability in Banach spaces

From now on, let $\mathcal{X}$ be a normed linear space and $\mathcal{Y}$ a Banach space. In 2007, C. Park, Y. S. Cho and M.-H. Han [6] proved the generalized Hyers-Ulam stability of the additive functional inequality

$$\big\|f(x) + f(y) + f(z)\big\| \le \big\|f(x+y+z)\big\|$$

in Banach spaces. In 2011, J. R. Lee, C. Park and D. Y. Shin [5] prove the generalized Hyers-Ulam stability of the additive functional inequality

$$\big\|f(2x) + f(2y) + 2f(z)\big\| \le \big\|2f(x+y+z)\big\|$$

in Banach spaces. In this paper, we prove the generalized Hyers-Ulam stability of the additive functional inequality

$$\left\|f(3x+2y+z)+f(x+2y+2z)+f(2x+2y+3z)\right\| \le \left\|6f(x+y+z)\right\|$$

in Banach spaces.

Lemma 2.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping with $f(0) = 0$. Then it is additive if and only if it satisfies*

$$\left\|f(3x+2y+z)+f(x+2y+2z)+f(2x+2y+3z)\right\| \le \left\|6f(x+y+z)\right\| \tag{1}$$

for all $x, y, z \in \mathcal{X}$.

Proof. If f is additive, then clearly

$$\left\|f(3x+2y+z)+f(x+2y+2z)+f(2x+2y+3z)\right\| = \left\|6f(x+y+z)\right\|$$

for all $x, y, z \in \mathcal{X}$.

Assume that f satisfies (1). Suppose $f(0) = 0$. Putting $z = 0$ and replacing y by $-x$ in (1), we get

$$\left\|f(x)+f(-x)\right\| \le \left\|6f(0)\right\| = 0$$

and so $f(-x) = -f(x)$ for all $x \in \mathcal{X}$. Replacing y by $-x-z$ in (1), we have

$$\left\|f(x-z)+f(-x)+f(z)\right\| \le 0$$

for all $x, z \in \mathcal{X}$. Replacing z by $-z$ in above inequality, then we obtain

$$f(x+z) = f(x)+f(z)$$

for all $x, z \in \mathcal{X}$. □

Theorem 2.2. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping with $f(0) = 0$. If there is a function $\varphi : X^3 \to [0, \infty)$ such that*

$$\left\|f(3x+2y+z)+f(x+2y+2z)+f(2x+2y+3z)\right\| \le \left\|6f(x+y+z)\right\|+\varphi(x,y,z) \tag{2}$$

and

$$\widetilde{\varphi}(x,y,z) := \sum_{j=0}^{\infty} \frac{1}{2^j}\varphi\big((-2)^j x, (-2)^j y, (-2)^j z\big) < \infty \tag{3}$$

for all $x, y, z \in \mathcal{X}$, then there exists a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\left\|f(x)-A(x)\right\| \le \frac{1}{2}\widetilde{\varphi}(-x,0,x) \tag{4}$$

for all $x \in \mathcal{X}$.

Proof. Replacing x, y, z by $-(-2)^n x, 0, (-2)^n x$, respectively, and dividing by 2^{n+1} in (2), since $f(0) = 0$, we get

$$\left\| \frac{f\big((-2)^{n+1}x\big)}{(-2)^{n+1}} - \frac{f\big((-2)^n x\big)}{(-2)^n} \right\| \le \frac{1}{2^{n+1}} \varphi\big(-(-2)^n x, 0, (-2)^n x\big)$$

for all $x \in \mathcal{X}$ and all nonnegative integers n. From the above inequality, we have

$$\begin{aligned} (5) \qquad \left\| \frac{f\big((-2)^n x\big)}{(-2)^n} - \frac{f\big((-2)^m x\big)}{(-2)^m} \right\| &\le \sum_{j=m}^{n-1} \left\| \frac{f\big((-2)^{j+1}x\big)}{(-2)^{j+1}} - \frac{f\big((-2)^j x\big)}{(-2)^j} \right\| \\ &\le \sum_{j=m}^{n-1} \frac{1}{2^{j+1}} \varphi\big(-(-2)^j x, 0, (-2)^j x\big) \end{aligned}$$

for all $x \in \mathcal{X}$ and all nonnegative integers m, n with $m < n$. By the condition (3), the sequence $\{\frac{f((-2)^n x)}{(-2)^n}\}$ is a Cauchy sequence for all $x \in \mathcal{X}$. Since $\mathcal{Y}$ is complete, the sequence $\{\frac{f((-2)^n x)}{(-2)^n}\}$ converges for all $x \in \mathcal{X}$. So one can define a mapping $A : \mathcal{X} \to \mathcal{Y}$ by

$$A(x) := \lim_{n\to\infty} \frac{f\big((-2)^n x\big)}{(-2)^n}$$

for all $x \in \mathcal{X}$. Taking $m = 0$ and letting n tend to ∞ in (5), we have the inequality (4).

Replacing x, y, z by $(-2)^n x, (-2)^n y, (-2)^n z$, respectively, and dividing by 2^n in (2), we obtain

$$\begin{aligned} &\left\| \frac{f\big((-2)^n(3x+2y+z)\big)}{(-2)^n} + \frac{f\big((-2)^n(x+2y+2z)\big)}{(-2)^n} + \frac{f\big((-2)^n(2x+2y+3z)\big)}{(-2)^n} \right\| \\ &\qquad \le \left\| \frac{6f\big((-2)^n(x+y+z)\big)}{(-2)^n} \right\| + \frac{1}{2^n}\varphi\big((-2)^n x, (-2)^n y, (-2)^n z\big) \end{aligned}$$

for all $x, y, z \in \mathcal{X}$ and all nonnegative integers n. Since (3) gives that

$$\lim_{n\to\infty} \frac{1}{2^n} \varphi\big((-2)^n x, (-2)^n y, (-2)^n z\big) = 0$$

for all $x, y, z \in \mathcal{X}$, letting n tend to ∞ in the above inequality, we see that A satisfies the inequality (1) and so it is additive by Lemma 2.1.

Let $A' : \mathcal{X} \to \mathcal{Y}$ be another additive mapping satisfying (4). Since both A and A' are additive, we have

$$\begin{aligned}\|A(x) - A'(x)\| &= \frac{1}{2^n}\|A\big((-2)^n x\big) - A'\big((-2)^n x\big)\| \\ &\le \frac{1}{2^n}\Big(\|A\big((-2)^n x\big) - f\big((-2)^n x\big)\| + \|f\big((-2)^n x\big) - A'\big((-2)^n x\big)\|\Big) \\ &\le \frac{1}{2^n}\widetilde{\varphi}\big(-(-2)^n x, 0, (-2)^n x\big) \\ &= \sum_{j=n}^{\infty}\frac{1}{2^j}\varphi\big(-(-2)^j x, 0, (-2)^j x\big)\end{aligned}$$

which goes to zero as $n \to \infty$ for all $x \in \mathcal{X}$ by (3). Therefore, A is a unique additive mapping satisfying (4), as desired. □

Corollary 2.3. *Let $\theta \in [0,\infty)$ and $p \in [0,1)$ and let $f : \mathcal{X} \to \mathcal{Y}$ be an odd mapping such that*

$$\begin{aligned}\|f(3x+2y+z) + f(x+2y+2z) + f(2x+2y+3z)\| \\ \le \|6f(x+y+z)\| + \theta\big(\|x\|^p + \|y\|^p + \|z\|^p\big)\end{aligned} \tag{6}$$

for all $x, y, z \in \mathcal{X}$. Then there exists a unique Cauchy additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - A(x)\| \le \frac{2\theta}{2-2^p}\|x\|^p$$

for all $x \in \mathcal{X}$.

Proof. In Theorem 2.2, take $\varphi(x,y,z) := \theta\big(\|x\|^p + \|y\|^p + \|z\|^p\big)$ for all $x, y, z \in \mathcal{X}$. Then we have the desired result. □

Theorem 2.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping with $f(0) = 0$. If there is a function $\varphi : X^3 \to [0,\infty)$ satisfying (2) such that*

$$\widetilde{\varphi}(x,y,z) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{(-2)^j}, \frac{y}{(-2)^j}, \frac{z}{(-2)^j}\right) < \infty \tag{7}$$

for all $x, y, z \in \mathcal{X}$, then there exists a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - A(x)\| \le \frac{1}{2}\widetilde{\varphi}(-x, 0, x) \tag{8}$$

for all $x \in \mathcal{X}$.

Proof. Replacing x, y, z by $0, \frac{-x}{(-2)^n}, \frac{x}{(-2)^n}$, respectively, and multiplying by 2^{n-1} in (2), since $f(0) = 0$, we have

$$\left\|(-2)^n f\left(\frac{x}{(-2)^n}\right) - (-2)^{n-1} f\left(\frac{x}{(-2)^{n-1}}\right)\right\| \le 2^{n-1}\varphi\left(\frac{-x}{(-2)^n}, 0, \frac{x}{(-2)^n}\right)$$

for all $x \in \mathcal{X}$ and all $n \in \mathbb{N}$. From the above inequality, we get

$$(9)\qquad \begin{aligned}&\left\|(-2)^n f\Big(\frac{x}{(-2)^n}\Big) - (-2)^m f\Big(\frac{x}{(-2)^m}\Big)\right\| \\ &\qquad \le \sum_{j=m+1}^{n} \left\|(-2)^j f\Big(\frac{x}{(-2)^j}\Big) - (-2)^{j-1} f\Big(\frac{x}{(-2)^{j-1}}\Big)\right\| \\ &\qquad \le \sum_{j=m+1}^{n} 2^{j-1}\varphi\Big(\frac{-x}{(-2)^j}, 0, \frac{x}{(-2)^j}\Big)\end{aligned}$$

for all $x \in \mathcal{X}$ and all nonnegative integers m, n with $m < n$. From (7), the sequence $\{(-2)^n f(\frac{x}{(-2)^n})\}$ is a Cauchy sequence for all $x \in \mathcal{X}$. Since $\mathcal{Y}$ is complete, the sequence $\{(-2)^n f(\frac{x}{(-2)^n})\}$ converges for all $x \in \mathcal{X}$. So one can define a mapping $A : \mathcal{X} \to \mathcal{Y}$ by

$$A(x) := \lim_{n\to\infty} (-2)^n f\Big(\frac{x}{(-2)^n}\Big)$$

for all $x \in \mathcal{X}$. To prove that A satisfies (8), putting $m = 0$ and letting $n \to \infty$ in (9), we have

$$\big\|f(x) - A(x)\big\| \le \sum_{j=1}^{\infty} 2^{j-1}\varphi\Big(\frac{-x}{(-2)^j}, 0, \frac{x}{(-2)^j}\Big) = \frac{1}{2}\widetilde{\varphi}(-x, 0, x)$$

for all $x \in \mathcal{X}$.

Replacing x, y, z by $\frac{x}{(-2)^n}, \frac{y}{(-2)^n}, \frac{z}{(-2)^n}$, respectively, and multiplying by 2^n in (2), we obtain

$$\begin{aligned}&\left\|(-2)^n f\Big(\frac{3x+2y+z}{(-2)^n}\Big) + (-2)^n f\Big(\frac{x+2y+2z}{(-2)^n}\Big) + (-2)^n f\Big(\frac{2x+2y+3z}{(-2)^n}\Big)\right\| \\ &\qquad \le \left\|(-2)^n 6 f\Big(\frac{x+y+z}{(-2)^n}\Big)\right\| + 2^n\varphi\Big(\frac{x}{(-2)^n}, \frac{y}{(-2)^n}, \frac{z}{(-2)^n}\Big)\end{aligned}$$

for all $x, y, z \in \mathcal{X}$ and all nonnegative integers n. Since (7) gives that

$$\lim_{n\to\infty} 2^n\varphi\Big(\frac{x}{(-2)^n}, \frac{y}{(-2)^n}, \frac{z}{(-2)^n}\Big) = 0$$

for all $x, y, z \in \mathcal{X}$, if we let $n \to \infty$ in the above inequality, then we have

$$\big\|A(3x+2y+z) + A(x+2y+2z) + A(2x+2y+3z)\big\| \le \big\|6A(x+y+z)\big\|$$

for all $x, y, z \in \mathcal{X}$. By Lemma 2.1, the mapping A is additive. The rest of the proof is similar to the corresponding part of the proof of Theorem 2.2. $\square$

Corollary 2.5. *Let $p > 1$ and θ be non-negative real numbers and let $f : \mathcal{X} \to \mathcal{Y}$ be an odd mapping satisfying* (6)*. Then there exists a unique Cauchy additive*

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mapping $A : \mathcal{X} \to \mathcal{Y}$ *such that*

$$\|f(x) - A(x)\| \le \frac{2\theta}{2^p - 2}\|x\|^p$$

for all $x \in \mathcal{X}$.

Proof. In Theorem 2.4, take $\varphi(x, y, z) := \theta\big(\|x\|^p+\|y\|^p+\|z\|^p\big)$ for all $x, y, z \in \mathcal{X}$. Then we have the desired result. $\square$

3. Hyers-Ulam stability using fixed point methods

Now, using fixed point methods, we investigate the generalized Hyers-Ulam stability of the functional inequality (1) in Banach spaces.

Theorem 3.1. *Let* $f : \mathcal{X} \to \mathcal{Y}$ *be an odd mapping and* $\phi : \mathcal{X}^3 \to [0, \infty)$ *a function such that*

$$\begin{aligned}(10)\qquad &\|f(3x + 2x + z) + f(x + 2y + 2z) + f(2x + 2y + 3z)\| \\ &\qquad \le \|6f(x + y + z)\| + \phi(x, y, z)\end{aligned}$$

for all $x, y, z \in \mathcal{X}$. *If there exists* $L \in (0, 1)$ *such that*

$$(11)\qquad \phi(x, y, z) \le 2L\phi\Big(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\Big)$$

for all $x, y, z \in \mathcal{X}$, *then there exists a unique Cauchy additive mapping* $A : \mathcal{X} \to \mathcal{Y}$ *satisfying*

$$(12)\qquad \|f(x) - A(x)\| \le \frac{1}{2 - 2L}\phi(-x, 0, x)$$

for all $x \in \mathcal{X}$.

Proof. Consider a set $S := \{g \,|\, g : \mathcal{X} \to \mathcal{Y}\}$ and introduce a generalized metric d on S as follows:

$$d(g, h) = d_\phi(g, h) := \inf S_\phi(g, h),$$

where

$$S_\phi(g, h) := \big\{C \in (0, \infty) : \|g(x) - h(x)\| \le C\phi(-x, 0, x) \text{ for all } x \in \mathcal{X}\big\}$$

for all $g, h \in S$. Now we show that (S, d) is complete. Let $\{h_n\}$ be a Cauchy sequence in (S, d). Then, for any $\varepsilon > 0$, there exists an integer $N_\varepsilon > 0$ such that $d(h_m, h_n) < \varepsilon$ for all $m, n \ge N_\varepsilon$. Since $d(h_m, h_n) = \inf S_\phi(h_m, h_n) < \varepsilon$, for all $m, n \ge N_\varepsilon$, there exists $C \in (0, \varepsilon)$ such that

$$(13)\qquad \|h_m(x) - h_n(x)\| \le C\phi(-x, 0, x) \le \varepsilon\phi(-x, 0, x)$$

for all $m, n \ge N_\varepsilon$ and all $x \in \mathcal{X}$. So $\{h_n(x)\}$ is a Cauchy sequence in $\mathcal{Y}$ for each $x \in \mathcal{X}$. Since $\mathcal{Y}$ is complete, $\{h_n(x)\}$ converges for each $x \in \mathcal{X}$. Thus a mapping $h : \mathcal{X} \to \mathcal{Y}$ can be defined by

$$h(x) := \lim_{n\to\infty} h_n(x)$$

for all $x \in \mathcal{X}$. Letting $n \to \infty$ in (13), we have

$$\begin{aligned} m \geq N_\varepsilon \Rightarrow & \left\| h_m(x) - h(x) \right\| \leq \varepsilon\phi(-x, 0, x) \quad \text{for all } x \in \mathcal{X} \\ \Rightarrow & \ \varepsilon \in S_\phi(h_m, h) \\ \Rightarrow & \ d(h_m, h) = \inf S_\phi(h_m, h) \leq \varepsilon. \end{aligned}$$

This means that the Cauchy sequence $\{h_n\}$ converges to h in (S, d). Hence (S, d) is complete.

Define a mapping $\Lambda : S \to S$ by

$$\Lambda h(x) := \frac{1}{2} h(2x)$$

for all $x \in \mathcal{X}$. We claim that Λ is strictly contractive on S. For any given $g, h \in S$, let $C_{gh} \in [0, \infty]$ be an arbitrary constant with $d(g, h) \leq C_{gh}$. Then, by (11),

$$\begin{aligned} & d(g, h) \leq C_{gh} \\ \Rightarrow \quad & \left\| g(x) - h(x) \right\| \leq C_{gh}\phi(-x, 0, x) \quad \text{for all } x \in \mathcal{X} \\ \Rightarrow \quad & \left\| \frac{1}{2} g(2x) - \frac{1}{2} h(2x) \right\| \leq \frac{1}{2} C_{gh}\phi(-2x, 0, 2x) \quad \text{for all } x \in \mathcal{X} \\ \Rightarrow \quad & \left\| \frac{1}{2} g(2x) - \frac{1}{2} h(2x) \right\| \leq L C_{gh}\phi(-x, 0, x) \quad \text{for all } x \in \mathcal{X}, \end{aligned}$$

that is, $d(\Lambda g, \Lambda h) \leq L C_{gh}$. Hence we see that $d(\Lambda g, \Lambda h) \leq L d(g, h)$ for any $g, h \in S$. Therefore, Λ is strictly contractive mapping on S with a Lipschitz constant $L \in (0, 1)$. Replacing x, y and z by $-x, 0$ and x in (10), respectively, we have

$$\left\| f(2x) - 2f(x) \right\| \leq \phi(-x, 0, x) \tag{14}$$

for all $x \in \mathcal{X}$. It follows from (11) and (14) that

$$\left\| f(x) - \frac{1}{2} f(2x) \right\| \leq \frac{1}{2}\phi(-x, 0, x) \tag{15}$$

for all $x \in \mathcal{X}$. Thus we obtain

$$d(f, \Lambda f) \leq \frac{1}{2}. \tag{16}$$

Therefore, it follows from Theorem 1.1 that the sequence $\{\Lambda^n f\}$ converges to a fixed point A of Λ, i.e.,

$$A : \mathcal{X} \to \mathcal{Y}, \quad A(x) = \lim_{n\to\infty} (\Lambda^n f)(x) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$$

and $A(2x) = 2A(x)$ for all $x \in \mathcal{X}$. Also, by Theorem 1.1 (c), there exists a positive integer n_0 such that A is the unique fixed point of Λ in the set

$$S^* = \{g \in S \mid d(\Lambda^{n_0} f, g) < \infty\}.$$

By (11) and (15), we see that

$$\begin{aligned}\left\|\Lambda^r f(x) - \Lambda^{r+1} f(x)\right\| &= \frac{1}{2^r}\left\| f(2^r x) - \frac{1}{2} f(2^{r+1}x)\right\| \\ &\le \frac{1}{2^{r+1}}\phi(-2^r x, 0, 2^r x) \le \frac{2L}{2^{r+1}}\phi(-2^{r-1}x, 0, 2^{r-1}x) \\ &\le \cdots \le \frac{(2L)^r}{2^{r+1}}\phi(-x,0,x) = \frac{L^r}{2}\phi(-x,0,x)\end{aligned}$$

for all $x \in \mathcal{X}$ and all $r \in \mathbb{N}$, that is, $d(\Lambda^r f, \Lambda^{r+1} f) \le \frac{L^r}{2} < \infty$ for all $r \in \mathbb{N}$. Since $f \in S$ and

$$d(f, \Lambda^{n_0} f) \le d(f, \Lambda f) + d(\Lambda f, \Lambda^2 f) + \cdots + d(\Lambda^{n_0-1} f, \Lambda^{n_0} f) < \infty,$$

we obtain $f \in S^*$. By Theorem 1.1 (d) and (16), we have

$$d(A, f) \le \frac{1}{1-L} d(\Lambda f, f) \le \frac{1}{2-2L},$$

i.e., the inequality (12) holds for all $x \in \mathcal{X}$. It follows from the definition of A and (10) that

$$\left\|A(3x + 2x + z) + A(x + 2y + 2z) + A(2x + 2y + 3z)\right\| \le \left\|6A(x + y + z)\right\|$$

for all $x, y, z \in \mathcal{X}$. By Lemma 2.1, the mapping $A : \mathcal{X} \to \mathcal{Y}$ is a Cauchy additive mapping. Therefore, there exists a unique Cauchy additive mapping $A : \mathcal{X} \to \mathcal{Y}$ satisfying (12). □

Let $\theta \in [0, \infty)$ and $p \in [0, 1)$ and let $f : \mathcal{X} \to \mathcal{Y}$ be an odd mapping such that

$$\begin{aligned}&\left\|f(3x + 2x + z) + f(x + 2y + 2z) + f(2x + 2y + 3z)\right\| \\ &\qquad\qquad \le \left\|6f(x + y + z)\right\| + \theta\big(\|x\|^p + \|y\|^p + \|z\|^p\big)\end{aligned}$$

for all $x, y, z \in \mathcal{X}$. In Theorem 3.1, take $\phi(x, y, z) := \theta\big(\|x\|^p + \|y\|^p + \|z\|^p\big)$ for all $x, y, z \in \mathcal{X}$ and choose $L = 2^{p-1}$, then we have the same result as Corollary 2.3.

Theorem 3.2. *Let $f : \mathcal{X} \to \mathcal{Y}$ be an odd mapping and $\phi : \mathcal{X}^3 \to [0, \infty)$ a function satisfying* (10). *If there exists $L \in (0, 1)$ such that*

$$\phi(x, y, z) \le \frac{L}{2}\phi(2x, 2y, 2z) \tag{17}$$

for all $x, y, z \in \mathcal{X}$. Then there exists a unique Cauchy additive mapping $A : \mathcal{X} \to \mathcal{Y}$ satisfying

$$\left\|f(x) - A(x)\right\| \le \frac{L}{2-2L}\phi(-x, 0, x)$$

for all $x \in \mathcal{X}$.

Proof. Consider the complete generalized metric space (S, d) given in the proof of Theorem 3.1. Now we consider the linear mapping $\Lambda : S \to S$ given by

$$\Lambda h(x) := 2h\Big(\frac{x}{2}\Big)$$

for all $x \in \mathcal{X}$. For any given $g, h \in S$, let $C_{gh} \in [0, \infty]$ be an arbitrary constant with $d(g,h) \le C_{gh}$. Then, by (17), we have

$$\begin{aligned} & d(g,h) \le C_{gh} \\ \Rightarrow\ & \|g(x) - h(x)\| \le C_{gh}\phi(-x,0,x) \ \text{ for all } x \in \mathcal{X} \\ \Rightarrow\ & \Big\|2g\Big(\frac{x}{2}\Big) - 2h\Big(\frac{x}{2}\Big)\Big\| \le 2C_{gh}\phi\Big(-\frac{x}{2},0,\frac{x}{2}\Big) \ \text{ for all } x \in \mathcal{X} \\ \Rightarrow\ & \Big\|2g\Big(\frac{x}{2}\Big) - 2h\Big(\frac{x}{2}\Big)\Big\| \le LC_{gh}\phi(-x,0,x) \ \text{ for all } x \in \mathcal{X}, \end{aligned}$$

that is, $d(\Lambda g, \Lambda h) \le LC_{gh}$. By the same arguments as the corresponding part of the proof of Theorem 3.1, we have the inequality (14). It follows from (14) and (17) that

$$\Big\|f(x) - 2f\Big(\frac{x}{2}\Big)\Big\| \le \phi\Big(-\frac{x}{2},0,\frac{x}{2}\Big) \le \frac{L}{2}\phi(-x,0,x)$$

for all $x \in \mathcal{X}$. Thus we obtain

$$d(f, \Lambda f) \le \frac{L}{2}.$$

The rest of the proof is similar to the corresponding part of the proof of Theorem 3.1. $\square$

Let $p > 1$ and θ be non-negative real numbers and let $f : \mathcal{X} \to \mathcal{Y}$ be an odd mapping such that

$$\begin{aligned} &\big\|f(3x + 2x + z) + f(x + 2y + 2z) + f(2x + 2y + 3z)\big\| \\ &\qquad \le \big\|6f(x + y + z)\big\| + \theta\big(\|x\|^p + \|y\|^p + \|z\|^p\big) \end{aligned}$$

for all $x, y, z \in \mathcal{X}$. In Theorem 3.2, take $\phi(x,y,z) := \theta\big(\|x\|^p + \|y\|^p + \|z\|^p\big)$ for all $x, y, z \in \mathcal{X}$ and choose $L = 2^{1-p}$, then we have the same result as Corollary 2.5.

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An improved algorithm for linear complementarity problems with interval data*

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Abstract

In this paper, the monotonicity of conjugate gradient method for the linear complementarity problem with S-matrix has been proved. With this property, we propose an algorithm for solving linear complementarity problem with interval data. As far as we know, the monotonicity of this algorithm has not been proved or stated explicit before. Numerical experiments are presented to show the monotonicity of the algorithm and illustrate the efficiency of the new algorithm.

Key words: Linear complementarity problem; Interval data; M-matrix; Monotonicity; Total step method with intersection; Single step method with intersection; Active index subset.

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1 Introduction

The linear complementarity problem (LCP) is of interest in a wide range of applications, such as free boundary problems [1], a Nash-equilibrium in bimatrix games [2], the interval hull of linear systems of interval equations [3], contact problems with friction [4], optimal stopping in Markov chains [5], circuit simulation [6], linear and quadratic programming [7] and economies with institutional restrictions upon prices [8].

In [9], they considered that the input data A and b were not precisely know, but can be enclosed in intervals. They proposed a total step method with intersection (TI) and a single step method with intersection (SI) for solving the linear complementarity problem with interval data. An important application of this problem is discretization of free boundary problem while neglecting the discretization error, for details in [10].

In this paper, in order to solve the linear complementarity problem with interval data $LCP([A],[b])$, we prove the monotonicity of the algorithm in [11]. The solution of

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$LCP([A],[b])$ can be found without discussing different cases of $[b]$. $LCP([A],[b])$ is simplified to compute the minimum solution $LCP([A],[b])$ which is the solution of $LCP([\bar{A}],[\underline{b}])$ and the maximum solution of $LCP([A],[b])$ which is the solution of $LCP([\underline{A}],[\bar{b}])$.

The reminder of this paper is organized as follows. In Section 2, we briefly review some important theorems, notations and algorithms of $LCP(A,b)$ and $LCP([A],[b])$. In Section 3, we propose a new algorithm for linear complementarity problem with interval data. Moreover, we prove the monotonicity of the original algorithm. Finally, several numerical experiments in Section 4 are presented to show the monotonicity of the original algorithm and illustrate the effectiveness of the new algorithm.

2 Notations and preparations

2.1 Definitions and lemmas

In this section, we introduce some definitions, lemmas and algorithms for $LCP(A,b)$ and $LCP([A],[b])$. If not stated here, they can be found in [9] or [11].

Defination 1 *Let $A \in \mathbb{R}^{n\times n}$ and $a_{ij} \le 0$ when $i \ne j$, then A is a Z-matrix.*

Defination 2 *Let $A \in \mathbb{Z}^{n\times n}$, A^{-1} exists and $A^{-1} \ge 0$, then A is an M-matrix.*

Defination 3 *Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ and $b = (b_i) \in \mathbb{R}^n$ be given. $LCP(A,b)$ is to find a vector $x \in \mathbb{R}^n$, such that*

$$Ax - b \ge 0, \quad x \ge 0, \quad x^{\mathrm{T}}(Ax - b) = 0.$$

Let x^* be the solution of $LCP(A,b)$, A be an M-matrix, then x^* is the unique solution of $LCP(A,b)$.

Suppose that index sets

$$J_0 = \{i|b_i > 0\}, \quad I_0 = N = \{1,2,\cdots,n\}, \quad J_0 = \{i|b_i \le 0\}.$$

Noting that $x = (x_1, x_2, \cdots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$. Let index set $I \subset N$, x_I is denoted as the subvector of x whose elements are $x_i, i \in I$. Let matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, index sets $I, J \subset N$, A_{IJ} is the submatrix of A whose elements are $a_{ij}, i \in I, j \in J$. If $I \subset N$ is a finite set, $|I|$ is the number of elements in I.

Clearly, if $J_0 = \emptyset$, $x^* = 0$ is the solution of $LCP(A,b)$ and has the following lemmas.

Lemma 1 *[11] The unique solution of $LCP(A,b)$ coincides with the unique solution of the following lower dimensional linear system*

$$\begin{cases} a_i^{\mathrm{T}}x - b_i = 0, & i \in J(x^*), \\ x_i = 0, & i \in I(x^*). \end{cases}$$

If index sets $I, J \subseteq N$ satisfies $I \cap J = \emptyset$ and $I \cup J = N$, the solution of the lower-dimensional linear system

$$\begin{cases} a_i^{\mathrm{T}}x - b_i = 0, & i \in J, \\ x_i = 0, & i \in I. \end{cases}$$

satisfies

$$x_J \ge 0, \quad A_{IJ}x_J - b_I \ge 0, \tag{1}$$

then x is the solution of $LCP(A,b)$.

Lemma 2 *[11] Let the index sets α, β satisfy $\alpha \subseteq J(x^*) = \{i \in N | x_i^* > 0\}$, $\alpha \cap \beta = N$ and*

$$\tilde{A_1} = \begin{pmatrix} E_{\alpha\alpha} & (A_{\alpha\alpha})^{-1}A_{\alpha\beta} \\ 0 & A_{\beta\beta} - A_{\beta\alpha}(A_{\alpha\alpha})^{-1}A_{\alpha\beta} \end{pmatrix},$$

$$\tilde{b_1} = \begin{pmatrix} (A_{\alpha\alpha})^{-1}b_\alpha \\ b_\beta - A_{\beta\alpha}(A_{\alpha\alpha})^{-1}b_\alpha \end{pmatrix},$$

where $E_{\alpha\alpha} \in R^{|\alpha|\times|\alpha|}$ is an identity matrix. Then the unique solution of $LCP(A, b)$ coincides with the unique solution of $LCP(\tilde{A_1}, \tilde{b_1})$.

Lemma 3 *[11] Let $J_0 = \{i | b^i > 0\}$, $J(x^*) = \{i | x_i^* > 0\}$, then it has*

$$J_0 \subseteq J(x^*).$$

Lemma 4 *[11] Let x^0 be the unique solution of $P(0)$ and (1) be not satisfied with $I = I_0$ and $J = J_0$. Definite the index set*

$$J_{1o} = \{i \in I_0 | a_i^{\mathrm{T}} x^0 - b_i < 0\}.$$

Then we have $J_{1o} \neq \emptyset$ and $J_{1o} \subset J([x^*])$.

Corollary 1 *[11] The solution of the lower-dimensional linear system*

$$A_{J_1J_1}X_{J_1} - b_{J_1} = 0$$

is positive.

In the following part, we introduce some notations, definitions and theorems in the linear complementarity problem with interval data which also can be found in [9].

Noting that $\mathbb{IR}, \mathbb{IR}^n, \mathbb{IR}^{n\times n}$ are the set of intervals, the set of interval vectors with n components, the set of $n \times n$ matrices with interval data, respectively. An interval always means a real compact interval. Interval vectors and interval matrices are vectors and matrices with interval entries, respectively. We write point intervals with brackets in which the element are contained. We use the notation $[a] = [\underline{a}, \overline{a}]$ for $[a] \in \mathbb{IR}$, $[x] = [\underline{x}, \overline{x}] = ([x_i]) = ([\underline{x}_i, \overline{x}_i]) \in \mathbb{IR}^n$, $i = 1, \cdots, n$ and $[A] = [\underline{A}, \overline{A}] = ([a_{ij}]) = ([\underline{a}_{ij}, \overline{a}_{ij}]) \in \mathbb{IR}^{n\times n}(i, j = 1, \cdots, n)$.

Defination 4 *[9] Let $[A] = ([a_{ij}]) \in \mathbb{IR}^{n\times n}$ and $[b] = ([b_i]) \in \mathbb{IR}^n$ be given. linear complementarity problem with interval data $LCP([A], [b])$ is to find a vector $[x] \in \mathbb{IR}^n$, such that*

$$[A][x] - [b] \geq 0, \quad [x] \geq 0, \quad [x]^{\mathrm{T}}([A][x] - [b]) = 0.$$

Defination 5 *[9] Internal matrix $[A] \in \mathbb{IR}^{n\times n}$ is called*

(1) Regular, if $\forall\ A \in [A]$ is nonsingular;

(2) An M-matrix, if $\forall\ A \in [A]$ is an M-matrix.

Lemma 5 *[9] Let $[A] \in \mathbb{IR}^{n\times n}$ be an M-matrix and $[b] \in \mathbb{IR}^n$, $[x] \in \mathbb{IR}^n$ be the solution of $[A][x] = [b]$, $u \in \mathbb{IR}^n$ be the solution of $\overline{A}x = \underline{b}$, and $v \in \mathbb{IR}^n$ be the solution of $\underline{A}x = \overline{b}$. Then it holds, $\inf([x]) = u$, and $\sup([x]) = v$.*

2.2 Active index subset for LCP

Let $A = (a_{i,j}) \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$ be given, the linear complementarity problem $LCP(A, b)$ is to find a vector $x \in \mathbb{R}^n$ such that

$$Ax - b \geq 0,\ x \geq 0,\ x^{\mathrm{T}}(Ax - b) = 0. \tag{2}$$

For converting LCP into lower dimension system of linear equations, we quote some notations in [11]. Let $N = \{1, 2, \cdots, n\}$. We define the index sets of x as

$$I(x) = \{i \in N | x_1 = 0\},\ \ J(x) = N \setminus I(x).$$

Suppose that x^* is the solution of $LCP(A, b)$, a_1^{T}, a_2^{T}, $\cdots$, a_n^{T} denote the rows of A. From Lemma 1, x^* is a solution of the following lower-dimensional linear system

$$\begin{cases} a_i^{\mathrm{T}}x - b_i = 0, & i \in J(x^*), \\ x_i = 0, & i \in I(x^*). \end{cases}$$

On the other hand, if $\tilde{x}$ is a solution of the linear system

$$\begin{cases} a_i^{\mathrm{T}}x - b_i = 0, & i \in J(x), \\ x_i = 0, & i \in I(x). \end{cases} \tag{3}$$

it has

$$\tilde{x}_i \geq 0, \quad \text{for} \quad \forall\, i \in J(\tilde{x}),$$

and

$$a_i^{\mathrm{T}}\tilde{x} - b_i \geq 0, \quad \text{for} \quad \forall\, i \in I(\tilde{x}),$$

where $\tilde{x}$ is a solution of the $LCP(A, b)$. Solving $LCP(A, b)$ is equivalent to find the solution of the linear equations (3).

Generally speaking, solving a linear system is easier than solving $LCP(A, b)$. With the same idea in [18], $LCP(A, b)$ can be locally converted into a linear system. The key point of this converting is the way of finding the index sets $I(x^*)$ and $J(x^*)$ depended on the solution x^*.

In [11], they constructed a finite sequence of linear systems for approaching the linear system (3). They gave a way to identify the index set $J(x^*)$ and constructed a finite sequence of index set $\{J_k\}_{k=0}^t$ satisfies

$$J_0 \subset J_1 \subset \cdots \subset J_t \subset J(x^*).$$

Evidently, they converted $LCP(A, b)$ into a linear system (3). The set J_k depends on the solution of the lower-dimension linear system

$$\begin{cases} a_i^{\mathrm{T}}x - b_i = 0, & i \in J_{k-1}, \\ x_i = 0, & i \in I_{k-1} \end{cases} \tag{$P(k-1)$}$$

for $\forall\, k \in \{1, 2, \cdots, t\}$. Then they got Algorithm 1 to solve $LCP(A, b)$.

Algorithm 1. ([11])

Step 1 Let $J_0 = \{i | b_i > 0\}$, $I_0 = N\ J_0$. If J_0 is empty, $I_0 = N = \{1, 2, \cdots, n\}$, $k = 0$.

Step 2 Solving the linear equations $A_{J_kJ_k}x_{J_k} - b_{J_k} = 0$, we obtain $x^*_{J_k}$.

Step 3 Let $x^* = \begin{pmatrix} x^*_{J_k} \\ 0_{I_k} \end{pmatrix}$, $I_k = N\backslash J_k$, if $A_{J_kJ_k}x^*_{J_k} -\ b_{J_k} \geq 0$, then x^* is the solution of $LCP(A, b)$. Otherwise, go to step 4.

Step 4 Let $J_{k_0} = \{i | \sum\limits_{i\in I_k} a_{ij}x^*_j - b_i < 0\}$, $J_{k+1} = J'_k \cup J_{k_0}$, $k = k + 1$, go to step 2.

3 Main results

3.1 An algorithm for LCP with interval data

Along the ideas in [11], we construct an algorithm to solve $LCP([A],[b])$.

Let $[A]=[\underline{A},\overline{A}]\in\mathbb{IR}^{n\times n}$, $[b]=[\underline{b},\overline{b}]\in\mathbb{IR}^{n}$ be interval matrix and interval vector. The linear complementarity problem with interval data $LCP([A],[b])$ is to find an interval vector $[x]\in\mathbb{IR}^{n}$ such that

$$[A][x]-[b]\geq 0,\quad [x]\geq 0,\quad [x]^{\mathrm{T}}([A][x]-[b])=0. \tag{4}$$

We define the index sets of $[x]\in\mathbb{IR}^{n}$ as

$$J_0^l=\{i|b_i>0\},\quad J_0'=\{i|\bar{b}_i>0\},\quad J_0''=\{i|\underline{b}_i>0\},$$

$$I_0^l=\{i|b_i\leq 0\},\quad I_0'=\{i|\bar{b}_i\leq 0\},\quad I_0''=\{i|\underline{b}_i\leq 0\}.$$

Let $[x^*]$ be the solution of $LCP([A],[b])$, and $[a_1]^{\mathrm{T}}$, $[a_2]^{\mathrm{T}}$, $\cdots$, $[a_n]^{\mathrm{T}}$ are the rows of $[A]$. To satisfy $[b_i]>0$ or $[b_i]\leq 0$, we depart $[b]$ into different cases as $[b]^1$, $[b]^2$, $\cdots$, $[b]^s$. Note that $[b]^l$, $l=\{1,2,\cdots,s\}$.

We give a method to identify the index set $J^l(x^*)$ of $LCP([A],[b]^l)$ by constructing a finite sequence of index sets $\{J_k\}_{k=0}^t$ satisfies

$$J_0^l\subset J_1^l\subset\cdots\subset J_t^l\subset J^l(x^*).$$

In fact, solving $LCP([A],[b]^l)$ is equivalent to find the solution $[x^*]$ of the following linear system with interval data

$$\begin{cases}[a]_i^{\mathrm{T}}x-[b]_i=0, & i\in J^l([x^*]),\\ [x_i]=0, & i\in I^l([x^*]).\end{cases}$$

If $[\tilde{x}]$ is a solution of the linear system

$$\begin{cases}[a_i]^{\mathrm{T}}[x]-[b_i]=0, & i\in J^l([x]),\\ [x_i]=0, & i\in I^l([x]),\end{cases} \tag{5}$$

with

$$[\tilde{x}_i]\geq 0,\quad \forall i\in J^l([\tilde{x}]),$$

and

$$[a_i^{\mathrm{T}}][\tilde{x}]-[b_i]\geq 0,\quad \forall i\in I^l([\tilde{x}]).$$

Then $[\tilde{x}]$ is a solution of the $LCP([A],[b]^l)$.

Therefore, $LCP([A],[b]^l)$ can be converted into the linear system (5). The set J_k^l depends on the solution of lower-dimension linear system

$$\begin{cases}[a_i^{\mathrm{T}}][x]-[b_i]=0, & i\in J_{k-1}^l,\\ [x_i]=0, & i\in I_{k-1}^l,\end{cases} \tag*{$Q(k-1)$}$$

for $\forall\ k\in\{1,2,\cdots,t\}$.

With the monotonicity of Algorithm 1 which has been proved in Section 3.3, we do not need to discuss the different cases of $[b]$. Then we get the following algorithm for $LCP([A],[b])$.

Algorithm 2.

Step 1 Let $J_0' = \{i|\bar{b}_i > 0\}$, if $J_0' = \emptyset$, go to step 2. $k = 0$.

Step 2 Solving the linear equation $\underline{A}_{J_k' J_k'} x_{J_k'} - \bar{b}_{J_k'} = 0$, we obtain $x^*_{J_k'}$.

Step 3 Let $x^* = \begin{pmatrix} x^*_{J_k'} \\ 0_{I_k'} \end{pmatrix}$, $I_k = N \backslash J_k'$, if $\underline{A}_{J_k' J_k'} x^*_{J_k'} - \bar{b}_{J_k'} \geq 0$, then x^* is the solution of $LCP(A,b)$. Go to step 5. Otherwise, go to step 4.

Step 4 Let $J_{k_0}' = \{i| \sum\limits_{i \in J_k'} \underline{A}_{ij} x_j^* - \bar{b}_i < 0\}$, $J_{k+1}' = J_k' \cup J_{k_0}'$, $k = k+1$, go to step 2.

Step 5 $x^* = \overline{x}$. Let $\underline{A} = \overline{A}$, $\bar{b} = \underline{b}$, go to step 1. We get x^* is $\underline{x}$. Then $[\underline{x}, \overline{x}]$ is the solution of the $LCP([A],[b])$.

3.2 Identification of positive variables with interval data

In this section we prove the solutions of the $LCP([A],[b])$ with our new algorithm are positive.

According to the condition of Lemma 1, it has to satisfy $[b_i]$, $i \in \{1, 2, \cdots, n\}$

$$[b_i] > 0, \quad \text{or} \quad [b_i] \leq 0.$$

When $\underline{b}_i \leq 0 \leq \bar{b}_i$, we have to depart $[b_i]$ into

$$[b_i]' = [\underline{b}_i, 0], \quad [b_i]'' = [0, \bar{b}_i].$$

Then $[b]$ is departed into different cases $[b]^1, [b]^2, \cdots, [b]^p$. If the number of i which satisfies $\underline{b}_i \leq 0 \leq \bar{b}_i$ is m, it has $p = 2^m$. Therefore, solving the $LCP([A],[b])$ is equivalent to find the solution of $LCP([A],[b]^l)$, $l \in \{1, 2, 3, \cdots, 2^m\}$. In the following part of this section, we will prove the solution of the $LCP([A],[b])$ with the method of Algorithm 1 is positive.

First, we prove the solution of $LCP([A],[b]^l)$ is positive.

Theorem 1 *If $J_0^l = \emptyset$, the solution of $LCP([A],[b]^l)$ is $[x] = 0$.*

Proof: Since $J_0^l = \emptyset$, it has $[b]^l \leq 0$. Suppose that $[x] \neq 0$, it follows that $0 < x \in [x]$, then $[A]x - [b]^l = 0$, $x \in [A]^{-1}\ [b]^l$. From $A \in [A]$ is an M-matrix, $[A]^{-1} \geq 0$, $[b] \leq 0$, we have $x \leq 0$. This is contradicted with $x > 0$. Thus, $[x] = 0$. □

Suppose that $J_0^l \neq \emptyset$. We have following lemmas.

Theorem 2 *The solution of $LCP([A],[b]^l)$ coincides with the solution of the lower-dimensional linear system with interval data*

$$\begin{cases} [a_i^{\mathrm{T}}][x] - [b_i^{\mathrm{T}}]^l = 0, & i \in J^l = \{i, [b_i]^l > 0\}, \\ [x_i] = 0, & i \in I^l. \end{cases}$$

On the other hand, the solution of the lower-dimensional linear system with interval data

$$\begin{cases} [a_i^{\mathrm{T}}][x] - [b_i^{\mathrm{T}}]^l = 0, & i \in J^l, \\ [x_i] = 0, & i \in I^l. \end{cases}$$

satisfies

$$X_J \geq 0 \quad and \quad [A_{IJ}][X_J] - [b_I] \geq 0. \tag{6}$$

Proof: The results can be obtained obviously. □

Theorem 3 *Let $J_0^l = \{i, [b_i]^l > 0\}$, $J^l(x^*)$ be the index set when the iteration terminates, $J^l(x^*)$ be the index set get from the last lower-dimension linear system with interval data. We have*

$$J_0^l \subseteq J^l(x^*).$$

Proof: For x^* is the solution of the $(LCP[A], [b]^l)$, $i \in N$, it has

$$[a_{ii}][x_i^*] \geq -\sum_{i \neq j} [a_{ij}][x_j^*] + [b_i]^l \geq [b_i]^l.$$

For $\forall\ i \in J_0^l$, $[b_i]^l > 0$, $[a_{ii}] > 0$, $[x_i^*] > 0$. We have $i \in J^l([x^*])$. The conclusion of this lemma has been proved. □

Theorem 4 *Let the index sets α, β satisfy $\alpha \subseteq J^l(x^*) = \{i \in N | [x_i^*] > 0\}$, $\alpha \bigcap \beta = \emptyset$, $\alpha \cap \beta = N$. Noting that*

$$[\tilde{A}] = \begin{pmatrix} [E_{\alpha\alpha}] & [(A_{\alpha\alpha})^{-1}][A_{\alpha\beta}] \\ 0 & [A_{\alpha\alpha}] - [A_{\alpha\beta}][(A_{\beta\beta})]^{-1}[A_{\beta\alpha}] \end{pmatrix},$$

and

$$[\tilde{b}]^l = \begin{pmatrix} [(A_{\alpha\alpha})]^{-1}[b_\alpha]^l \\ [b_\beta]^l - [a_{\beta\alpha}][(a_{\alpha\alpha})]^{-1}[b_\alpha]^l \end{pmatrix},$$

where $E_{\alpha\alpha} \in \mathbb{R}^{|\alpha| \times |\alpha|}$ is an identity matrix, $|\alpha|$ is the number of the elements in index set α.

Proof: Prove as the process of Lemma 2.2 in [1]. □

Theorem 5 *Let $[x^0]$ be the unique solution of lower-dimension linear system $LCP([A], [b]^l)$, $Q(0)$. (6) is not satisfied by $I^l \neq I_0^l$ and $J^l \neq J_0^l$. Suppose that the index set*

$$J_{1o}^l = \{i \in I_0^l | [a_i^{\mathrm{T}}][x_0] - [b_i]^l < 0\}.$$

It then follows that $J_{1o}^l \neq \emptyset$ and $J_{1o}^l \subset J^l([x^*])$.

Proof: It is clear that $J_{1o}^l \neq \emptyset$. We only need to prove $J_{1o}^l \subset J^l([x^*])$. Let

$$\tilde{A} = \begin{pmatrix} E_{J_0^l J_0^l} & (A_{J_0^l J_0^l})^{-1} A_{J_0^l I_0^l} \\ 0 & A_{I_0^l I_0^l} - A_{I_0^l J_0^l}(A_{J_0^l J_0^l})^{-1} A_{J_0^l I_0^l} \end{pmatrix}, \tilde{b} = \begin{pmatrix} (A_{J_0^l J_0^l})^{-1} b_{J_0^l} \\ b_{I_0^l} - A_{I_0^l J_0^l}(A_{J_0^l J_0^l})^{-1} b_{J_0^l} \end{pmatrix}, \tag{7}$$

From Theorem 4, we have $LCP([A], [b]^l)$ and $LCP([\tilde{A}], [\tilde{b}]^l)$ are equivalent. Note that $i \in J_{1o}^l$ if and only if $i \in I_0^l$ and $[\tilde{b}]^l > 0$. From Theorem 3, it has $i \in J^l([x^*])$. □

The fact that $J_0^l \bigcap J_{1o}^l = \emptyset$ is obvious. Let $J_1 = J_o^l \cap J_{1o}^l$, we get $J_1 \supset J_0$. From Theorem 4, $J_1^l \subseteq J^l(x^*)$. For interval matrix $[A]$ is an M-matrix and $[\tilde{b}_i]^l > 0$ which consists of the index i by J_1^l. Index set J_1^l in $LCP([\tilde{A}], [\tilde{b}])$ plays the same role as J_0^l in $LCP([A], [b])$. Then it has the following corollary.

Corollary 2 *The solution of the lower-dimensional linear system with interval data*

$$[A_{J_1^l J_1^l}][X_{J_1^l}] - [b_{J_1^l}] = 0$$

is positive.

Repeating the process above, it has a sequence of index sets

$$J_0^l \subset J_1^l \subset \cdots \subset J_t^l = J^l(x^*).$$

Here, l is finite, $t \leq n$ and $J_t^l = J^l(x^*)$. The solution of lower-dimension linear system with interval data is the solution of $LCP([A],[b]^l)$. Then the solution of $LCP([A],[b])$ is positive.

3.3 The monotonicity of Algorithm 1

In this section, we prove the monotonicity of Algorithm 1 with several theorems as follows.

Theorem 6 *In Algorithm 1, $x^i_{J_k}$ is the solution of kth iteration, $x^i_{J_{k+1}}$ is the solution of $(k+1)$th iteration, then it has*

$$x^i_{J_k} < x^i_{J_{k+1}}, \quad 1 \leq i \leq J_k,$$

where $J_{k+1} = J_k \bigcap J_{ko}$.

Proof: Since $x^i_{J_k}$ is the solution of kth iteration, we have

$$A_{J_kJ_k}x_{J_k} = b_{J_k}, \quad A_{J_{k0}J_k}x_{J_{k0}} - q_{J_{k0}} < 0.$$

Suppose that $x_{J_{k+1}} = (x'_{J_{k+1}}, x'_{J_{ko}})^{\mathrm{T}}$, then

$$(A_{J_kJ_k}, A_{J_kJ_{k0}})\begin{pmatrix} x'_{J_{k+1}} \\ x'_{J_{k0}} \end{pmatrix} = (b_{J_k}).$$

We have

$$A_{J_kJ_k}x'_{J_{k+1}} = b_{J_k} - A_{J_kJ_{k0}}x'_{J_{k0}}.$$

From Corollary 2, it has $x'_{J_{k+1}}, x'_{J_{ko}} > 0$. According to A is an M-matrix, it has $A_{J_kJ_{k0}} < 0$, then

$$x_{J_{k+1}} > x_{J_k}.$$

□

Theorem 6 shows that the solution of $LCP(A,b)$ increases with the step increasing.

Theorem 7 *Let $A^1 \geq A^2$, $b^1 \leq b^2$, which mean $a^1_{ij} \geq a^2_{ij}, 1 \leq i,j \leq n$, $b^1_i \leq b^2_i, 1 \leq i \leq n$. In Algorithm 1, at kth step of iteration, $J^1_k \supseteq J^2_k$. Then at $(k+1)$th step of iteration, it has*

$$J^1_{k+1} \supseteq J^2_{k+1}.$$

Proof: When $k=1$, it is clear that $J^1_{k+1} \supset J^2_{k+1}$.

When $k \neq 1$, we suppose that x_1, x_2 are the solutions of A_1, b_1 and A_2, b_2, respectively. From Lemma 5, it has

$$x_1 = (A^1_{J^2_k J^2_k})^{-1} b^1_{J^2_k} > (A^2_{J^2_k J^2_k})^{-1} b^2_{J^2_k} = x_2.$$

J^1_{ko}, J^2_{ko} satisfy

$$J^1_{ko} = \{i | (a^1_i)^{\mathrm{T}} x_1 - b^1_i \leq 0\},$$
$$J^2_{ko} = \{i | (a^2_i)^{\mathrm{T}} x_1 - b^2_i \leq 0\},$$
$$J^1_{k+1} = J^1_k \bigcap J^1_{ko} \quad and \quad J^2_{k+1} = J^2_k \bigcap J^2_{ko}.$$

According to the condition of this theorem $A^1 \geq A^2$, $b^1 \leq b^2$, it has

$$A^2 x_2 - b^2 \leq A^2 x_2 - b^1.$$

Let $p \in J^2_{ko}$, $x^2_p = x^1_p = 0$, we obtain

$$a^2_{pp} x^2_p + (\sum_{i=1}^{p-1} a^2_{ip} x^2_i) > a^1_{pp} x^1_p + (\sum_{i=1}^{p-1} a^1_{ip} x^1_i).$$

Evidently, it has

$$A^2 x_2 - b^2 \leq A^2 x_2 - b^1 \leq A^1 x_1 - b^1.$$

We get $J^1_{ko} \subseteq J^2_{ko}$, then the conclusion of this theorem has been proved. □

Theorem 8 *For $\forall$ A, $A' \in \mathbb{R}^{n\times n}$ are M-matrices, $A < A'$, $b > b'$, $x_i > 0$. In Algorithm 1, if $J_k = \{1, \cdots k\}$, it has*

$$\begin{pmatrix} a_{11} & \cdots & -a_{1k} \\ \cdots & \cdots & \cdots \\ -a_{k1} & \cdots & a_{kk} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_k \end{pmatrix}, \tag{8}$$

and

$$\begin{pmatrix} a'_{11} & \cdots & -a'_{1k} \\ \cdots & \cdots & \cdots \\ -a'_{k1} & \cdots & a'_{kk} \end{pmatrix} \begin{pmatrix} x'_1 \\ \vdots \\ x'_k \end{pmatrix} = \begin{pmatrix} b'_1 \\ \vdots \\ b'_k \end{pmatrix}. \tag{9}$$

In (8) and (9), we obtain $(x_1, \cdots, x_k)^{\mathrm{T}} > (x'_1, \cdots, x'_k)^{\mathrm{T}}$. After one step iteration if it has $J_{k+1} = \{1, \cdots, k, k+1\}$, $(x_1, \cdots, x_k, x_k+1)^{\mathrm{T}}$ and $(x'_1, \cdots, x'_k, x_k+1')^{\mathrm{T}}$ are the solutions of

$$\begin{pmatrix} a_{11} & \cdots & -a_{1k} & -a_{1k+1} \\ \cdots & \cdots & \cdots & \cdots \\ -a_{k1} & \cdots & a_{kk} & -a_{k,k+1} \\ -a_{k+1,1} & \cdots & -a_{k+1,k} & a_{k+1,k+1} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_k \\ x_{k+1} \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_k \\ b_{k1} \end{pmatrix}, \tag{10}$$

and

$$\begin{pmatrix} a'_{11} & \cdots & -a'_{1k} & -a'_{1k+1} \\ \cdots & \cdots & \cdots & \cdots \\ -a'_{k1} & \cdots & a'_{kk} & -a'_{k,k+1} \\ -a'_{k+1,1} & \cdots & -a'_{k+1,k} & a'_{k+1,k+1} \end{pmatrix} \begin{pmatrix} x'_1 \\ \vdots \\ x'_k \\ x'_{k+1} \end{pmatrix} = \begin{pmatrix} b'_1 \\ \vdots \\ b'_k \\ b'_{k+1} \end{pmatrix}, \tag{11}$$

respectively. Then we have

$$(x_1, \cdots, x_k, x_{k+1})^{\mathrm{T}} > (x'_1, \cdots, x'_k, x'_{k+1})^{\mathrm{T}}.$$

Proof: Let

$$A_1 = \begin{pmatrix} a_{11} & \cdots & -a_{1k} & 0 \\ \cdots & \cdots & \cdots & \cdots \\ -a_{k1} & \cdots & a_{kk} & 0 \\ 0 & \cdots & 0 & a_{k+1,k+1} \end{pmatrix}, \ A'_1 = \begin{pmatrix} a'_{11} & \cdots & -a'_{1k} & 0 \\ \cdots & \cdots & \cdots & \cdots \\ -a'_{k1} & \cdots & a'_{kk} & 0 \\ 0 & \cdots & 0 & a'_{k+1,k+1} \end{pmatrix},$$

we have

$$\begin{aligned} A_1 x &= \begin{pmatrix} a_{11} & \cdots & -a_{1k} & 0 \\ \cdots & \cdots & \cdots & \cdots \\ -a_{k1} & \cdots & a_{kk} & 0 \\ 0 & \cdots & 0 & a_{k+1,k+1} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_k \\ x_{k+1} \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_k \\ b_{k+1} \end{pmatrix} \\ &\Rightarrow \begin{cases} \begin{pmatrix} a_{11} & \cdots & -a_{1k} \\ \cdots & \cdots & \cdots \\ -a_{k1} & \cdots & a_{kk} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_k \end{pmatrix} \\ a_{k+1,k+1} x_{k+1} = b_{k+1} \end{cases} \\ &\Rightarrow (x_1, \cdots, x_k, x_{k+1})^{\mathrm{T}}. \end{aligned}$$

$$\begin{aligned} A'_1 x' &= \begin{pmatrix} a'_{11} & \cdots & -a'_{1k} & 0 \\ \cdots & \cdots & \cdots & \cdots \\ -a'_{k1} & \cdots & a'_{kk} & 0 \\ 0 & \cdots & 0 & a'_{k+1,k+1} \end{pmatrix} \begin{pmatrix} x'_1 \\ \vdots \\ x'_k \\ x'_{k+1} \end{pmatrix} = \begin{pmatrix} b'_1 \\ \vdots \\ b'_k \\ b'_{k+1} \end{pmatrix} \\ &\Rightarrow \begin{cases} \begin{pmatrix} a'_{11} & \cdots & -a'_{1k} \\ \cdots & \cdots & \cdots \\ -a'_{k1} & \cdots & a'_{kk} \end{pmatrix} \begin{pmatrix} x'_1 \\ \vdots \\ x'_k \end{pmatrix} = \begin{pmatrix} b'_1 \\ \vdots \\ b'_k \end{pmatrix} \\ a'_{k+1,k+1} x'_{k+1} = b'_{k+1} \end{cases} \\ &\Rightarrow (x'_1, \cdots, x'_k, x'_{k+1})^{\mathrm{T}}. \end{aligned}$$

Since $(x_1, \cdots, x_k)^{\mathrm{T}} > (x'_1, \cdots, x'_k)^{\mathrm{T}}$, $a_{k+1,k+1} < a'_{k+1,k+1}$, $b_{k+1} > b'_{k+1}$, it has

$$x_{k+1} > x'_{k+1}.$$

In the form of A_1, $A_1{}'$, we obtain

$$(x_1, \cdots, x_k, x_{k+1})^{\mathrm{T}} > (x'_1, \cdots, x'_k, x'_{k+1})^{\mathrm{T}}.$$

Now we discuss more general form of A_1, ${A_1}'$. Let

$$A_2 = \begin{pmatrix} a_{11} & \cdots & -a_{1k} & -a_{1,k+1} \\ \cdots & \cdots & \cdots & \cdots \\ -a_{k1} & \cdots & a_{kk} & 0 \\ 0 & \cdots & 0 & a_{k+1,k+1} \end{pmatrix}, \ {A_2}' = \begin{pmatrix} a'_{11} & \cdots & -a'_{1k} & -a'_{1,k+1} \\ \cdots & \cdots & \cdots & \cdots \\ -a'_{k1} & \cdots & a'_{kk} & 0 \\ 0 & \cdots & 0 & a'_{k+1,k+1} \end{pmatrix},$$

we have

$$\begin{aligned} A_2x &= \begin{pmatrix} a_{11} & \cdots & -a_{1k} & -a_{1,k+1} \\ \cdots & \cdots & \cdots & \cdots \\ -a_{k1} & \cdots & a_{kk} & 0 \\ 0 & \cdots & 0 & a_{k+1,k+1} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_k \\ x_{k+1} \end{pmatrix} \\ &= \begin{pmatrix} b_1 \\ \vdots \\ b_k \\ b_{k+1} \end{pmatrix} \begin{pmatrix} a_{11} & \cdots & -a_{1k} & 0 \\ \cdots & \cdots & \cdots & \cdots \\ -a_{k1} & \cdots & a_{kk} & 0 \\ 0 & \cdots & 0 & a_{k+1,k+1} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_k \\ x'_{k+1} \end{pmatrix} + \begin{pmatrix} 0 & \cdots & 0 & -a_{1,k+1} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_k \\ x_{k+1} \end{pmatrix} \\ &= \begin{pmatrix} b_1 \\ \vdots \\ b_k \\ b_{k+1} \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} {A_2}'x' &= \begin{pmatrix} a'_{11} & \cdots & -a'_{1k} & -a'_{1,k+1} \\ \cdots & \cdots & \cdots & \cdots \\ -a'_{k1} & \cdots & a'_{kk} & 0 \\ 0 & \cdots & 0 & a'_{k+1,k+1} \end{pmatrix} \begin{pmatrix} x'_1 \\ \vdots \\ x'_k \\ x'_{k+1} \end{pmatrix} \\ &= \begin{pmatrix} b'_1 \\ \vdots \\ b'_k \\ b'_{k+1} \end{pmatrix} \begin{pmatrix} a'_{11} & \cdots & -a'_{1k} & 0 \\ \cdots & \cdots & \cdots & \cdots \\ -a'_{k1} & \cdots & a'_{kk} & 0 \\ 0 & \cdots & 0 & a'_{k+1,k+1} \end{pmatrix} \begin{pmatrix} x'_1 \\ \vdots \\ x'_k \\ x'_{k+1} \end{pmatrix} + \begin{pmatrix} 0 & \cdots & 0 & -a'_{1,k+1} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} x'_1 \\ \vdots \\ x'_k \\ x'_{k+1} \end{pmatrix} \\ &= \begin{pmatrix} b'_1 \\ \vdots \\ b'_k \\ b'_{k+1} \end{pmatrix}. \end{aligned}$$

Decomposing $(b_1, \cdots, b_k, b_{k+1})^{\mathrm{T}}$, it follows that

$$\begin{pmatrix} a_{11} & \cdots & -a_{1k} & 0 \\ \cdots & \cdots & \cdots & \cdots \\ -a_{k1} & \cdots & a_{kk} & 0 \\ 0 & \cdots & 0 & a_{k+1,k+1} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_k \\ x_{k+1} \end{pmatrix} = \begin{pmatrix} b_1^{(1)} \\ \vdots \\ b_k^{(1)} \\ b_{k+1}^{(1)} \end{pmatrix}, \tag{12}$$

and

$$\begin{pmatrix} 0 & \cdots & 0 & -a_{1,k+1} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_k \\ x_{k+1} \end{pmatrix} = \begin{pmatrix} b_1^{(2)} \\ \vdots \\ b_k^{(2)} \\ b_{k+1}^{(2)} \end{pmatrix}. \tag{13}$$

Decomposing $(b'_1, \cdots, b'_k, b'_{k+1})^{\mathrm{T}}$, it has

$$\begin{pmatrix} a'_{11} & \cdots & -a'_{1k} & 0 \\ \cdots & \cdots & \cdots & \cdots \\ -a'_{k1} & \cdots & a'_{kk} & 0 \\ 0 & \cdots & 0 & a'_{k+1,k+1} \end{pmatrix} \begin{pmatrix} x'_1 \\ \vdots \\ x'_k \\ x'_{k+1} \end{pmatrix} = \begin{pmatrix} b'^{(1)}_1 \\ \vdots \\ b'^{(1)}_k \\ b'^{(1)}_{k+1} \end{pmatrix}, \tag{14}$$

and

$$\begin{pmatrix} 0 & \cdots & 0 & -a'_{1,k+1} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} x'_1 \\ \vdots \\ x'_k \\ x'_{k+1} \end{pmatrix} = \begin{pmatrix} b'^{(2)}_1 \\ \vdots \\ b'^{(2)}_k \\ b'^{(2)}_{k+1} \end{pmatrix}. \tag{15}$$

From the above conclusion, we can infer that if $J_{k+1} = \{1, \cdots, k, k+1\}$, $J_k = \{1, \cdots k\}$, the solution in (12) is bigger than in (14), and the solution in (13) is bigger than in (15), then

$$(x_1, \cdots, x_k, x_{k+1})^{\mathrm{T}} > (x'_1, \cdots, x'_k, x'_{k+1})^{\mathrm{T}}.$$

□

Corollary 3 *For $\forall\ A,\ A' \in \mathbb{R}^{n\times n}$ are $M-$matrices, $A < A'$, $b > b'$, $x_i > 0$. In Algorithm 1, if $J_k \supset J_k{}'$ and*

$$A_{J_kJ_k} x_{J_k} = b_{J_k} \quad \text{and} \quad A'_{J_kJ_k} x'_{J_k} = b'_{J_k},$$

it has $x_{J_k} > x'_{J_k}$. Then after one step iteration it has $J_{k+1} \supset J'_{k+1}$. If $x_{J_{k+1}}$, $x_{J_{k+1}}$ are the solutions of

$$A_{J_{k+1}J_{k+1}} x_{J_{k+1}} = b_{J_{k+1}} \quad \text{and} \quad A'_{J_{k+1}J_{k+1}} x'_{J_{k+1}} = b'_{J_{k+1}},$$

we have

$$x_{J_{k+1}} > x'_{J_{k+1}}.$$

Proof: With the Theorems 6, 7, 8, we obtain the conclusion of this corollary. □

4 Numerical experiments

In this section, we report on the numerical results obtained with a Matlab 7.0.1 implementation on window XP with 2.39 GHz 64-bit processor. The matrices $[A]$ in the experiments are M-matrices. Our main goal is to test the monotonicity of Algorithm 1 and the effectiveness of Algorithm 2.

Example 1. Let

$$A = \begin{pmatrix} [10,20] & [-4,-1] & & \\ [-5,-1] & [10,20] & [-4,-1] & \\ & \ddots & \ddots & \ddots \\ & & [-5,-1] & [10,20] \end{pmatrix}_{6\times 6},$$

$$[b] = \begin{pmatrix} [1,1.5] & [1,1.5] & [1,1.5] & [1,1.5] & [-0.2,-0.1] & [-0.02,-0.01] \end{pmatrix}^{\mathrm{T}}.$$

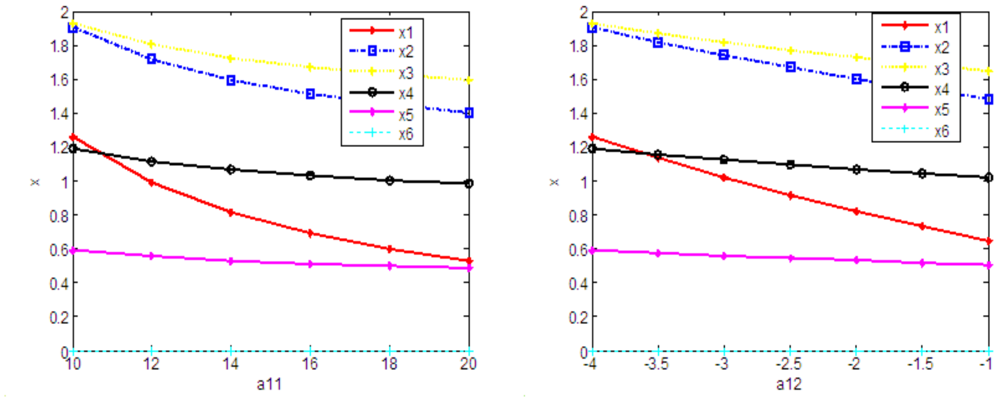

Figure 1: Changing the elements a_{11} and a_{12} in Example 1, we get different solutions of $LCP([\underline{A}], [\overline{b}])$.

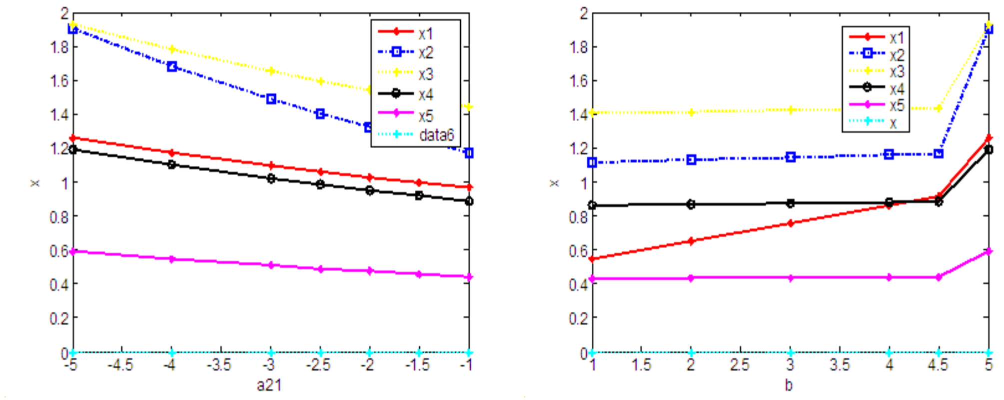

Figure 2: Changing the elements a_{21} and b_1 in Example 1, we get different solutions of $LCP([\underline{A}], [\overline{b}])$.

With Algorithm 2, we obtain

$$[x] = \begin{pmatrix} [0.0528, 1.2604] & [0.0553, 1.9011] & [0.0528, 1.9271] & [0, 1.1914] & [0, 0.5947] & [0, 0] \end{pmatrix}^{\mathrm{T}}.$$

In order to test the monotonicity of Algorithm 1. We change the elements in $[\underline{A}]$ and $[\overline{b}]$ when they are mentioned. Then we get Figure 1 and Figure 2.

In Figure 1 and Figure 2, if a_{11} is reduced, the solution x increases. a_{11} is a represent of a_{ij} for $\forall\, i = j$. When a_{12} is reduced, the solution x increases. a_{12} is a represent of a_{ij} for $\forall\, i \neq j$. When b_1 increases, the solution x increases. In $[b]$, b_1 is a represent of b_i, for $\forall\, i \in \{1, \cdots, n\}$. The results of Figure 1 and Figure 2 coincide with the Corollary 3.

Example 2. [9] Let

$$[A] = \begin{pmatrix} [1,1.5] & -0.5 & \cdots & \cdots & -0.5 \\ 0 & \ddots & \ddots & \cdots & \vdots \\ \vdots & \ddots & \ddots & -0.5 & -0.5 \\ \vdots & \cdots & 0 & [1,1.5] & -0.5 \\ 0 & \cdots & \cdots & 0 & [1,1.5] \end{pmatrix} \in \mathbb{IR}^{10\times 10}.$$

and

$$[b_i] = \left\{ \begin{array}{ll} [0.2, 0.3] & if\ i = 2k+1 \\ [-1, -0.9] & if\ i = 2k \end{array} \right\} \ i = 1, \cdots, 10.$$

From Total step method with intersection (TI) and Single step method with intersection (SI) proposed by Alefeld and Schafer in [11], they get the inclusion

$$[x_1] = \begin{pmatrix} [3.081847279378140E+000, 1.951054687500001E+001] \\ [2.911385459533605E+000, 1.380703125000001E+001] \\ [1.583539094650204E+000, 8.404678500000004E+000] \\ [1.787654320987653E+000, 6.403125000000002E+000] \\ [7.407407407407402E-001, 3.468750000000001E+000] \\ [1.155555555555555E+000, 3.125000000000001E+000] \\ [2.666666666666664E-001, 1.275000000000001E+000] \\ [7.999999999999998E-001, 1.650000000000001E+000] \\ [0.000000000000000E+000, 3.000000000000001E-001] \\ [5.999999999999998E-001, 1.000000000000000E+000] \end{pmatrix}$$

after 10 steps using TI, after 10 steps using SI, respectively.

Let $A \in [A]$, $b \in [b]$ in $LCP(A, b)$ has

$$A = \begin{pmatrix} 1.5 & -0.5 & \cdots & \cdots & -0.5 \\ 0 & \ddots & \ddots & \cdots & \vdots \\ \vdots & \ddots & \ddots & -0.5 & -0.5 \\ \vdots & \cdots & 0 & 1.5 & -0.5 \\ 0 & \cdots & \cdots & 0 & 1.5 \end{pmatrix},$$

$$b_i = \left\{ \begin{array}{ll} 0.2 & if\ i = 2k+1 \\ -1 & if\ i = 2k \end{array} \right\} \ i = 1, \cdots, 10.$$

We get

$$\tilde{x} = (0.4214, 0, 0.31605, 0, 0.23704, 0, 0.17778, 0, 0.13333, 0)^{\mathrm{T}}.$$

From Algorithm 2, we get

$$[x_2] = \begin{pmatrix} [0.4140, 1.6781] \\ [0, 0.3188] \\ [0.3161, 1.0125] \\ [0, 0] \\ [0.2370, 0.6750] \\ [0, 0] \\ [0.1778, 0.4500] \\ [0, 0] \\ [0.1333, 0.3000] \\ [0, 0] \end{pmatrix}$$

after 1 step.

According to the above solutions of $[A]$ and $[b]$, $\tilde{x}$ is concluded by $[x_2]$, but not concluded by $[x_1]$. The solution of Algorithm 2 is more precise than TI and SI in [9]. From the results of the number of iteration step, Algorithm 2 is more efficient than TI and SI. Moreover, comparing with TI and SI, Algorithm 2 do not need to find a initial point.

Example 3. In this experiment, we present a finite different discretization problem of one side obstacle [12].

$$\langle -\Delta u - f, v - u\rangle \geq 0, \forall v \in K,$$

where $K = \{v \in H_0^1(\Omega) : v \geq 0\}$, $f = 4\sin(4xy)$, $\Omega = (0, a) \times (0, a)$, a is an interval data, $a = [0.9, 1.1]$. We can turn it into this form by discretization.

$$Ax - b \geq 0,\ x \geq 0,\ x^{\mathrm{T}}(Ax - b) = 0,$$

where $h = \frac{[0.9,1.1]}{r}$, $n = r^2$, $b = (4h^2 \sin(\frac{4ij}{r^2}))_{ij}$, $i, j = 1, \cdots r$, and

$$A = \begin{pmatrix} B & -I & & & \\ -I & B & -I & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -I \\ & & & -I & B \end{pmatrix} \in \mathbb{R}^{n\times n}.$$

Here, $I \in \mathbb{R}^{r\times r}$ is an identity matrix and

$$B = \begin{pmatrix} 4 & -1 & & & \\ -1 & 4 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 4 \end{pmatrix} \in \mathbb{R}^{r\times r}.$$

The inner and outer iterative stop criterion are $\|A_{J_kJ_k} - b_{J_k}\| < 10^{-10}$ and $\|\min\{Ax + b, x\}\| < 10^{-10}$. In Table 1, "$n$" denotes the order of the problem in Experiment 3, "d_1" and "d_2" stand for the number of elements in J_0 when we compute the maximum and minimum respectively, "n_1" and "n_2" are the number of the iteration when we compute the

maximum and minimum respectively, "t" is the total time we take in Algorithm 2. From Table 1, computational estate increase with the dimension of the problem increase but the iteration number increase slow with the dimension of the problem increase. Algorithm 2 is efficient when the dimension of the problem is bigger.

n	n_1	n_2	d_1	d_2	t
16	1	1	15	15	5.025
64	1	1	61	61	5.438
144	2	2	138	138	4.954
256	3	3	246	246	8.486
400	4	4	385	385	12.765
576	5	5	555	555	29.526
784	6	6	756	756	60.281
1024	7	7	990	990	121.266
1296	7	7	1255	1255	212.751
1600	8	8	1551	1551	392.231

Table 1: Numerical results for a finite different discretization problem of one side obstacle with different dimensions where inner and outer iterative stop criterion are $\|A_{J_kJ_k} - b_{J_k}\| < 10^{-10}$ and $\|\min\{Ax + b, x\}\| < 10^{-10}$.

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An approach based on fuzzy contra-harmonic mean operators to group decision making

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Abstract

Contra-harmonic mean is widely used as a tool to aggregate central tendency data and is useful whenever a few individual observations contribute a disproportionate amount to the arithmetic mean. In this paper, we investigate the situations in which the input data are expressed in fuzzy values and develop some fuzzy contra-harmonic mean operators, such as fuzzy weighted contra-harmonic mean operator, fuzzy ordered weighted contra-harmonic mean operator, and fuzzy hybrid contra-harmonic mean operator. Especially, all these operators can reduce to aggregate interval or real numbers. Then based on the developed operators, we present an approach to multiple attribute group decision making and illustrate it with a practical example.

1 Introduction

Decision making is an important subject in business, manufacturing, and service. Group decision making is a typical decision making activity where utilizing several experts alleviate some of the decision making difficulties due to the problem's complexity and uncertainty. In the real world, the uncertainty, constraints, and even unclear knowledge of the experts imply that decision makers cannot provide exact numbers to express their opinions. Fuzzy sets [33] are a very useful tool to express a decision maker's preference information over objects in process of decision making under uncertain or vague environments. In order to

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get a decision result, an important step is the aggregation of fuzzy data. Many techniques have been developed to aggregate fuzzy data information. Lee and Kuo [10] developed the fuzzy number OWA (FN-OWA) operator, which we will refer to it as the fuzzy OWA (FOWA) operator, and its application to image enhancement. The main characteristic of FOWA operator is that it uses uncertain information in the aggregation represented by fuzzy numbers. Mitchell and Estrakh [15] presented the OWA operator with fuzzy ranks. Yager [27] proposed a method for including importance in the OWA aggregations using fuzzy systems modeling. The FOWA operator has been studied by different authors [3, 5, 6, 12, 30, 34, 20, 17]. Chen and Chen [4] used fuzzy numbers to extend the induced OWA (IOWA) [31] operator to present the fuzzy number IOWA (FN-IOWA) operator, wherein fuzzy numbers are used to describe the argument values and the weights of the IOWA operator, and the aggregation results are obtained by using fuzzy number arithmetic operations. Based on the FN-IOWA operator, they developed a new algorithm to deal with multicriteria fuzzy decision making problems. Xu [20] introduced the fuzzy ordered weighted geometric (FOWG) operator. Xu and Wu [25] proposed the fuzzy induced ordered weighted averaging (FIOWA) operator. Xu and Da [24] developed the fuzzy induced ordered weighted geometric (FIOWG) operator. Wang and Luo [18] introduced the generalized fuzzy weighted mean (GFWM) operator. It includes the fuzzy weighted average (FWA), the fuzzy weighted geometric mean (FWGM), the fuzzy weighted harmonic mean (FWHM), the fuzzy weighted quadratic mean (FWQM) and the fuzzy weighted root-power mean (FWRM) operators as its special cases. They developed linear programming models for solving the GFWM and its special cases and investigated the order relationships among the FWA, the FWGM and the FWHM operators. Xu [22] also extended the traditional harmonic mean to fuzzy environments and introduced the some fuzzy harmonic mean operators.Based on the developed operators, he presented an approach to multiple attribute group decision making. Wei [19] developed a fuzzy induced ordered weighted harmonic mean (FIOWHM) operator, studied some desirable properties of the FIOWHM operators, such as commutativity, idempotency and monotonicity, and applied the FIOWHM and FWHM operators to group decision making with triangular fuzzy information. Merigó and Casanovas [14] presented the fuzzy generalized ordered weighted averaging (FGOWA) operator. It is an extension of the generalized ordered weighted averaging (GOWA) operator for uncertain situations where the available information is given in the form of fuzzy numbers. In [13], they further introduced the fuzzy OWAWA (FOWAWA) operator, which unifies the FOWA operator and the FWA operator in the same formulation, and studied some of its main properties and particular cases including a wide range of new aggregation operators such as the FOWA, fuzzy arithmetic weighted average (FAWA) and the fuzzy arithmetic OWA (FAOWA) operators.

Contra-harmonic mean is the arithmetic mean of the squares divided by the arithmetic mean, which is a conservative average to be used to provide for aggregation lying between the max and min operators. Contra-harmonic mean is widely used as a tool to aggregate central tendency data and is useful when-

ever a few individual observations contribute a disproportionate amount to the arithmetic mean. Consider that, in the existing literature, the contra-harmonic mean is generally considered as a fusion technique of numerical data, in the real-life situations, the input data sometimes cannot be obtained exactly, but fuzzy data can be given. Therefore, how to aggregate fuzzy data by using the contra-harmonic mean? is an interesting research topic and is worth paying attention to. In this paper, we develop some fuzzy contra-harmonic mean (FCHM) operators. To do so, the remainder of this paper is arranged in the following sections. Section 2 reviews some basic aggregation operators. Section 3 develops some FCHM operators, such as fuzzy weighted contra-harmonic mean (FWCHM) operator, fuzzy ordered weighted contra-harmonic mean (FOWCHM) operator, fuzzy hybrid contra-harmonic mean (FHCHM) operator, and so on. Section 4 presents an approach to multiple attribute group decision making based on the developed operators. Section 5 illustrates the presented approach with a practical example. Section 6 ends the paper with some concluding remarks.

2 Basic aggregation operators

In this section, we review some basic aggregation techniques and some of their fundamental characteristics.

Definition 1. [8] Let WAA : $R^n \to R$, if

$$\mathrm{WAA}(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} w_j a_j, \tag{1}$$

where a_j $(j = 1, 2, \ldots, n)$ is a collection of positive real numbers, and $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of a_j $(j = 1, 2, \ldots, n)$, with $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$, then WAA is called the weighted arithmetic averaging (WAA) operator. Especially, if $w_i = 1$, $w_j = 0$, $j \neq i$, then $\mathrm{WAA}(a_1, a_2, \ldots, a_n) = a_i$; if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then the WAA operator is reduced to the arithmetic averaging (AA) operator, i.e.,

$$\mathrm{AA}(a_1, a_2, \ldots, a_n) = \frac{1}{n} \sum_{j=1}^{n} a_j. \tag{2}$$

Definition 2. [2] Let WCHM : $(R^+)^n \to R^+$, if

$$\mathrm{WCHM}(a_1, a_2, \ldots, a_n) = \frac{\sum_{j=1}^{n} w_j a_j^2}{\sum_{j=1}^{n} w_j a_i}, \tag{3}$$

where a_j $(j = 1, 2, \ldots, n)$ is a collection of positive real numbers, and $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of a_j $(j = 1, 2, \ldots, n)$, with $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$, then WCHM is called the weighted contra-harmonic mean (WCHM) operator. Especially, if $w_i = 1$, $w_j = 0$, $j \neq i$, then $\mathrm{WCHM}(a_1, a_2,$

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$\ldots, a_n) = a_i$; if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then the WCHM operator is reduced to the contra-harmonic mean (CHM) operator, i.e.,

$$\mathrm{CHM}(a_1, a_2, \ldots, a_n) = \frac{\sum_{j=1}^{n} a_j^2}{\sum_{j=1}^{n} a_j}. \tag{4}$$

The WAA and WCHM operators first weight all the given data, and then aggregate all these weighted data into a collective one. Yager [26, 27, 28, 29] introduced and studied the OWA operator that weights the ordered positions of the data instead of weighting the data themselves.

Definition 3. [26] An OWA operator of dimension n is a mapping OWA : $R^n \to R$ that has an associated vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ such that $\omega_j \geq 0$ and $\sum_{j=1}^{n} \omega_j = 1$. Furthermore,

$$\mathrm{OWA}(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} \omega_j b_j, \tag{5}$$

where b_j is the jth largest of a_i $(i = 1, 2, \ldots, n)$. Especially, if $\omega_i = 1$, $\omega_j = 0$, $j \neq i$, then $b_n \leq \mathrm{OWA}(a_1, a_2, \ldots, a_n) = b_i \leq b_1$; if $\omega = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then

$$\mathrm{OWA}(a_1, a_2, \ldots, a_n) = \frac{1}{n} \sum_{j=1}^{n} b_j = \frac{1}{n} \sum_{j=1}^{n} a_j = \mathrm{AA}(a_1, a_2, \ldots, a_n). \tag{6}$$

3 Fuzzy contra-harmonic mean operators

The above-mentioned aggregation techniques can only deal with the situation that the arguments are represented by the exact numerical values, but are invalid if the aggregation information is given in other forms, such as triangular fuzzy number [9], which is a widely used tool to deal with uncertainty and fuzzyness, described as follows:

Definition 4. [9] A triangular fuzzy number $\hat{a}$ can be defined by a triplet $[a^L, a^M, a^U]$. The membership function $\mu_{\hat{a}}(x)$ is defined as:

$$\mu_{\hat{a}}(x) = \begin{cases} 0, & x < a^L; \\ \frac{x - a^L}{a^M - a^L}, & a^L \leq x \leq a^M; \\ \frac{x - a^U}{a^M - a^U}, & a^M \leq x \leq a^U; \\ 0, & x > a^U, \end{cases}$$

where $a^U \geq a^M \geq a^L \geq 0$, a^L and a^U stand for the lower and upper values of $\hat{a}$, respectively, and a^M stands for the modal value [9]. Especially, if any two of a^L, a^M and a^U are equal, then $\hat{a}$ is reduced to an interval number; if all a^L, a^M and a^U are equal, then $\hat{a}$ is reduced to a real number. For convenience, we let Ω be the set of all triangular fuzzy numbers.

Let $\hat{a} = [a^L, a^M, a^U]$ and $\hat{b} = [b^L, b^M, b^U]$ be two triangular fuzzy numbers, then some operational laws defined as follows [9]:

1) $\hat{a} + \hat{b} = [a^L, a^M, a^U] + [b^L, b^M, b^U] = [a^L + b^L, a^M + b^M, a^U + b^U]$;
2) $\lambda\hat{a} = \lambda[a^L, a^M, a^U] = [\lambda a^L, \lambda a^M, \lambda a^U]$;
3) $\hat{a} \times \hat{b} = [a^L, a^M, a^U] \times [b^L, b^M, b^U] = [a^L b^L, a^M b^M, a^U b^U]$;
4) $\frac{1}{\hat{a}} = \frac{1}{[a^L, a^M, a^U]} = \left[\frac{1}{a^U}, \frac{1}{a^M}, \frac{1}{a^L}\right]$.

In order to compare two triangular fuzzy numbers, Xu [22] provided the following definition:

Definition 5. Let $\hat{a} = [a^L, a^M, a^U]$ and $\hat{b} = [b^L, b^M, b^U]$ be two triangular fuzzy numbers, then the degree of possibility of $\hat{a} \geq \hat{b}$ is defined as follows:

$$\begin{aligned} p(\hat{a} \geq \hat{b}) = \delta \max &\left\{1 - \max\left(\frac{b^M - a^L}{a^M - a^L + b^M - b^L}, 0\right), 0\right\} \\ +(1-\delta)\max &\left\{1 - \max\left(\frac{b^U - a^M}{a^U - a^M + b^U - b^M}, 0\right), 0\right\}, \delta \in [0,1] \end{aligned} \tag{7}$$

which satisfies the following properties:

$$0 \leq p(\hat{a} \geq \hat{b}) \leq 1,\ p(\hat{a} \geq \hat{a}) = 0.5,\ p(\hat{a} \geq \hat{b}) + p(\hat{b} \geq \hat{a}) = 1. \tag{8}$$

In the following, we shall give a simple procedure for ranking of the triangular fuzzy numbers $\hat{a}_i$ $(i = 1, 2, \ldots, n)$. First, by using Equation (4), we compare each $\hat{a}_i$ with all the $\hat{a}_j$ $(j = 1, 2, \ldots, n)$, for simplicity, let $p_{ij} = p(\hat{a}_i \geq \hat{a}_j)$, then we develop a possibility matrix [7, 23] as

$$P = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix}, \tag{9}$$

where $p_{ij} \geq 0$, $p_{ij} + p_{ji} = 1$, $p_{ii} = \frac{1}{2}$, $i, j = 1, 2, \ldots, n$. Summing all elements in each line of matrix P, we have $p_i = \sum_{j=1}^n p_{ij}$, $i = 1, 2, \ldots, n$. Then, in accordance with the values of p_i $(i = 1, 2, \ldots, n)$, we rank the $\hat{a}_i$ $(i = 1, 2, \ldots, n)$ in descending order.

Based on operational laws of triangular fuzzy numbers, we extend the WCHM operator (3) to fuzzy environment:

Definition 6. Let $\hat{a}_j = [a_j^L, a_j^M, a_j^U]$ $(j = 1, 2, \ldots, n)$ be a collection of triangular fuzzy numbers, and let FWCHM : $\Omega^n \to \Omega$, if

$$\begin{aligned} \mathrm{FWCHM}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) &= \frac{\sum_{j=1}^n w_j \hat{a}_j^2}{\sum_{j=1}^n w_j \hat{a}_j} \\ &= \left[\frac{\sum_{j=1}^n w_j (a_j^L)^2}{\sum_{j=1}^n w_j a_j^U}, \frac{\sum_{j=1}^n w_j (a_j^M)^2}{\sum_{j=1}^n w_j a_j^M}, \frac{\sum_{j=1}^n w_j (a_j^U)^2}{\sum_{j=1}^n w_j a_j^L}\right] \end{aligned} \tag{10}$$

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where $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of $\hat{a}_j$ $(j = 1, 2, \ldots, n)$, with $w_j \geq 0$ and $\sum_{j=1}^n w_j = 1$, then FWCHM is called a fuzzy weighted contra-harmonic mean (FWCHM) operator.

Especially, if $w_i = 1$, $w_j = 0$, $j \neq i$, then $\text{FWCHM}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) = \hat{a}_i$; if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then the FWCHM operator reduces to the fuzzy contra-harmonic mean (FCHM) operator:

$$\begin{aligned} \text{FCHM}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) &= \frac{\sum_{j=1}^n \hat{a}_j^2}{\sum_{j=1}^n \hat{a}_j} \\ &= \left[\frac{\sum_{j=1}^n (a_j^L)^2}{\sum_{j=1}^n a_j^U}, \frac{\sum_{j=1}^n (a_j^M)^2}{\sum_{j=1}^n a_j^M}, \frac{\sum_{j=1}^n (a_j^U)^2}{\sum_{j=1}^n a_j^L} \right]. \quad (11) \end{aligned}$$

If the triangular fuzzy numbers $\hat{a}_j = [a_j^L, a_j^M, a_j^U]$ $(j = 1, 2, \ldots, n)$ reduce to the interval numbers $\tilde{a}_j = [a_j^L, a_j^U]$ $(j = 1, 2, \ldots, n)$, then the FWCHM operator (10) reduces to the uncertain weighted contra-harmonic mean (UWCHM) operator:

$$\begin{aligned} \text{UWCHM}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) &= \frac{\sum_{j=1}^n w_j \tilde{a}_j^2}{\sum_{j=1}^n w_j \tilde{a}_j} \\ &= \left[\frac{\sum_{j=1}^n w_j (a_j^L)^2}{\sum_{j=1}^n w_j a_j^U}, \frac{\sum_{j=1}^n w_j (a_j^U)^2}{\sum_{j=1}^n w_j a_j^L} \right]. \quad (12) \end{aligned}$$

If $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then the UWCHM operator reduces to the uncertain contra-harmonic mean (UCHM) operator:

$$\text{UCHM}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \frac{\sum_{j=1}^n \tilde{a}_j^2}{\sum_{j=1}^n \tilde{a}_j} = \left[\frac{\sum_{j=1}^n (a_j^L)^2}{\sum_{j=1}^n a_j^U}, \frac{\sum_{j=1}^n (a_j^U)^2}{\sum_{j=1}^n a_j^L} \right]. \quad (13)$$

If $a_j^L = a_j^U = a_j$, for all j, then Eqs. (12) and (13), respectively, reduces to the WCHM operator (3) and the CHM operator (4).

Example 1. Given a collection of triangular fuzzy numbers: $\hat{a}_1 = [2, 3, 4]$, $\hat{a}_2 = [1, 2, 4]$, $\hat{a}_3 = [2, 4, 6]$, $\hat{a}_4 = [1, 3, 5]$, let $w = (0.3, 0.1, 0.2, 0.4)^T$ be the weight vector of $\hat{a}_i$ $(i = 1, 2, 3, 4)$, then by Eq. (10), we have

$$\text{FWCHM}(\hat{a}_1, \hat{a}_2, \hat{a}_3, \hat{a}_4) = [0.5208, 3.1935, 15.7333].$$

Based on the OWA and FCHM operators, we define a fuzzy ordered weighted contra-hamonic mean (FOWCHM) operator as below:

Definition 7. Let $\hat{a}_j = [a_j^L, a_j^M, a_j^U]$ $(j = 1, 2, \ldots, n)$ be a collection of triangular fuzzy numbers. A fuzzy ordered weighted contra-hamonic mean (FOWCHM) operator of dimension n is a mapping FOWCHM : $\Omega^n \to \Omega$,

that has an associated vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ such that $\omega_j \geq 0$ and $\sum_{j=1}^n \omega_j = 1$. Furthermore,

$$\begin{aligned}\mathrm{FOWCHM}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) &= \frac{\sum_{j=1}^n \omega_j \hat{a}^2_{\sigma(j)}}{\sum_{j=1}^n \omega_j \hat{a}_{\sigma(j)}} \\ &= \left[\frac{\sum_{j=1}^n \omega_j (a^L_{\sigma(j)})^2}{\sum_{j=1}^n \omega_j a^U_{\sigma(j)}}, \frac{\sum_{j=1}^n \omega_j (a^M_{\sigma(j)})^2}{\sum_{j=1}^n \omega_j a^M_{\sigma(j)}}, \frac{\sum_{j=1}^n \omega_j (a^U_{\sigma(j)})^2}{\sum_{j=1}^n \omega_j a^L_{\sigma(j)}} \right], \quad (14)\end{aligned}$$

where $\hat{a}_{\sigma(j)} = [a^L_{\sigma(j)}, a^M_{\sigma(j)}, a^U_{\sigma(j)}]$ $(j = 1, 2, \ldots, n)$, and $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is a permutation of $(1, 2, \ldots, n)$ such that $\hat{a}_{\sigma(j-1)} \geq \hat{a}_{\sigma(j)}$ for all $j = 2, 3, \ldots, n$.

If there is a tie between $\hat{a}_i$ and $\hat{a}_j$, then we replace each of $\hat{a}_i$ and $\hat{a}_j$ by their average $(\hat{a}_i + \hat{a}_j)/2$ in process of aggregation. If k items are tied, then we replace these by k replicas of their average. The weighting vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ can be determined by using some weight determining methods like the normal distribution based method, see Refs [11, 21, 29] for more details.

Similar to the OWA operator, the FOWCHM operator has the following properties:

Theorem 1. Let $\hat{a}_j = [a^L_j, a^M_j, a^U_j]$ $(j = 1, 2, \ldots, n)$ be a collection of triangular fuzzy numbers, the following are valid:

(1) **Idempotency**: If all $\hat{a}_j$ $(j = 1, 2, \ldots, n)$ are equal, i.e., $\hat{a}_j = \hat{a}$, for all i, then

$$\mathrm{FOWCHM}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) = \hat{a}.$$

(2) **Boundedness**: Let $\hat{a}^- = \left[\min_j\{a^L_j\}, \min_j\{a^M_j\}, \min_j\{a^U_i\}\right]$ and $\hat{a}^+ = \left[\max_j\{a^L_j\}, \max_j\{a^M_j\}, \max_j\{a^U_j\}\right]$, then

$$\hat{a}^- \leq \mathrm{FOWCHM}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) \leq \hat{a}^+.$$

(3) **Monotonicity**: Let $\hat{a}^*_j = [a^{L*}_j, a^{M*}_j, a^{U*}_j]$ $(j = 1, 2, \ldots, n)$ be a collection of triangular fuzzy numbers, then if $a^L_j \leq a^{L*}_j$, $a^M_j \leq a^{M*}_j$ and $a^U_j \leq a^{U*}_j$ for all j, then

$$\mathrm{FOWCHM}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) \leq \mathrm{FOWCHM}(\hat{a}^*_1, \hat{a}^*_2, \ldots, \hat{a}^*_n).$$

(4) **Commutativity**: Let $\hat{a}'_j = [a^{L\prime}_j, a^{M\prime}_j, a^{U\prime}_j]$ $(j = 1, 2, \ldots, n)$ be a collection of triangular fuzzy numbers, then

$$\mathrm{FOWCHM}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) = \mathrm{FOWCHM}(\hat{a}'_1, \hat{a}'_2, \ldots, \hat{a}'_n),$$

where $(\hat{a}'_1, \hat{a}'_2, \ldots, \hat{a}'_n)$ is any permutation of $(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n)$.

Especially, if $\omega = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then the FOWCHM operator reduces to the FCHM operator; if the triangular fuzzy numbers $\hat{a}_j = [a^L_j, a^M_j, a^U_j]$ $(j = 1, 2, \ldots, n)$ reduce to the interval numbers $\tilde{a}_j = [a^L_j, a^U_j]$ $(j = 1, 2, \ldots, n)$,

then the FOWCHM operator reduces to the uncertain ordered weighted contra-harmonic mean (UOWCHM) operator:

$$\begin{aligned}\mathrm{UOWCHM}(\tilde{a}_i, \tilde{a}_2, \ldots, \tilde{a}_n) &= \frac{\sum_{j=1}^n \omega_j \tilde{a}_{\sigma(j)}^2}{\sum_{j=1}^n \omega_j \tilde{a}_{\sigma(j)}} \\ &= \left[\frac{\sum_{j=1}^n \omega_j (a_{\sigma(j)}^L)^2}{\sum_{j=1}^n \omega_j a_{\sigma(j)}^U}, \frac{\sum_{j=1}^n \omega_j (a_{\sigma(j)}^U)^2}{\sum_{j=1}^n \omega_j a_{\sigma(j)}^L}\right], \quad (15)\end{aligned}$$

where $\tilde{a}_{\sigma(j)} = [a_{\sigma(j)}^L, a_{\sigma(j)}^U]$, $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is a permutation of $(1, 2, \ldots, n)$ such that $\tilde{a}_{\sigma(j-1)} \geq \tilde{a}_{\sigma(j)}$ for all $j = 2, 3, \ldots, n$.

If $a_j^L = a_j^U = a_j$, for all j, then the UOWCHM operator reduces to the ordered weighted contra-harmonic mean (OWCHM) operator:

$$\mathrm{OWCHM}(a_1, a_2, \ldots, a_n) = \frac{\sum_{j=1}^n \omega_j b_j^2}{\sum_{j=1}^n \omega_j b_j}, \quad (16)$$

where b_j is the jth largest of a_j $(j = 1, 2, \ldots, n)$. The OWCHM operator (16) has some special cases:

(1) If $\omega = (1, 0, \ldots, 0)^T$, then

$$\mathrm{OWCHM}(a_1, a_2, \ldots, a_n) = \max\{a_i\} = b_1. \quad (17)$$

(2) If $\omega = (0, 0, \ldots, 1)^T$, then

$$\mathrm{OWCHM}(a_1, a_2, \ldots, a_n) = \min\{a_i\} = b_n. \quad (18)$$

(3) If $\omega_j = 1$, $\omega_i = 0$, $i \neq j$, then

$$b_n \leq \mathrm{OWCHM}(a_1, a_2, \ldots, a_n) = b_j \leq b_1. \quad (19)$$

(4) If $\omega = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then

$$\mathrm{OWCHM}(a_1, a_2, \ldots, a_n) = \frac{\sum_{j=1}^n b_j^2}{\sum_{j=1}^n b_j} = \frac{\sum_{j=1}^n a_j^2}{\sum_{j=1}^n a_j} = \mathrm{CHM}(a_1, a_2, \ldots, a_n). \quad (20)$$

Example 2. Given a collection of triangular fuzzy numbers: $\hat{a}_1 = [3, 4, 6]$, $\hat{a}_2 = [1, 2, 4]$, $\hat{a}_3 = [2, 4, 5]$, $\hat{a}_4 = [1, 3, 5]$, and $\hat{a}_5 = [2, 5, 7]$. To rank these triangular fuzzy numbers, we first compare each triangular fuzzy number $\hat{a}_i$ with all triangular fuzzy numbers $\hat{a}_j$ $(j = 1, 2, 3, 4, 5)$ by using Eq. (9) (without loss of generality, set $\delta = 0.5$), let $p_{ij} = p(\hat{a}_i \geq \hat{a}_j)$ $(j = 1, 2, 3, 4, 5)$, then we utilize these possibility degrees to construct the following matrix $P = (p_{ij})_{5\times 5}$:

$$P = \begin{pmatrix} 0.5000 & 1 & 0.6667 & 0.8750 & 0.3750 \\ 0 & 0.5000 & 0 & 0.2917 & 0 \\ 0.3333 & 1 & 0.5000 & 0.7083 & 0.2000 \\ 0.1250 & 0.7083 & 0.2917 & 0.5000 & 0.1000 \\ 0.6250 & 1 & 0.8000 & 0.9000 & 0.5000 \end{pmatrix}.$$

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Summing all elements in each line of matrix P, we have

$$p_1 = 3.4167,\ p_2 = 0.7917,\ p_3 = 2.7417,\ p_4 = 1.7250,\ p_5 = 3.8250$$

and then we rank the triangular fuzzy number $\hat{a}_i$ $(i = 1, 2, 3, 4, 5)$ in descending order in accordance with the values of p_i $(i = 1, 2, 3, 4, 5)$:

$$\hat{a}_{\sigma(1)} = \hat{a}_5,\ \hat{a}_{\sigma(2)} = \hat{a}_1,\ \hat{a}_{\sigma(3)} = \hat{a}_3,\ \hat{a}_{\sigma(4)} = \hat{a}_4,\ \hat{a}_{\sigma(5)} = \hat{a}_2.$$

Suppose that the weighting vector $\omega = (\omega_1, \omega_2, \omega_3, \omega_4, \omega_5)^T$ of the FOWCHM operator is $\omega = (0.1117, 0.2365, 0.3036, 0.3265, 0.1117)^T$ (derived by the normal distribution based method [21]), then by Eq. (14), we get

$$\mathrm{FOWCHM}(\hat{a}_1, \hat{a}_2, \hat{a}_3, \hat{a}_4, \hat{a}_5) == [0.7292, 3.7787, 15.9364].$$

Clearly, the fundamental characteristic of the FWCHM operator is that it considers the importance of each given triangular fuzzy number, whereas the fundamental characteristic of the FOWCHM operator is the reordering step, and it weights all the ordered positions of the triangular fuzzy numbers instead of weighting the given triangular fuzzy numbers themselves. By combining the advantages of the FWCHM and FOWCHM operators, in the following, we develop a fuzzy hybrid contra-harmonic mean (FHCHM) operator that weights both the given triangular fuzzy numbers and their ordered positions.

Definition 8. Let $\hat{a}_j = [a_j^L, a_j^M, a_j^U]$ $(j = 1, 2, \ldots, n)$ be a collection of triangular fuzzy numbers. A fuzzy hybrid contra-harmonic mean (FHCHM) operator of dimension n is a mapping $\mathrm{FHCHM} : \Omega^n \to \Omega$, which has an associated vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ with $\omega_j \geq 0$ and $\sum_{j=1}^n \omega_j = 1$, such that

$$\begin{aligned}\mathrm{FHCHM}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) &= \frac{\sum_{j=1}^n \omega_j \dot{\hat{a}}_{\sigma(j)}^2}{\sum_{j=1}^n \omega_j \dot{\hat{a}}_{\sigma(j)}} \\ &= \left[\frac{\sum_{j=1}^n \omega_j (\dot{a}_{\sigma(j)}^L)^2}{\sum_{j=1}^n \omega_j \dot{a}_{\sigma(j)}^U}, \frac{\sum_{j=1}^n \omega_j (\dot{a}_{\sigma(j)}^M)^2}{\sum_{j=1}^n \omega_j \dot{a}_{\sigma(j)}^M}, \frac{\sum_{j=1}^n \omega_j (\dot{a}_{\sigma(j)}^U)^2}{\sum_{j=1}^n \omega_j \dot{a}_{\sigma(j)}^L}\right], \quad (21)\end{aligned}$$

where $\dot{\hat{a}}_{\sigma(j)} = [\dot{a}_{\sigma(j)}^L, \dot{a}_{\sigma(j)}^M, \dot{a}_{\sigma(j)}^U]$ is the jth largest of the weighted triangular fuzzy numbers $\dot{\hat{a}}_j$ $(\dot{\hat{a}}_j = nw_j\hat{a}_j,\ j = 1, 2, \ldots, n)$, $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of $\hat{a}_j$ $(j = 1, 2, \ldots, n)$ with $w_j \geq 0$ and $\sum_{j=1}^n w_j = 1$, and n is the balancing coefficient.

Especially, if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then $\dot{\hat{a}}_j = \hat{a}_j$, for all j, in this case, the FHCHM operator reduces to the FOWCHM operator. Moreover, if the triangular fuzzy numbers $\hat{a}_j = [a_j^L, a_j^M, a_j^U]$ $(j = 1, 2, \ldots, n)$ reduce to the interval numbers $\tilde{a}_j = [a_j^L, a_j^U]$ $(j = 1, 2, \ldots, n)$, then the FHCHM operator reduces to the uncertain hybrid contra-harmonic mean (UHCHM) operator:

$$\mathrm{UHCHM}(\tilde{a}_i, \tilde{a}_2, \ldots, \tilde{a}_n) = \frac{\sum_{j=1}^n \omega_j \dot{\tilde{a}}_{\sigma(j)}^2}{\sum_{j=1}^n \omega_j \dot{\tilde{a}}_{\sigma(j)}}, \quad (22)$$

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where $\dot{\tilde{a}}_{\sigma(j)}$ is the jth largest of the weighted interval numbers $\dot{\tilde{a}}_j$ ($\dot{\tilde{a}}_j = nw_j\tilde{a}_j$, $j = 1, 2, \ldots, n$), $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of $\tilde{a}_j$ ($j = 1, 2, \ldots, n$) with $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$, and n is the balancing coefficient. Especially, if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then $\dot{\tilde{a}}_j = \tilde{a}_j$, $j = 1, 2, \ldots, n$, in this case, the UHCHM operator reduces to the UOWCHM operator.

If $a_j^L = a_j^U = a_j$, for all j, then the UHCHM operator reduces to the hybrid contra-harmonic mean (HCHM) operator:

$$\mathrm{HCHM}(a_i, a_2, \ldots, a_n) = \frac{\sum_{j=1}^{n} \omega_j \dot{a}_{\sigma(j)}^2}{\sum_{j=1}^{n} \omega_j \dot{a}_{\sigma(j)}}, \tag{23}$$

where $\dot{a}_{\sigma(j)}$ is the jth largest of the weighted interval numbers $\dot{a}_j$ ($\dot{a}_j = nw_j a_j$, $j = 1, 2, \ldots, n$), $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of a_j ($j = 1, 2, \ldots, n$) with $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$, and n is the balancing coefficient. Especially, if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then $\dot{a}_j = a_j$, $j = 1, 2, \ldots, n$, in this case, the HCHM operator reduces to the OWCHM operator.

Example 3. Given a collection of triangular fuzzy numbers: $\hat{a}_1 = [2, 4, 5]$, $\hat{a}_2 = [1, 3, 4]$, $\hat{a}_3 = [2, 3, 5]$, $\hat{a}_4 = [3, 4, 5]$, and $\hat{a}_5 = [2, 5, 8]$, and let $w = (0.20, 0.25, 0.15, 0.25, 0.15)^T$ be the weight vector of $\hat{a}_j$ ($j = 1, 2, 3, 4, 5$). Then we get the weighted triangular fuzzy numbers:

$$\begin{aligned} &\dot{\hat{a}}_1 = [2, 4, 5],\ \dot{\hat{a}}_2 = [1.25, 3.75, 5],\ \dot{\hat{a}}_3 = [1.5, 2.25, 3.75], \\ &\dot{\hat{a}}_4 = [3.75, 5, 6.25],\ \dot{\hat{a}}_5 = [1.5, 3.75, 6]. \end{aligned}$$

By using Eq. (9) (without loss of generality, set $\delta = 0.5$), we construct the following matrix:

$$P = \begin{pmatrix} 0.5000 & 0.5833 & 0.9545 & 0.0385 & 0.4864 \\ 0.4167 & 0.5000 & 0.8462 & 0 & 0.4154 \\ 0.0455 & 0.1538 & 0.5000 & 0 & 0.1250 \\ 0.9615 & 1 & 1 & 0.5000 & 0.8571 \\ 0.5136 & 0.5846 & 0.8750 & 0.1429 & 0.5000 \end{pmatrix}.$$

Summing all elements in each line of matrix P, we have

$$p_1 = 2.5628,\ p_2 = 2.1782,\ p_3 = 0.8243,\ p_4 = 4.3187,\ p_5 = 2.6160$$

and then we rank the triangular fuzzy number $\dot{\hat{a}}_j$ ($j = 1, 2, 3, 4, 5$) in descending order in accordance with the values of p_i ($i = 1, 2, 3, 4, 5$):

$$\dot{\hat{a}}_{\sigma(1)} = \dot{\hat{a}}_4,\ \dot{\hat{a}}_{\sigma(2)} = \dot{\hat{a}}_5,\ \dot{\hat{a}}_{\sigma(3)} = \dot{\hat{a}}_1,\ \dot{\hat{a}}_{\sigma(4)} = \dot{\hat{a}}_2,\ \dot{\hat{a}}_{\sigma(5)} = \dot{\hat{a}}_3.$$

Suppose that $w = (0.1117, 0.2365, 0.3036, 0.3265, 0.1117)^T$ is the weighting vector of the FHCHM operator, then by Eq. (21), we get

$$\begin{aligned} \mathrm{FHCHM}(\hat{a}_1, \hat{a}_2, \hat{a}_3, \hat{a}_4, \hat{a}_5) &= \frac{\sum_{j=1}^{5} \omega_j \dot{\hat{a}}_{\sigma(j)}^2}{\sum_{j=1}^{5} \omega_j \dot{\hat{a}}_{\sigma(j)}} \\ &= [0.7173, 3.9011, 15.4360]. \end{aligned}$$

4 Approaches to multiple attribute group decision making with triangular fuzzy information

For a group decision making with triangular fuzzy information, let $X = \{x_1, x_2, \ldots, x_n\}$ be a discrete set of n alternatives, and $G = \{G_1, G_2, \ldots, G_m\}$ be the set of m attributes, whose weight vector is $w = (w_1, w_2, \ldots, w_m)^T$ with $w_i \geq 0$ and $\sum_{i=1}^m w_i = 1$, and let $D = \{d_1, d_2, \ldots, d_s\}$ be the set of decision makers, whose weight vector is $v = (v_1, v_2, \ldots, v_s)^T$, where $v_k \geq 0$ and $\sum_{k=1}^s v_k = 1$. Suppose that $A^{(k)} = (\hat{a}_{ij}^{(k)})_{m\times n}$ is the decision matrix, where $\hat{a}_{ij}^{(k)} = [a_{ij}^{L(k)}, a_{ij}^{M(k)}, a_{ij}^{U(k)}]$ is an attribute value, which takes the form of triangular fuzzy number, of the alternative $x_j \in X$ with respect to the attribute $G_i \in G$.

Then, we utilize the FWCHM and FHCHM operators to propose an approach to multiple attribute group decision making with triangular fuzzy data, which involves the following steps:

Step 1. Normalize each attribute value $\hat{a}_{ij}^{(k)}$ in the matrix $A^{(k)}$ into a corresponding element in the matrix $R^{(k)} = (\hat{r}_{ij}^{(k)})_{m\times n}$ ($\hat{r}_{ij}^{(k)} = [r_{ij}^{L(k)}, r_{ij}^{M(k)}, r_{ij}^{U(k)}]$) using the following formulas:

$$\hat{r}_{ij}^{(k)} = \frac{\hat{a}_{ij}^{(k)}}{\sum_{j=1}^n \hat{a}_{ij}^{(k)}} = \left[\frac{a_{ij}^{L(k)}}{\sum_{j=1}^n a_{ij}^{U(k)}}, \frac{a_{ij}^{M(k)}}{\sum_{j=1}^n a_{ij}^{M(k)}}, \frac{a_{ij}^{U(k)}}{\sum_{j=1}^n a_{ij}^{L(k)}}\right], \quad \text{for benefit attribute } G_i, \tag{24}$$

$$\begin{aligned}\hat{r}_{ij}^{(k)} &= \frac{1/\hat{a}_{ij}^{(k)}}{\sum_{j=1}^n (1/\hat{a}_{ij}^{(k)})} \\ &= \left[\frac{1/a_{ij}^{U(k)}}{\sum_{j=1}^n (1/a_{ij}^{L(k)})}, \frac{1/a_{ij}^{M(k)}}{\sum_{j=1}^n (1/a_{ij}^{M(k)})}, \frac{1/a_{ij}^{L(k)}}{\sum_{j=1}^n (1/a_{ij}^{U(k)})}\right], \\ &\quad \text{for cost attribute } G_i,\end{aligned} \tag{25}$$

where $i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n,\ k = 1, 2, \ldots, s$.

Step 2. Utilize the FWCHM operator:

$$\begin{aligned}\hat{r}_j^{(k)} &= \text{FWCHM}(\hat{r}_{1j}^{(k)}, \hat{r}_{2j}^{(k)}, \ldots, \hat{r}_{mj}^{(k)}) = \frac{\sum_{i=1}^m w_i (\hat{r}_{ij}^{(k)})^2}{\sum_{i=1}^m w_i \hat{r}_{ij}^{(k)}} \\ &= \left[\frac{\sum_{i=1}^m w_i (r_{ij}^{L(k)})^2}{\sum_{i=1}^m w_i r_{ij}^{L(k)}}, \frac{\sum_{i=1}^m w_i (r_{ij}^{M(k)})^2}{\sum_{i=1}^m w_i r_{ij}^{M(k)}}, \frac{\sum_{i=1}^m w_i (r_{ij}^{U(k)})^2}{\sum_{i=1}^m w_i r_{ij}^{U(k)}}\right]\end{aligned} \tag{26}$$

to aggregate all the elements in the jth column of $R^{(k)}$ and get the overall attribute value $\hat{r}_j^{(k)}$ of the alternative x_j corresponding to the decision maker d_k.

Step 3. Utilize the FHCHM operator:

$$\hat{r}_j = \text{FHCHM}(\hat{r}_j^{(1)}, \hat{r}_j^{(2)}, \ldots, \hat{r}_j^{(s)}) = \frac{\sum_{k=1}^{s} \omega_k (\dot{\hat{r}}_j^{(\sigma(k))})^2}{\sum_{k=1}^{s} \omega_k \dot{\hat{r}}_j^{(\sigma(k))}}$$
$$= \left[\frac{\sum_{k=1}^{s} \omega_k (\dot{r}_j^{L(\sigma(k))})^2}{\sum_{k=1}^{s} \omega_k \dot{r}_j^{U(\sigma(k))}}, \frac{\sum_{k=1}^{s} \omega_k (\dot{r}_j^{M(\sigma(k))})^2}{\sum_{k=1}^{s} \omega_k \dot{r}_j^{M(\sigma(k))}}, \frac{\sum_{k=1}^{s} \omega_k (\dot{r}_j^{U(\sigma(k))})^2}{\sum_{k=1}^{s} \omega_k \dot{r}_j^{L(\sigma(k))}} \right] \quad (27)$$

to aggregate the overall attribute values $\hat{r}_j^{(k)}$ $(k = 1, 2, \ldots, s)$ corresponding to the decision maker d_k $(k = 1, 2, \ldots, s)$ and get the collective overall attribute value $\hat{r}_j$, where $\dot{\hat{r}}_j^{(\sigma(k))} = \left[\dot{r}_j^{L(\sigma(k))}, \dot{r}_j^{M(\sigma(k))}, \dot{r}_j^{U(\sigma(k))}\right]$ is the kth largest of the weighted data $\dot{\hat{r}}_j^{(k)}$ $(\dot{\hat{r}}_j^{(k)} = sv_k \hat{r}_j^{(k)}, k = 1, 2, \ldots, s)$, and $\omega = (\omega_1, \omega_2, \ldots, \omega_s)^T$ is the weighting vector of the FHCHM operator, with $\omega_k \geq 0$ and $\sum_{k=1}^{s} \omega_k = 1$.

Step 4. Compare each $\hat{r}_j$ with all $\hat{r}_i$ $(i = 1, 2, \ldots n)$ by using Eq. (9), and let $p_{ij} = p(\hat{r}_i \geq \hat{r}_j)$, and then construct a possibility matrix $P = (p_{ij})_{n \times n}$, where $p_{ij} \geq 0$, $p_{ij} + p_{ji} = 1$, $p_{ii} = 0.5$, $i, j = 1, 2, \ldots n$. Summing all elements in each line of matrix P, we have $p_i = \sum_{j=1}^{n} p_{ij}$, $i = 1, 2, \ldots, n$, and then reorder $\hat{r}_j$ $(j = 1, 2, \ldots, n)$ in descending order in accordance with the values of p_j $(j = 1, 2, \ldots, n)$.

Step 5. Rank all the alternatives x_j $(j = 1, 2, \ldots, n)$ by the ranking of $\hat{r}_j$ $(j = 1, 2, \ldots, n)$, and then select the most desirable one.

Step 6. End.

5 Illustrative example

In this section, we use a multiple attribute group decision making problem of determining what kind of air-conditioning systems should be installed in a library (adopted from [22, 32]) to illustrate the proposed approach.

A city is planning to build a municipal library. One of the problems facing the city development commissioner is to determine what kind of air-conditioning systems should be installed in the library. The contractor offers five feasible alternatives, which might be adapted to the physical structure of the library. The alternatives x_j $(j = 1, 2, 3, 4, 5)$ are to be evaluated using triangular fuzzy numbers by the three decision makers d_k $(k = 1, 2, 3)$ (whose weight vector is $v = (0.4, 0.3, 0.3)^T$) under three major impacts: economic, functional, and operational. Two monetary attributes and six nonmonetary attributes (that is, G_1: owning cost ($/ft^2), G_2: operating cost ($/ft^2), G_3: performance (*), G_4: noise level (Db), G_5: maintainability (*), G_6: reliability (%), G_7: flexibility (*), G_8: safety (*), where * unit is from $0 - 1$ scale, three attributes G_1, G_2, and G_4 are cost attributes, and the other five attributes are benefit attributes, suppose that the weight vector od the attributes G_i $(i = 1, 2, \ldots, 8)$ is $w = (0.05, 0.08, 0.14, 0.12, 0.18, 0.21, 0.05, 0.17)^T$) emerged from three impacts is Tables 1-3.

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Table 1: Triangular fuzzy number decision matrix $A^{(1)}$

	x_1	x_2	x_3	x_4	x_5
G_1	$[3.5, 4.0, 4.7]$	$[1.7, 2.0, 2.3]$	$[3.5, 3.8, 4.2]$	$[3.5, 3.8, 4.5]$	$[3.3, 3.8, 4.0]$
G_2	$[5.5, 6.0, 6.5]$	$[4.8, 5.1, 5.5]$	$[4.5, 5.2, 5.5]$	$[4.5, 4.7, 5.0]$	$[5.5, 5.7, 6.0]$
G_3	$[0.7, 0.8, 0.9]$	$[0.5, 0.56, 0.6]$	$[0.5, 0.6, 0.7]$	$[0.7, 0.85, 0.9]$	$[0.6, 0.7, 0.8]$
G_4	$[35, 40, 45]$	$[70, 73, 75]$	$[65, 68, 70]$	$[40, 42, 45]$	$[50, 55, 60]$
G_5	$[0.4, 0.45, 0.5]$	$[0.4, 0.44, 0.6]$	$[0.7, 0.76, 0.8]$	$[0.9, 0.97, 1.0]$	$[0.5, 0.54, 0.6]$
G_6	$[95, 98, 100]$	$[70, 73, 75]$	$[80, 83, 90]$	$[90, 93, 95]$	$[85, 90, 95]$
G_7	$[0.3, 0.35, 0.5]$	$[0.7, 0.75, 0.8]$	$[0.8, 0.9, 1.0]$	$[0.6, 0.75, 0.8]$	$[0.4, 0.5, 0.6]$
G_8	$[0.7, 0.74, 0.8]$	$[0.5, 0.53, 0.6]$	$[0.6, 0.68, 0.7]$	$[0.7, 0.8, 0.9]$	$[0.8, 0.85, 0.9]$

Table 2: Triangular fuzzy number decision matrix $A^{(2)}$

	x_1	x_2	x_3	x_4	x_5
G_1	$[4.0, 4.3, 4.5]$	$[2.1, 2.2, 2.4]$	$[5.0, 5.1, 5.2]$	$[4.3, 4.4, 4.5]$	$[3.0, 3.3, 3.5]$
G_2	$[6.0, 6.3, 6.5]$	$[5.0, 5.1, 5.2]$	$[4.5, 4.7, 5.0]$	$[5.0, 5.1, 5.3]$	$[7.0, 7.5, 8.0]$
G_3	$[0.7, 0.8, 0.9]$	$[0.4, 0.5, 0.6]$	$[0.5, 0.55, 0.6]$	$[0.7, 0.75, 0.8]$	$[0.7, 0.8, 0.9]$
G_4	$[37, 38, 39]$	$[70, 73, 75]$	$[65, 66, 67]$	$[40, 42, 45]$	$[50, 52, 55]$
G_5	$[0.4, 0.5, 0.6]$	$[0.5, 0.55, 0.6]$	$[0.8, 0.85, 0.9]$	$[0.8, 0.95, 1.0]$	$[0.4, 0.44, 0.5]$
G_6	$[92, 93, 95]$	$[70, 75, 80]$	$[83, 84, 85]$	$[90, 91, 92]$	$[90, 93, 95]$
G_7	$[0.4, 0.45, 0.5]$	$[0.8, 0.85, 0.9]$	$[0.7, 0.73, 0.8]$	$[[0.7, 0.85, 0.9]$	$[0.4, 0.45, 0.5]$
G_8	$[0.6, 0.7, 0.8]$	$[0.6, 0.65, 0.7]$	$[0.5, 0.6, 0.7]$	$[0.7, 0.76, 0.8]$	$[0.7, 0.8, 0.9]$

Table 3: Triangular fuzzy number decision matrix $A^{(3)}$

	x_1	x_2	x_3	x_4	x_5
G_1	$[4.3, 4.4, 4.6]$	$[2.2, 2.4, 2.5]$	$[4.5, 4.8, 5.0]$	$[4.7, 4.9, 5.0]$	$[3.1, 3.2, 3.4]$
G_2	$[6.4, 6.7, 7.0]$	$[5.0, 5.2, 5.5]$	$[4.7, 4.8, 4.9]$	$[5.5, 5.7, 6.0]$	$[6.0, 6.5, 7.0]$
G_3	$[0.8, 0.85, 0.9]$	$[0.5, 0.6, 0.7]$	$[0.6, 0.7, 0.8]$	$[0.7, 0.8, 0.9]$	$[0.7, 0.75, 0.8]$
G_4	$[36, 38, 40]$	$[72, 73, 75]$	$[67, 68, 70]$	$[45, 48, 50]$	$[55, 57, 60]$
G_5	$[0.4, 0.46, 0.5]$	$[0.4, 0.45, 0.6]$	$[0.8, 0.95, 1.0]$	$[0.8, 0.85, 0.9]$	$[0.5, 0.55, 0.6]$
G_6	$[93, 94, 95]$	$[77, 78, 80]$	$[85, 87, 90]$	$[90, 94, 95]$	$[90, 96, 100]$
G_7	$[0.4, 0.5, 0.6]$	$[0.8, 0.9, 1.0]$	$[0.8, 0.86, 0.9]$	$[0.6, 0.7, 0.8]$	$[0.5, 0.57, 0.6]$
G_8	$[0.7, 0.78, 0.8]$	$[0.5, 0.55, 0.6]$	$[0.6, 0.68, 0.7]$	$[0.8, 0.85, 0.9]$	$[0.8, 0.85, 0.9]$

To select the best air-conditioning system, we utilize the approach based on the FWCHM and FHCHM operators, the main steps are as follows:

Step 1. By using Eqs. (24) and (25), we normalize each attribute value $\hat{a}_{ij}^{(k)}$ in the matrices $A^{(k)}$ $(k = 1, 2, 3)$ into the corresponding element in the matrices $R^{(k)} = (\hat{r}_{ij})_{8\times5}$ $(k = 1, 2, 3)$ (Tables 4-6):

Step 2: Utilize the FWCHM operator (26) to aggregate all elements in the jth column $R^{(k)}$ and get the overall attribute value $\hat{r}_j^{(k)}$:

$$\hat{r}_1^{(1)} = [0.1263, 0.2076, 0.3530]\,,\ \hat{r}_2^{(1)} = [0.1040, 0.1804, 0.3272]\,,$$
$$\hat{r}_3^{(1)} = [0.1234, 0.2013, 0.3325]\,,\ \hat{r}_4^{(1)} = [0.1528, 0.2452, 0.3777]\,,$$
$$\hat{r}_5^{(1)} = [0.1208, 0.2007, 0.3376]\,,\ \hat{r}_1^{(2)} = [0.1316, 0.2084, 0.3413]\,,$$
$$\hat{r}_2^{(2)} = [0.1229, 0.1928, 0.3057]\,,\ \hat{r}_3^{(2)} = [0.1346, 0.2016, 0.3138]\,,$$
$$\hat{r}_4^{(2)} = [0.1483, 0.2365, 0.3638]\,,\ \hat{r}_5^{(2)} = [0.1293, 0.2021, 0.3229]\,,$$

Table 4: Normalized triangular fuzzy number decision matrix $R^{(1)}$

	x_1	x_2	x_3	x_4	x_5
G_1	$[0.12, 0.16, 0.21]$	$[0.25, 0.32, 0.43]$	$[0.14, 0.17, 0.21]$	$[0.13, 0.17, 0.21]$	$[0.14, 0.17, 0.22]$
G_2	$[0.15, 0.18, 0.21]$	$[0.18, 0.21, 0.24]$	$[0.18, 0.20, 0.25]$	$[0.20, 0.23, 0.25]$	$[0.16, 0.19, 0.21]$
G_3	$[0.18, 0.23, 0.30]$	$[0.13, 0.16, 0.20]$	$[0.13, 0.17, 0.23]$	$[0.18, 0.24, 0.30]$	$[0.15, 0.20, 0.27]$
G_4	$[0.22, 0.26, 0.32]$	$[0.13, 0.14, 0.16]$	$[0.14, 0.15, 0.17]$	$[0.22, 0.25, 0.28]$	$[0.16, 0.19, 0.23]$
G_5	$[0.11, 0.14, 0.17]$	$[0.11, 0.14, 0.21]$	$[0.20, 0.24, 0.28]$	$[0.26, 0.31, 0.34]$	$[0.14, 0.17, 0.21]$
G_6	$[0.21, 0.22, 0.24]$	$[0.15, 0.17, 0.18]$	$[0.18, 0.19, 0.21]$	$[0.20, 0.21, 0.23]$	$[0.19, 0.21, 0.23]$
G_7	$[0.08, 0.11, 0.18]$	$[0.19, 0.23, 0.29]$	$[0.22, 0.28, 0.36]$	$[0.16, 0.23, 0.29]$	$[0.11, 0.15, 0.21]$
G_8	$[0.18, 0.21, 0.24]$	$[0.13, 0.15, 0.18]$	$[0.15, 0.19, 0.21]$	$[0.18, 0.22, 0.27]$	$[0.21, 0.24, 0.27]$

Table 5: Normalized triangular fuzzy number decision matrix $R^{(2)}$

	x_1	x_2	x_3	x_4	x_5
G_1	$[0.15, 0.16, 0.19]$	$[0.28, 0.32, 0.36]$	$[0.13, 0.14, 0.15]$	$[0.15, 0.16, 0.17]$	$[0.19, 0.21, 0.25]$
G_2	$[0.17, 0.18, 0.19]$	$[0.21, 0.22, 0.23]$	$[0.21, 0.24, 0.26]$	$[0.20, 0.22, 0.23]$	$[0.13, 0.15, 0.17]$
G_3	$[0.18, 0.24, 0.30]$	$[0.11, 0.15, 0.20]$	$[0.13, 0.16, 0.20]$	$[0.18, 0.22, 0.27]$	$[0.18, 0.24, 0.30]$
G_4	$[0.25, 0.27, 0.29]$	$[0.13, 0.14, 0.15]$	$[0.15, 0.15, 0.16]$	$[0.22, 0.24, 0.27]$	$[0.18, 0.20, 0.21]$
G_5	$[0.11, 0.15, 0.21]$	$[0.14, 0.17, 0.21]$	$[0.22, 0.26, 0.31]$	$[0.22, 0.29, 0.34]$	$[0.11, 0.13, 0.17]$
G_6	$[0.21, 0.21, 0.22]$	$[0.16, 0.17, 0.19]$	$[0.19, 0.19, 0.20]$	$[0.20, 0.21, 0.22]$	$[0.20, 0.21, 0.22]$
G_7	$[0.11, 0.14, 0.17]$	$[0.22, 0.26, 0.30]$	$[0.19, 0.22, 0.27]$	$[0.19, 0.26, 0.30]$	$[0.19, 0.14, 0.17]$
G_8	$[0.15, 0.20, 0.26]$	$[0.15, 0.19, 0.23]$	$[0.13, 0.17, 0.23]$	$[0.18, 0.22, 0.26]$	$[0.18, 0.23, 0.29]$

Table 6: Normalized triangular fuzzy number decision matrix $R^{(3)}$

	x_1	x_2	x_3	x_4	x_5
G_1	$[0.15, 0.17, 0.18]$	$[0.28, 0.30, 0.35]$	$[0.14, 0.15, 0.17]$	$[0.14, 0.15, 0.16]$	$[0.20, 0.23, 0.25]$
G_2	$[0.16, 0.17, 0.19]$	$[0.20, 0.22, 0.24]$	$[0.22, 0.24, 0.25]$	$[0.18, 0.20, 0.22]$	$[0.16, 0.17, 0.20]$
G_3	$[0.20, 0.23, 0.27]$	$[0.12, 0.16, 0.21]$	$[0.15, 0.19, 0.24]$	$[0.17, 0.22, 0.27]$	$[0.17, 0.20, 0.24]$
G_4	$[0.26, 0.28, 0.31]$	$[0.14, 0.15, 0.16]$	$[0.15, 0.16, 0.17]$	$[0.21, 0.22, 0.25]$	$[0.17, 0.19, 0.20]$
G_5	$[0.11, 0.14, 0.17]$	$[0.11, 0.14, 0.21]$	$[0.22, 0.29, 0.34]$	$[0.22, 0.26, 0.31]$	$[0.14, 0.17, 0.21]$
G_6	$[0.20, 0.21, 0.22]$	$[0.17, 0.17, 0.18]$	$[0.18, 0.19, 0.21]$	$[0.20, 0.21, 0.22]$	$[0.20, 0.21, 0.23]$
G_7	$[0.10, 0.14, 0.19]$	$[0.21, 0.25, 0.32]$	$[0.21, 0.24, 0.29]$	$[0.15, 0.20, 0.26]$	$[0.13, 0.16, 0.19]$
G_8	$[0.18, 0.21, 0.24]$	$[0.13, 0.15, 0.18]$	$[0.15, 0.18, 0.21]$	$[0.21, 0.23, 0.26]$	$[0.21, 0.23, 0.26]$

$$\hat{r}_1^{(3)} = [0.1479, 0.2093, 0.3008]\,,\ \hat{r}_2^{(3)} = [0.1196, 0.1815, 0.2989]\,,$$
$$\hat{r}_3^{(3)} = [0.1355, 0.2170, 0.3376]\,,\ \hat{r}_4^{(3)} = [0.1530, 0.2235, 0.3340]\,,$$
$$\hat{r}_5^{(3)} = [0.1409, 0.2004, 0.2921]\,.$$

Step 3. Utilize the FHCHM operator (27) (suppose that its weight vector is $\omega = (0.243, 0.514, 0.243)^T$, let $\delta = 0.5$) to aggregate the overall attribute value $\hat{r}_j^{(k)}$ $(k = 1, 2, 3)$ corresponding to the decision maker d_k $(k = 1, 2, 3)$, and get the collective overall attribute value $\hat{r}_j$:

$$\hat{r}_1 = [0.0634, 0.2542, 1.0615]\,,\ \hat{r}_2 = [0.0507, 0.2243, 1.0426]\,,$$
$$\hat{r}_3 = [0.0631, 0.2484, 0.9940]\,,\ \hat{r}_4 = [0.0774, 0.2948, 1.0861]\,,$$
$$\hat{r}_5 = [0.0618, 0.2457, 1.0001]\,.$$

Step 4. Compare each $\hat{r}_j$ with all $\hat{r}_i$ $(i = 1, 2, 3, 4, 5)$ by using Eq. (9) (without loss of generality, set $\delta = 0.5$), and let $p_{ij} = p(\hat{r}_i \geq \hat{r}_j)$, and then construct a possibility matrix:

$$P = \begin{pmatrix} 0.5 & 0.5367 & 0.5158 & 0.4563 & 0.5179 \\ 0.4633 & 0.5 & 0.4785 & 0.4201 & 0.4806 \\ 0.4842 & 0.5215 & 0.5 & 0.4397 & 0.5021 \\ 0.5437 & 0.5799 & 0.5603 & 0.5 & 0.5622 \\ 0.4821 & 0.5194 & 0.4979 & 0.4378 & 0.5 \end{pmatrix}.$$

Summing all elements in each line of matrix P, we have

$$p_1 = 2.5267, p_2 = 2.3426, p_3 = 2.4475, p_4 = 2.7460, p_5 = 2.4372$$

and then reorder $\hat{r}_j$ $(j = 1, 2, 3, 4, 5)$ in descending order in accordance with the values of p_j $(j = 1, 2, 3, 4, 5)$:

$$\hat{r}_4 > \hat{r}_1 > \hat{r}_3 > \hat{r}_5 > \hat{r}_2.$$

Step 5. Rank all the alternatives x_j $(j = 1, 2, 3, 4, 5)$ by the ranking of $\hat{r}_j$ $(j = 1, 2, 3, 4, 5)$:

$$x_4 \succ x_1 \succ x_3 \succ x_5 \succ x_2$$

and thus the most desirable alternative is x_4.

From the above analysis, the results obtained by the proposed approach are slightly different to the ones obtained Xu's [22] and Park and Park's [16] approaches (see Table 7). Each of methods has its advantages and disadvantages and none of them can always perform better than the others in any situations. It perfectly depends on how we look at things, and not on how they are themselves. As we can see, depending on aggregation operators used, the ranking of the alternatives is slightly different. Therefore, depending on aggregation operators used, the results may lead to different decisions. However, the best alternative is x_4.

Table 7: Comparison of the proposed approach with other approaches

	Xu' approach [22]	Park and Park's approach [16]	Proposed approach
Solution method			
Aggregation stage	FWHM operator	GFWBHM operator	FWCHM operator
Exploitation stage	FHHM operator	GFOWBHM operator	FHCHM operator
Ranking of alternatives	$x_4 \succ x_5 \succ x_3 \succ x_1 \succ x_2$	$x_4 \succ x_3 \succ x_5 \succ x_1 \succ x_2$	$x_4 \succ x_1 \succ x_3 \succ x_5 \succ x_2$

6 Conclusions

In this paper, we have extended the traditional harmonic mean to fuzzy environments and introduced the FWCHM operator. Based on the FWCHM operator

and Yager's OWA operator [26], we have developed the FOWCHM operator and the FHCHM operator. It has been shown that both the FOWCHM and FWCHM operators are the special cases of the FHCHM operator. It has also been pointed out that if all the input fuzzy data are reduced to the interval or numerical data, then the FHCHM operator is reduced to the UHCHM operator and the HCHM operator, respectively. In these situations, the WCHM operator and the OWCHM operator are the two special cases of the HCHM operator; the UWCHM operator and the UOWCHM operator are the two special cases of the UHCHM operator.Moreover, we have applied these operators to multiple attribute group decision making. The application of the developed operators in other fields is a promising direction for future research.

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Difference of a new integral-type operator from H^∞ to the Bloch type space in the unit ball

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[1]

Abstract

Let g be a holomorphic function and φ a holomorphic self-map of the unit ball $\mathbb{B}$. In this paper, we characterize the difference of a new integral-type operator P_φ^g from the space H^∞ of all bounded holomorphic functions to the Bloch type space on the unit ball, and give some necessary and sufficient conditions for the difference to be bounded and compact.

1 Introduction

Let $\mathbb{B}$ be the unit ball of $\mathbb{C}^n$ with boundary S. If $n = 1$ we will denote the unit disk $\mathbb{B}^1$ simply by $\mathbb{D}$, with boundary $\partial\mathbb{D}$. The class of all holomorphic functions on $\mathbb{B}$ will be denoted by $H(\mathbb{B})$, and by $S(\mathbb{B})$ denote the collection of all holomorphic self-maps of $\mathbb{B}$.

A positive continuous function v on $[0, 1)$ is called normal (see, [21]), if there exist three constants $0 \le \delta < 1$, and $0 < a < b < \infty$, such that for $r \in [\delta, 1)$

$$\frac{v(r)}{(1-r)^a} \downarrow 0, \qquad \frac{v(r)}{(1-r)^b} \uparrow \infty$$

as $r \to 1$.

Assume v is normal, we recall that the weighted-type space H_v^∞ consists of all $f \in H(\mathbb{B})$ such that

$$\|f\|_{H_v^\infty} = \sup_{z\in\mathbb{B}} v(z)|f(z)| < \infty.$$

When $v(z) = 1$, we know that $H_v^\infty = H^\infty$, that is

$$H^\infty = \{f \in H(\mathbb{B}), \sup_{z\in\mathbb{B}} |f(z)| < \infty\}.$$

For $z = (z_1, ..., z_n)$ and $w = (w_1, ..., w_n)$ in $\mathbb{C}^n$, we denote the inner product of z and w by

$$\langle z, w\rangle = z_1\overline{w}_1 + ... + z_n\overline{w}_n,$$

and we write $|z| = \sqrt{\langle z, z\rangle} = \sqrt{|z_1|^2 + ... + |z_n|^2}$. For $f \in H(\mathbb{B})$, let

$$\nabla f(z) = \Big(\frac{\partial f}{\partial z_1}(z), ..., \frac{\partial f}{\partial z_n}(z)\Big),\ \Re f(z) = \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}(z)$$

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be the complex gradient of f and the radial derivative of f.

For a real number $\alpha > 0$,the $\alpha-$Bloch space $\mathcal{B}^\alpha(\mathbb{B})$, (see. e.g., [25, 28, 27]) is the space consisting of all $f \in H(\mathbb{B})$ such that

$$b_\alpha(f) = \sup_{z\in\mathbb{B}}(1-|z|^2)^\alpha|\Re f(z)| \asymp \sup_{z\in\mathbb{B}}(1-|z|^2)^\alpha|\nabla f(z)| < \infty.$$

where $A \asymp B$ means that there is a positive constant C such that $B/C \leq A \leq CB$.

It is well known that $\mathcal{B}^\alpha$ is a Banach space under the norm defined by

$$\|f\|_{\mathcal{B}} = |f(0)| + b_\alpha(f).$$

For $\alpha = 1, \mathcal{B}^1 = \mathcal{B}$ is the classical Bloch space. Let $\mathcal{B}_0^\alpha$ denote the subspace of $\mathcal{B}^\alpha$ consisting of $f \in \mathcal{B}^\alpha$ for which

$$(1-|z|^2)^\alpha|\Re f(z)| \to 0 \ as \ |z| \to 1.$$

This space is called the little $\alpha-$Bloch space.

Every $\varphi \in S(\mathbb{B})$ induces a composition operator C_φ defined by $C_\varphi f = f\circ\varphi$ for $f \in H(\mathbb{B})$, $z \in \mathbb{B}$. For some results on composition operators, see, e.g. [1] and the references therein.

Let $g \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. The product of integral and composition operator on $H(\mathbb{D})$, was introduced and studied by S. Li and S. Stević in [9] and [10]. The operator is defined as follows:

$$J_gC_\varphi f(z) = \int_0^z f(\varphi(\xi))g(\xi)d\xi.$$

In [17], S. Stević has extended the operator to the unit ball setting as follows: let $\varphi \in S(\mathbb{B})$, $g \in H(\mathbb{B})$ and $g(0) = 0$, the product of composition and integral operator in the unit ball $\mathbb{B}$ is defined in this way:

$$P_\varphi^g(f)(z) = \int_0^1 f(\varphi(tz))g(tz)\frac{dt}{t}$$

for $f \in H(\mathbb{B})$, $z \in \mathbb{B}$.

If $n = 1$, then $g \in H(\mathbb{D})$ and $g(0) = 0$, so that $g(z) = zg_0(z)$, for some $g_0 \in H(\mathbb{D})$. By the change of variable $\xi = tz$, it follows that

$$P_\varphi^g(f)(z) = \int_0^1 f(\varphi(tz))tzg_0(tz)\frac{dt}{t} = \int_0^z f(\varphi(\xi))g_0(\xi)d\xi.$$

Thus operator is a natural extension of the operator J_gC_φ. For some recent results on this operator, see, e.g. [17, 18, 19, 20, 24] and so on.

Recently, many papers focused on studying the mapping properties of the difference of two composition operators, i.e., of an operator of the form

$$T = C_\varphi - C_\psi.$$

In [12], MacCluer, Ohno and Zhao, among other results, characterized the compactness of the difference of two composition operators on $H^\infty(\mathbb{D})$ in terms of the Poincaré distance. A fewer years later, these results were extended to the setting of $H^\infty(\mathbb{B})$ by Toews [22] and independently by Gorkin et al. [7]. In [11], Moorhouse showed that if the pseudo-hyperbolic distance between the image values φ and ψ converges to zero as $z \to \zeta$ for every point ζ at which φ and ψ have finite angular derivative then the difference $C_\varphi - C_\psi$ yields a compact operator. Differences of composition operators on the Bloch type space are studied in [13]. In 2011, Hosokawa and Ohno [8] studied the boundedness and compactness of the differences of two weighted composition operators acting from the Bloch space $\mathcal{B}$ to the space H^∞ of bounded analytic functions on the open unit disk. In 2012, Zhou and Liang [26] characterized

the boundedness and compactness of the differences of weighted composition operators from Hardy space to weighted-type spaces on the unit ball. As expected, the compact difference may be characterized by the pseudo-distance.

Building on those foundation, this paper continues the research of this part, and discusses the difference of two integral-type operators from H^∞ to the Bloch type space in the ball.

2 Notation and Lemmas

First, we will introduce some notation and state a couple of lemmas.

The pseudo-distance ρ on $\mathbb{B}$ is defined by

$$\rho(z,w) := |\Phi_z(w)|$$

for $z, w \in \mathbb{B}$, where Φ_z is the involutive automorphism in $\mathbb{B}$ that interchanges the point 0 and z.

Lemma 1. *[18, Lemma 2] Suppose $f, g \in H(\mathbb{B})$ and $g(0) = 0$. Then*

$$\Re P_\varphi^g(f)(z) = f(\varphi(z)g(z)).$$

Lemma 2. *If $f \in H_\alpha^\infty(\mathbb{B})$, then for all z and w in $\mathbb{B}$ we have*

$$\left|(1-|z|^2)^\alpha f(z) - (1-|w|^2)^\alpha f(w)\right| \le C||f||_\alpha \cdot \rho(z,w).$$

Proof. An exercise in [28] shows that:

$$\left|(1-|z|^2)^\alpha f(z) - (1-|w|^2)^\alpha f(w)\right| \le C||f||_\alpha \cdot \beta(z,w),$$

where

$$\beta(z,w) = \frac{1}{2}\ln\frac{1+\rho(z,w)}{1-\rho(z,w)}.$$

If $\rho(z,w) < \frac{1}{2}$, note that for $0 \le x < \frac{1}{2}$, $\ln\frac{1+x}{1-x} \le 3x$, the lemma is true.
If $\rho(z,w) \ge \frac{1}{2}$, then we have

$$\left|(1-|z|^2)^\alpha f(z) - (1-|w|^2)^\alpha f(w)\right| \le 2||f||_\alpha \le 4||f||_\alpha \cdot \rho(z,w).$$

The lemma follows by combining the two cases above. □

Lemma 3. *[28, Lemma 1.2] For each $a, z \in \mathbb{B}$, we have*

$$1-\rho^2(a,z) = \frac{(1-|a|^2)(1-|z|^2)}{|1-<z,a>|^2}.$$

Lemma 4. *[23, Lemma 3.1] There exists a constant $C > 0$ such that if $f \in H_\nu^\infty$, then*

$$|f(p) - f(q)| \le C\,\|f\|_\nu^\infty \cdot \rho(z,w)$$

for all $z, w \in \mathbb{D}$ and $f \in H_\nu^\infty$.

The following lemma is an easy modification of Proposition 3.11 in [1].

Lemma 5. *Let $0 < \alpha < \infty$, $g \in H(\mathbb{B})$ and φ be a holomorphic self-map of $\mathbb{B}$. Then $P_\varphi^g : H^\infty \to \mathcal{B}^\alpha$ is compact if and only if $P_\varphi^g : H^\infty \to \mathcal{B}^\alpha$ is bounded and for any bounded sequence $(f_k)_{k\in N}$ in H^∞ which converges to zero uniformly on $\mathbb{B}$ as $k \to \infty$, we have $||(P_{\varphi_1}^{g_1} - P_{\varphi_2}^{g_2})f_k||_{\mathcal{B}^\alpha} \to 0$ as $k \to \infty$.*

Constants are denoted by C in this paper, they are positive and not necessarily the same in each appearance.

3 Main theorems

Theorem 1. *Let $\varphi_1, \varphi_2 \in S(\mathbb{B})$ and $g_1, g_2 \in H(\mathbb{B})$ with $g_1(0) = 0, g_2(0) = 0$. Then the following statements are equivalent.*

(i) $P_{\varphi_1}^{g_1} - P_{\varphi_2}^{g_2} : H^\infty \to \mathcal{B}^\alpha$ *is bounded;*

(ii)

$$\sup_{z\in\mathbb{B}}(1-|z|^2)^\alpha \, |g_1(z)| \, \rho(\varphi_1(z), \varphi_2(z)) < \infty \tag{1}$$

$$\sup_{z\in\mathbb{B}}(1-|z|^2)^\alpha \, |g_2(z)| \, \rho(\varphi_1(z), \varphi_2(z)) < \infty \tag{2}$$

$$\sup_{z\in\mathbb{B}}\Big|(1-|z|^2)^\alpha g_1(z) - (1-|z|^2)^\alpha g_2(z)\Big| < \infty. \tag{3}$$

Proof. We first prove $(ii) \Rightarrow (i)$. We assume that $(1), (2), (3)$ hold.

Obviously, $\big|(P_{\varphi_1}^{g_1} - P_{\varphi_2}^{g_2})f(0) = 0\big|$. By Lemmas 1, 2 and 3, for every $f \in H^\infty$, we have

$$\begin{aligned}
||P_{\varphi_1}^{g_1} - P_{\varphi_2}^{g_2}||_{\mathcal{B}^\alpha} &= \sup_{z\in\mathbb{B}}(1-|z|^2)^\alpha \, |f(\varphi_1(z))g_1(z) - f(\varphi_2(z))g_2(z)| \\
&\le \sup_{z\in\mathbb{B}}(1-|z|^2)^\alpha \, |f(\varphi_1(z))| \, |g_1(z) - g_2(z)| \\
&+ \sup_{z\in\mathbb{B}}(1-|z|^2)^\alpha |g_2(z)| \, |f(\varphi_1(z)) - f(\varphi_2(z))| \\
&\le \sup_{z\in\mathbb{B}}(1-|z|^2)^\alpha \, |g_1(z) - g_2(z)| \, \|f\|_\infty \\
&+ C\sup_{z\in\mathbb{B}}(1-|z|^2)^\alpha |g_2(z)| \rho(\varphi_1(z), \varphi_2(z)) \, \|f\|_\infty \\
&\le C \, \|f\|_\infty .
\end{aligned}$$

This shows $P_{\varphi_1}^{g_1} - P_{\varphi_2}^{g_2}$ is bounded.

Next we show that (i) implies (ii). We assume $P_{\varphi_1}^{g_1} - P_{\varphi_2}^{g_2} : H^\infty \to \mathcal{B}^\alpha$ is bounded.

For every $\omega \in \mathbb{B}$, we take the test function

$$f_\omega(z) = \frac{\langle \phi_{\varphi_2(\omega)}(z), \phi_{\varphi_2(\omega)}(\varphi_1(\omega)) \rangle}{\big|\phi_{\varphi_2(\omega)}(\varphi_1(\omega))\big|}$$

when $\varphi_1(\omega) \ne \varphi_2(\omega)$; $f_\omega(z) = 0$, when $\varphi_1(\omega) \doteq \varphi_2(\omega)$. Some easy calculations show that $\sup_{\omega\in\mathbb{B}} ||f_\omega||_\infty \le 1$.

$$\begin{aligned}
C \ge ||P_{\varphi_1}^{g_1} - P_{\varphi_2}^{g_2}||_{\mathcal{B}^\alpha} &= \sup_{z\in\mathbb{B}}(1-|z|^2)^\alpha \, |f_\omega(\varphi_1(z))g_1(z) - f_\omega(\varphi_2(z))g_2(z)| \\
&\ge (1-|\omega|^2)^\alpha \, \big|\phi_{\varphi_2(\omega)}(\varphi_1(\omega))\big| \, |g_1(\omega)| \\
&= (1-|\omega|^2)^\alpha \, |g_1(\omega)| \, \rho(\phi_1(\omega), \varphi_2(\omega)).
\end{aligned}$$

Since the arbitrary of ω, we have

$$\sup_{z\in\mathbb{B}}(1-|z|^2)^\alpha \, |g_1(z)| \, \rho(\varphi_1(z), \varphi_2(z)) < \infty. \tag{4}$$

Similarly,

$$\sup_{z\in\mathbb{B}}(1-|z|^2)^\alpha \, |g_2(z)| \, \rho(\varphi_1(z), \varphi_2(z)) < \infty. \tag{5}$$

We take $f(z) = 1$, it follows that

$$\sup_{z\in\mathbb{B}}(1-|z|^2)^\alpha \, |g_1(z) - g_2(z)| < \infty. \tag{6}$$

This completes the proof of this theorem. □

Theorem 2. *Let $\varphi_1, \varphi_2 \in S(\mathbb{B})$ and $g_1, g_2 \in H(\mathbb{B})$ with $g_1(0) = 0, g_2(0) = 0$. Suppose that $P^{g_1}_{\varphi_1}$,$P^{g_2}_{\varphi_2} : H^\infty \to \mathcal{B}^\alpha$ are bounded, then the following statements are equivalent.*

(i) $P^{g_1}_{\varphi_1} - P^{g_2}_{\varphi_2} : H^\infty \to \mathcal{B}^\alpha$ *is compact;*

(ii)

$$\lim_{|\varphi_1(z)|\to 1}(1-|z|^2)^\alpha |g_1(z)| \rho(\varphi_1(z), \varphi_2(z)) = 0 \tag{7}$$

$$\lim_{|\varphi_2(z)|\to 1}(1-|z|^2)^\alpha |g_2(z)| \rho(\varphi_1(z), \varphi_2(z)) = 0 \tag{8}$$

$$\lim_{|\varphi_1(z)|\to 1, |\varphi_2(z)|\to 1} \Big|(1-|z|^2)^\alpha g_1(z) - (1-|z|^2)^\alpha g_2(z)\Big| = 0. \tag{9}$$

Proof. We first prove $(ii) \Rightarrow (i)$. We assume that $(1), (2), (3)$ hold.

Obviously, for any $\varepsilon > 0$, there is a $0 < \delta < 1$, such that

$$(1-|z|^2)^\alpha |g_1(z)| \rho(\varphi_1(z), \varphi_2(z)) < \varepsilon$$

when $\delta < \varphi_1(z) < 1$;

$$(1-|z|^2)^\alpha |g_1(z)| \rho(\varphi_1(z), \varphi_2(z)) < \varepsilon$$

when $\delta < \varphi_2(z) < 1$;

$$\Big|(1-|z|^2)^\alpha g_1(z) - (1-|z|^2)^\alpha g_2(z)\Big| < \varepsilon$$

when $\delta < \varphi_1(z) < 1, \delta < \varphi_2(z) < 1$.

Assume that (f_k) is a sequence in H^∞ and converges to 0 uniformly on compact subsets of $\mathbb{B}$ as $k \to \infty$. Then, we have

$$\begin{aligned}
||(P^{g_1}_{\varphi_1} - P^{g_2}_{\varphi_2})f_k||_{\mathcal{B}^\alpha} &= \sup_{z\in\mathbb{B}}(1-|z|^2)^\alpha |f_k(\varphi_1(z))g_1(z) - f_k(\varphi_2(z))g_2(z)| \\
&\leq \sup_{z\in K_1}(1-|z|^2)^\alpha |f_k(\varphi_1(z))g_1(z) - f_k(\varphi_2(z))g_2(z)| \\
&+ \sup_{z\in K_2}(1-|z|^2)^\alpha |f_k(\varphi_1(z))g_1(z) - f_k(\varphi_2(z))g_2(z)| \\
&+ \sup_{z\in K_3}(1-|z|^2)^\alpha |f_k(\varphi_1(z))g_1(z) - f_k(\varphi_2(z))g_2(z)| \\
&+ \sup_{z\in K_4}(1-|z|^2)^\alpha |f_k(\varphi_1(z))g_1(z) - f_k(\varphi_2(z))g_2(z)| \\
&= I_1 + I_2 + I_3 + I_4
\end{aligned}$$

where $K_1 = \{z \in \mathbb{B} : |\varphi_1(z)| \leq \delta \ |\varphi_2(z)| \leq \delta\}$, $K_2 = \{z \in \mathbb{B} : |\varphi_1(z)| > \delta \ |\varphi_2(z)| \leq \delta\}$, $K_3 = \{z \in \mathbb{B} : |\varphi_1(z)| \leq \delta \ |\varphi_2(z)| > \delta\}$, $K_4 = \{z \in \mathbb{B} : |\varphi_1(z)| > \delta \ |\varphi_2(z)| > \delta\}$. Obviously, $I_1 \to 0$, as $P^{g_1}_{\varphi_1}, P^{g_2}_{\varphi_2}$ are bounded and f_k converges to 0 on K_1.

Noticing that $P^{g_1}_{\varphi_1}$,$P^{g_2}_{\varphi_2}$ are bounded and by Lemma 4, we have

$$\begin{aligned}
I_2 &= \sup_{z\in K_2}(1-|z|^2)^\alpha |f_k(\varphi_1(z))g_1(z) - f_k(\varphi_2(z))g_2(z)| \\
&\leq \sup_{z\in K_2}(1-|z|^2)^\alpha |g_1(z) - g_2(z)| \, |f_k(\varphi_2(z))| \\
&+ \sup_{z\in K_2}(1-|z|^2)^\alpha |f_k(\varphi_1(z)) - f_k(\varphi_2(z))| \, |g_1(z)| \\
&\leq \sup_{z\in\mathbb{B}}(1-|z|^2)^\alpha |g_1(z) - g_2(z)| \, |f_k(\varphi_2(z))| \\
&+ C\sup_{z\in K_2}(1-|z|^2)^\alpha |g_1(z)| \rho(\varphi_1(z), \varphi_2(z))||f_k||_\infty \\
&\leq C\,|f_k(\varphi_2(z))| + C_1\varepsilon.
\end{aligned}$$

So $I_2 \to 0$, as $k \to \infty$. Similarly $I_3 \to 0$, as $k \to \infty$. Lastly, we will show $I_4 \to 0$, as $k \to \infty$.

$$\begin{aligned} I_4 &= \sup_{z\in K_4} (1-|z|^2)^\alpha \left|f_k(\varphi_1(z))g_1(z) - f_k(\varphi_2(z))g_2(z)\right| \\ &\le \sup_{z\in K_4} (1-|z|^2)^\alpha |g_1(z)-g_2(z)|\,|f_k(\varphi_2(z))| \\ &+ \sup_{z\in K_4} (1-|z|^2)^\alpha |f_k(\varphi_1(z)) - f_k(\varphi_2(z))|\,|g_1(z)| \\ &\le \sup_{z\in\mathbb{B}} (1-|z|^2)^\alpha |g_1(z)-g_2(z)|\,||f_k||_\infty \\ &+ C\sup_{z\in K_2} (1-|z|^2)^\alpha |g_1(z)|\,\rho(\varphi_1(z),\varphi_2(z))||f||_\infty \\ &\le C\varepsilon. \end{aligned}$$

So $I_4 \to 0$, as $k \to \infty$. That is (i) holds.

Next we show that (i) implies (ii). Assume that $P^{g_1}_{\varphi_1} - P^{g_2}_{\varphi_2} : H^\infty \to \mathcal{B}^\alpha$ is compact. Let $\{z_n\}_{n\in N}$ be a sequence in $\mathbb{B}$ such that $|\varphi_1(z_n)| \to 1$ as $n \to \infty$. We take the test function

$$f_n(z) = \frac{1-|\varphi_1(z_n)|^2}{1-\langle z,\varphi_1(z_n)\rangle}\,\frac{\langle \phi_{\varphi_2(\omega)}(z), \phi_{\varphi_2(\omega)}(\varphi_1(\omega))\rangle}{\left|\phi_{\varphi_2(\omega)}(\varphi_1(\omega))\right|},$$

when $\varphi_1(z_n) \ne \varphi_2(z_n)$; $f_n(z) = 0$, when $\varphi_1(z_n) \doteq \varphi_2(z_n)$ It is easy to check that $\sup_{n\in N} ||f_n||_\infty \le 1$, and $f_n \to 0$ uniformly on compacts of $\mathbb{B}$ as $n \to \infty$.

By using Lemma 5, we have $||(P^{g_1}_{\varphi_1} - P^{g_2}_{\varphi_2})f_n||_{\mathcal{B}^\alpha} \to 0$ as $n \to \infty$.

Therefor we have

$$\begin{aligned} ||(P^{g_1}_{\varphi_1} - P^{g_2}_{\varphi_2})f_n||_{\mathcal{B}^\alpha} &\ge (1-|z|^2)^\alpha |g_1(z_n)|\,\rho(\phi_1(z_n),\varphi_2(z_n)) \\ &\ge 0. \end{aligned}$$

So we get

$$\lim_{|\varphi_1(z)|\to 1} (1-|z|^2)^\alpha |g_1(z)|\,\rho(\varphi_1(z),\varphi_2(z)) = 0. \tag{10}$$

Using the same way, we can get

$$\lim_{|\varphi_2(z)|\to 1} (1-|z|^2)^\alpha |g_2(z)|\,\rho(\varphi_1(z),\varphi_2(z)) = 0. \tag{11}$$

When $|\varphi_2(z_n)| \to 1$ as $n \to \infty$,

Lastly, we assume $\{z_n\}_{n\in N}$ be a sequence in $\mathbb{B}$ such that $|\varphi_1(z_n)| \to 1$ and $|\varphi_2(z_n)| \to 1$as $n \to \infty$.

Let the test function $f_n(z) = \frac{1-|\varphi_1(z_n)|^2}{1-<z,\varphi_1(z_n)>}$, we can obtain that $\sup_{n\in N} ||f_n||_\infty \le 1$, and $f_n \to 0$ uniformly on compacts of $\mathbb{B}$ as $n \to \infty$.

By using Lemma 4, we have

$$\begin{aligned} ||(P^{g_1}_{\varphi_1} - P^{g_2}_{\varphi_2})f_n||_{\mathcal{B}^\alpha} &\ge (1-|z|^2)^\alpha \left| g_1(z_n) - \frac{1-|\varphi_1(z_n)|^2}{1-<\varphi_2(z_n),\varphi_1(z_n)>} g_2(z_n)\right| \\ &\ge (1-|z_n|^2)^\alpha |g_1(z_n)-g_2(z_n)| \\ &- (1-|z_n|^2)^\alpha \left|\frac{1-|\varphi_1(z_n)|^2}{1-<\varphi_2(z_n),\varphi_1(z_n)>}\right| |g_2(z_n)| \\ &\ge (1-|z_n|^2)^\alpha |g_1(z_n)-g_2(z_n)| \\ &- C(1-|z_n|^2)^\alpha |g_2(z_n)|\,\rho(\varphi_1(z_n),\varphi_2(z_n))||f_n||_\infty. \end{aligned}$$

So we have,

$$\begin{aligned}(1-|z_n|^2)^\alpha\,|g_1(z_n)-g_2(z_n)| &= \|(P_{\varphi_1}^{g_1}-P_{\varphi_2}^{g_2})f_n\|_{\mathcal{B}^\alpha}\\ &+ C(1-|z_n|^2)^\alpha\,|g_2(z_n)|\,\rho(\varphi_1(z_n),\varphi_2(z_n))\|f_n\|_\infty.\end{aligned}$$

Therefore, by Lemmas 5 and 8 we have

$$\sup_{z\in\mathbb{B}}\Big|(1-|z|^2)^\alpha g_1(z)-(1-|z|^2)^\alpha g_2(z)\Big|<\infty. \tag{12}$$

This completes the proof of this theorem. □

Using the same way, we can get the following theorem.

Theorem 3. *Let $\varphi_1,\varphi_2\in S(\mathbb{B})$. Then the following statements are equivalent.*
(i) $P_{\varphi_1}^{g_1}-P_{\varphi_2}^{g_2}:H^\infty\to\mathcal{B}_0^\alpha$ *is bounded;*
(ii) $P_{\varphi_1}^{g_1}-P_{\varphi_2}^{g_2}:H^\infty\to\mathcal{B}_0^\alpha$ *is compact;*
(iii)

$$\lim_{|z|\to1}(1-|z|^2)^\alpha\,|g_1(z)|\,\rho(\varphi_1(z),\varphi_2(z))=0 \tag{13}$$

$$\lim_{|z|\to1}(1-|z|^2)^\alpha\,|g_2(z)|\,\rho(\varphi_1(z),\varphi_2(z))=0 \tag{14}$$

$$\lim_{|z|\to1}\Big|(1-|z|^2)^\alpha g_1(z)-(1-|z|^2)^\alpha g_2(z)\Big|=0. \tag{15}$$

Remark If $\alpha=1$, then $\mathcal{B}^\alpha$ will be Bloch space $\mathcal{B}$. The similar results from H^∞ to the Bloch space $\mathcal{B}$ corresponding to Theorems 1 and 2 also hold.

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A new construction of quadrature formulas for Cauchy singular integral*

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Abstract

This paper presents quadrature formulas for Cauchy singular integrals. The asymptotic expansions of the errors with the power $h^{2\mu}(\mu = 1, \cdots, m)$ are obtained by Euler-Maclaurin expansions. In the first part of this work, we derive formulas for One-dimension Cauchy singular integral and their corresponding Euler-Maclaurin expansion on the basis of the midpoint rule and Sidi-Israeli′s quadrature formulas for the boundary integral equations with weak singularities. In the second part of this work, we propose the quadrature formulas for the Two-dimension singular integral on the basis of quadrature formulas for the One-dimension Cauchy singular integral. For calculating singular integrals the algorithms are very simple and straightforward, without calculation any derivative value of function, the accuracy order of the algorithms is very high.

Keyword: Two-dimension Cauchy singular integral · Euler-Maclaurin expansion · Quadrature formula

AMS subject classification: 65D10 · 65D32

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1 Introduction

Consider One-dimension cauchy singular integral

$$I(g_1) = \int_a^b \frac{g_1(x)}{(x-s)}dx, \tag{1.1}$$

and Two-dimension cauchy singular integral

$$I(g_2) = \int_a^b \int_c^d \frac{g_2(x,y)}{(x-s)(y-t)}dxdy. \tag{1.2}$$

Cauchy singular integral and Cauchy singular integral equations have applications in wavelet problems with integro-differential equations. It is very important to explore the effective methods to calculate singular integral and solve Cauchy singular integral equation. However, exact solution can not be acquired easily through direct calculation. Therefore, numerical method is usually used to this kind of problem. So far the Euler-Maclaurin expansion of the error of a trapezoidal rule has extended to many fields such as Cauchy singular and weakly singular integrals, e.g., Navot[2] constructed modifying trapezoidal rules of integrals with algebraic and logarithmic singularties at end point, and proved that the errors have Euler-Maclaurin expansions. Sidi and Israeli[1] presented Euler-Maclaurin expansions modifying trapezoidal rules of integrals with algebraic, logarithmic, and Cauchy singularities at the interior points of the interval. Based on another modifying trapezoidal formula and periodization methods, this paper a high accuracy algorithm for Cauchy singular integral will be developed. We derive quadrature formulas for Cauchy singular integral, and give their Euler-Maclaurin expansion with the power $h^{2\mu}$ ($\mu = 1, ..., m$).

The rest of the article is organized as follows. In Section 2, we briefly explain some notation and some Lemma which are used to state and to prove our results. In Section 3, we sate our result with its proof.

2 Preliminaries

Some notation and Lemmas as follows are needed in this article.

From the definition of the Cauchy principal value integrals[9], we have

$$\begin{aligned} p.v. \int_c^d \frac{1}{(t-y)}dy &= \lim_{\varepsilon\to 0}[\int_c^{t-\varepsilon} \frac{1}{(t-y)}dy + \int_{t+\varepsilon}^d \frac{1}{(t-y)}dy] \\ &= \ln\frac{d-t}{t-c}, \end{aligned} \tag{2.1}$$

and

$$p.v \int_c^d \frac{g(y)}{(t-y)}dy = \int_c^d \frac{g(y)-g(t)}{(t-y)}dy + g(t)\ln\frac{d-t}{t-c}, \tag{2.2}$$

where $t \in (c,d)$ $g(y) \in C^l[c,d]$, and $l \in N$. Let n be the number of equally spaced nodes used in the quadrature, $h = (b-a)/n$, $x_j = a + jh(j = 0, 1, \cdots, n)$ and assume the singular point $x = t$ is one of the abscissas, that is, $t \in \{x_j\}_{j=1}^{n-1}$.

To simplify notations, the double prime on a summation means that coefficients of the first and last terms in the summation are 1/2, that is, we let

$$\sum_{j=n_1}^{n_2}{}'' \omega_j = \frac{1}{2}\omega_{n_1} + \omega_{n_1+1} + \cdots + \omega_{n_2-1} + \frac{1}{2}\omega_{n_2}.$$

The following Lemma will be useful to prove the main results.

Lemma 2.1.[1] *Let $f(x)$ be $2l$ times differentiable on $[a,b]$, $F(x) = f(x)/(x-s)$ and $I(g) = \int_a^b F(x)dx$. Then*

$$\begin{aligned} E_l(h) =& I(g) - Q(h) \\ =& hf'(s) + \sum_{\mu=1}^{l-1} \frac{B_{2\mu}}{(2\mu)!}[F^{(2\mu-1)}(a) - F^{(2\mu-1)}(b)]h^{2\mu} \\ &+ O(h^{2l}), \quad as\ \ h \to 0 \end{aligned} \tag{2.3}$$

where

$$Q(h) = h \sum_{i=1, x_i \neq s}^{n}{}'' F(x_i)$$

is the quadrature formulas of cauchy singular integral, $B_{2\mu}$ is the Bernoulli numbers.

Lemma 2.2.[1] *Assume that the functions $g(x)$ and $\tilde{g}(x)$ are $2m$ times differentiable on $[a,b]$, and that the functions $G(x)$ are periodic function with period $T = b-a$. Moreover, if $G(x)$ are also $2m$ times differentiable on $(-\infty.\infty)/\{t+kT\}_{k=-\infty}^{\infty}$, the following conclusions hold:*
(a) *for $G(x) = \frac{g(x)}{(x-t)} + \tilde{g}(x)$, there exists the error estimate*

$$E_n[G] = [\tilde{g}(t) + g'(t)]h + O(h^{2m}), \quad as\ \ h \to 0$$

where

$$Q_n[G] = h \sum_{j=1, x_j \neq t}^{n} G(x_j); \tag{2-4}$$

(b) *for $G(x) = |x-t|^s g(x) + \tilde{g}(x)$, $s > -1$, then*

$$\begin{aligned} E_n[G] = & -2\sum_{\nu=1}^{m-1} \frac{\zeta(-s-2\nu)}{(2\nu)!} g^{(2\nu)}(t) h^{2\nu+s+1} \\ & + O(h^{2m}), \quad as\ \ h \to 0 \end{aligned}$$

where

$$Q_n[G] = h \sum_{j=1,x_j\neq t}^{n} G(x_j) + \tilde{g}(t)h - 2\zeta(-s)g(t)h^{s+1}; \tag{2-5}$$

(c) *for* $G(x) = |x-t|^s ln|x-t|g(x) + \tilde{g}(x)$, $s > -1$, *then*

$$E_n[G] = 2\sum_{\nu=1}^{m-1}[\zeta'(-s-2\nu) - \zeta(-s-2\nu)lnh]\frac{g^{2\nu}(t)}{(2\nu)!}h^{2\nu+s+1} + O(h^{2m}), \quad as \;\; h \to 0$$

where

$$Q_n[G] = h \sum_{j=1,x_j=t}^{n} G(x_j) + \tilde{g}(t)h + 2[\zeta'(-s) - \zeta(-s)lnh]g(t)h^{s+1}. \tag{2-6}$$

In particular, if $s = 0$ *in (c),* $\zeta'(0) = -\frac{1}{2}ln(2\pi)$, *then*

$$E_n[G] = 2\sum_{\nu=1}^{m-1}\frac{\zeta'(-2\nu)}{(2\nu)!}g^{(2\nu)}(t)h^{2\nu+1} + O(h^{2m}), \quad as \;\; h \to 0$$

where

$$Q_n[G] = h \sum_{j=1,x_j\neq t}^{n} G(x_j) + \tilde{g}(t)h + ln(\frac{h}{2\pi})g(t)h, \tag{2-7}$$

$$E_n[G] = \int_a^b G(x)dx - Q_n[G],$$

$B_{2\mu}$ *is the Bernoulli numbers and* $\zeta(z)$ *is the Riemann zeta function.*

3 Main results

Our main goal in this section is to establish the following results with proof.

Theorem 3.1. *Let* $f(x)$ *be* $2l$ *times differentiable on* $[a,b]$, $F(x) = f(x)/(x-s)$ *and* $I(g) = \int_a^b F(x)dx$. *Then*

$$\begin{aligned} E_l(h) =& I(g) - Q(h) \\ =& \sum_{\mu=1}^{l-1}\frac{(2^{1-2\mu}-1)B_{2\mu}}{(2\mu)!}[F^{(2\mu-1)}(a) - F^{(2\mu-1)}(b)]h^{2\mu} \\ &+ O(h^{2l}), \quad as \;\; h \to 0 \end{aligned} \tag{3.1}$$

where

$$Q(h) = h \sum_{i=1, x_i \neq s}^{n} F(a + \frac{2i-1}{2} h) \tag{3.2}$$

is the new quadrature formulas of cauchy singular integral.

Proof. From Sidi formula[1], as $h \to 0$, we have

$$\begin{aligned}\int_a^b F(x)dx =& h \sum_{j=0, x_j \neq s}^{n} {}'' F(x_j) + hg'(s) \\ &+ \sum_{\mu=1}^{l-1} \frac{B_{2\mu}}{(2\mu)!} [F^{2\mu-1}(a) - F^{2\mu-1}(b)] h^{2\mu} + O(h^{2l}),\end{aligned} \tag{3.3}$$

$$\begin{aligned}\int_a^b F(x)dx =& \frac{h}{2} \sum_{j=0, x_j \neq t}^{2n} {}'' F(x_j) + \frac{h}{2} g'(s) \\ &+ \sum_{\mu=1}^{l-1} \frac{B_{2\mu}}{(2\mu)!} [F^{2\mu-1}(a) - F^{2\mu-1}(b)] (\frac{h}{2})^{2\mu} + O(h^{2l}),\end{aligned} \tag{3.4}$$

Using $(3.4) \times 2 - (3.3)$, we have

$$\begin{aligned}\int_a^b G(x)dx =& h[\sum_{j=0, x_j \neq t}^{2n} {}'' G(x_j) - \sum_{j=0, x_j \neq t}^{n} {}'' G(x_j)] \\ &+ \sum_{\mu=1}^{l-1} \frac{B_{2\mu}}{(2\mu)!} [G^{2\mu-1}(a) - G^{2\mu-1}(b)] \times [2^{1-2\mu} - 1] h^{2\mu} + O(h^{2l}) \\ =& h \sum_{i=1, x_i \neq s}^{n} F(a + \frac{2i-1}{2} h_m) \\ &+ \sum_{\mu=1}^{l-1} \frac{B_{2\mu}}{(2\mu)!} [G^{2\mu-1}(a) - G^{2\mu-1}(b)] \times [2^{1-2\mu} - 1] h^{2\mu} + O(h^{2l}).\end{aligned}$$

This completes the proof. $\square$

We use the first part

$$Q_l[G] = h \sum_{i=1, x_i \neq s}^{n} F(a + \frac{2i-1}{2} h) \tag{3.5}$$

as the quadrature formulas of Cauchy singular integrals with the $O(h^2)$. Comparing with current accuracy of the quadrature formulas $O(h)$, it is greatly improved. The formula (3.1) is the asymptotic expansions of the errors with the power $h^{2\mu}$ obtained by Euler-Maclaurin expansions. We improve the accuracy obviously by the extrapolation.

Corollary 3.2. *Assume that the function $f(x)$ is $2l$ times differentiable on $[a,b]$. Assume also that the function $F(x)$ is periodic with period $T=b-a$, and that they are $2l$ times differentiable on $(-\infty,+\infty)\backslash\{x+kT\}_{k=-\infty}^{+\infty}$, as $h\to 0$. Then we have*

$$\int_a^b F(x)dx = h\sum_{i=1,x_i\neq s}^{n} F(a+\frac{2i-1}{2}h)+O(h^{2l}). \tag{3.6}$$

Remark 3.3 *These formulas are advantageous in the following ways:*

(a) *Computing the discrete elements of singular integrals and singular integral equations is only assignment. Calculation does not need any singular integrals, greatly reducing computation cost;*

(b) *It has the error asymptotic expansion;*

(c) *The condition number for solving integral equations is very small;*

(d) *It possesses a posteriori error estimate;*

(e) *Extrapolation method can be used to improve greatly the accuracy.*

Based on the quadrature formulas of one-dimension cauchy singular integral, we can derive the quadrature formulas of Two-dimension Cauchy singular integrals and their asymptotic expansions.

Let n be a positive integer, $h_m=(b-a)/m,\ h_n=(d-c)/n$, $x_i=a+ih_m,\ y_j=a+jh_n\ (i,\ j=0,1,...,n)$ and $s\in\{x_i|\ 0\le i\le n\},\ t\in\{y_j|\ 0\le j\le n\}$.

Theorem 3.4. *Let $f(x,y)$ be $2l$ times differentiable for x and y on $[a,b]\times[c,d]$, and let $F(x,y)=\frac{f(x,y)}{(x-s)(y-t)}$, $F_1(x,y)=\frac{f(x,y)}{(x-s)}$, $F_2(x,y)=\frac{f(x,y)}{(y-t)}$ and $I(x,y)=$*

$\int_a^b \int_c^d F(x,y)dxdy$. *Then as* $h_m \to 0$ *and* $h_n \to 0$,

$$
\begin{aligned}
E_{m,n}(h_m,h_n) =& I(x,y) - Q(h_m,h_n) \\
=& h_n \sum_{j=1,y_j\neq t}^{n} \frac{1}{c+\frac{2j-1}{2}h_n - t} \sum_{\mu=1}^{l-1} \{ \frac{h_m^{2\mu} B_{2\mu}(2^{1-2\mu}-1)}{(2\mu)!} \\
& \times [\frac{\partial^{2\mu-1}}{\partial x^{2\mu-1}} \frac{f(x,c+\frac{2j-1}{2}h_n)}{(x-s)}|_{x=a} - \frac{\partial^{2\mu-1}}{\partial x^{2\mu-1}} \frac{f(x,c+\frac{2j-1}{2}h_n)}{(x-s)}|_{x=b}]\} \\
& + \sum_{\mu=1}^{l-1} \frac{h_n^{2\mu} B_{2\mu}(2^{1-2\mu}-1)}{(2\mu)!} b_\mu(s,t) + O(h^{2l}). \qquad (3.7)
\end{aligned}
$$

where

$$
Q(h_m,h_n) = h_m h_n \sum_{i=1,x_i\neq s}^{m} \sum_{j=1,y_j\neq t}^{n} \frac{f(a+\frac{2i-1}{2}h_m, c+\frac{2j-1}{2}h_n)}{(a+\frac{2i-1}{2}h_m - s)(c+\frac{2j-1}{2}h_n - t)} \qquad (3.8)
$$

is a new quadrature formula of Two-dimension Cauchy singular integral,

$$
b_\mu(s,t) = \frac{\partial^{2\mu-1}}{\partial y^{2\mu-1}} \int_a^b F(x,y)dx|_{y=c} - \frac{\partial^{2\mu-1}}{\partial y^{2\mu-1}} \int_a^b F(x,y)dx||_{y=d},
$$

and $B_{2\mu}$ *is the Bernoulli numbers,* $h = max(h_m,h_n)$.

Proof. From Theorem 3.1, we have

$$
\begin{aligned}
I(x,y) =& \int_a^b \int_c^d \frac{f(x,y)dxdy}{(x-s)(y-t)} = \int_a^b \frac{1}{(x-s)} [\int_c^d \frac{f(x,y)dy}{(y-t)}]dx \\
=& \int_a^b \frac{1}{(x-s)} \{ h_n \sum_{j=1,y_j\neq t}^{n} \frac{f(x,c+\frac{2j-1}{2}h_n)}{c+\frac{2j-1}{2}h_n - t} + \sum_{\mu=1}^{l-1} \frac{h_n^{2\mu} B_{2\mu}(2^{1-2\mu}-1)}{(2\mu)!} \\
& \times [\frac{\partial^{2\mu-1}}{\partial y^{2\mu-1}} \frac{f(x,y)}{(y-t)}|_{y=c} - \frac{\partial^{2\mu-1}}{\partial y^{2\mu-1}} \frac{f(x,y)}{(y-t)}|_{y=d}] + O(h_n^{2l})\}dx \\
=& I_1(x,y) + I_2(x,y) + I_3(x,y). \qquad (3.9)
\end{aligned}
$$

Here,

$$\begin{aligned}
I_1(x,y) &= h_n \sum_{j=1,y_j\neq t}^{n} \frac{1}{c+\frac{2j-1}{2}h_n - t}\int_a^b \frac{f(x,c+\frac{2j-1}{2}h_n)}{(x-s)}dx \\
&= h_n \sum_{j=1,y_j\neq t}^{n} \frac{1}{c+\frac{2j-1}{2}h_n - t}\{h_m \sum_{i=1,x_i\neq s}^{m} \frac{f(a+\frac{2i-1}{2}h_m, c+\frac{2j-1}{2}h_n)}{a+\frac{2i-1}{2}h_m - s} \\
&\quad + \sum_{\mu=1}^{l-1} \frac{h_m^{2\mu}B_{2\mu}(2^{1-2\mu}-1)}{(2\mu)!}[\frac{\partial^{2\mu-1}}{\partial x^{2\mu-1}}\frac{f(x,c+\frac{2j-1}{2}h_n)}{(x-s)}|_{x=a} \\
&\quad - \frac{\partial^{2\mu-1}}{\partial x^{2\mu-1}}\frac{f(x,c+\frac{2j-1}{2}h_n)}{(x-s)}|_{x=b}] + O(h_m^{2l})\} \\
&= Q(h_m,h_n) + I_{11}(x,y) + I_{12}(x,y),
\end{aligned} \tag{3.10}$$

where

$$Q(h_m,h_n) = h_m h_n \sum_{i=1,x_i\neq s}^{m}\sum_{j=1,y_j\neq t}^{n} \frac{f(a+\frac{2i-1}{2}h_m, c+\frac{2j-1}{2}h_n)}{(a+\frac{2i-1}{2}h_m - s)(c+\frac{2j-1}{2}h_n - t)}, \tag{3.11}$$

$$\begin{aligned}
I_{11}(x,y) =& h_n \sum_{j=1,y_j\neq t}^{n} \frac{1}{c+\frac{2j-1}{2}h_n - t}\sum_{\mu=1}^{l-1}\{\frac{h_m^{2\mu}B_{2\mu}(2^{1-2\mu}-1)}{(2\mu)!} \\
&\times [\frac{\partial^{2\mu-1}}{\partial x^{2\mu-1}}\frac{f(x,c+\frac{2j-1}{2}h_n)}{(x-s)}|_{x=a} - \frac{\partial^{2\mu-1}}{\partial x^{2\mu-1}}\frac{f(x,c+\frac{2j-1}{2}h_n)}{(x-s)}|_{x=b}]\},
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
I_{12}(x,y) =& h_n \sum_{j=1,y_j\neq t}^{n} \frac{1}{c+\frac{2j-1}{2}h_n - t} O(h_m^{2l}) \\
=& O(h_m^{2l})\{\int_c^d \frac{dy}{y-t} - \sum_{\mu=1}^{l-1}\frac{h_n^{2\mu}(2^{1-2\mu}-1)B_{2\mu}}{(2\mu)!}[(\frac{1}{y-t})^{(2\mu-1)}|_{y=c} \\
&- (\frac{1}{y-t})^{(2\mu-1)}|_{y=d}] + O(h_n^{2l})\} \\
=& \ln|\frac{d-t}{t-c}|O(h_m^{2l}) - \sum_{\mu=1}^{l-1}\frac{h_n^{2\mu}(2^{1-2\mu}-1)B_{2\mu}}{(2\mu)!}w_{2\mu}(t)O(h_m^{2l}) + O(h_n^2)O(h_m^{2l}) \\
=& O(h_m^{2l}).
\end{aligned} \tag{3.13}$$

Using Theorem 3.1 again, we have

$$
\begin{aligned}
h_n \sum_{j=1, y_j \neq t}^{n} \frac{1}{c+\frac{2j-1}{2}h_n - t} &= \int_c^d \frac{dy}{y-t} - \sum_{\mu=1}^{l-1} \frac{h_n^{2\mu}(2^{1-2\mu}-1)B_{2\mu}}{(2\mu)!}[(\frac{1}{y-t})^{(2\mu-1)}|_{y=c} \\
&\quad - (\frac{1}{y-t})^{(2\mu-1)}|_{y=d}] + O(h_n^{2l}) \\
&= \ln|\frac{d-t}{t-c}| - \sum_{\mu=1}^{l-1} \frac{h_n^{2\mu}(2^{1-2\mu}-1)B_{2\mu}(-1)^{2\mu-1}(2\mu-1)!}{(2\mu)!} \\
&\quad \times [\frac{1}{(c-t)^{2\mu}} - \frac{1}{(d-t)^{2\mu}}] + O(h_n^{2l}). \qquad (3.14)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
I_{11}(s,t) = &[\ln|\frac{d-t}{t-c}| - \sum_{\mu=1}^{l-1} \frac{h_n^{2\mu}(2^{1-2\mu}-1)B_{2\mu}(-1)^{2\mu-1}(2\mu-1)!}{(2\mu)!} w_{2\mu}(t) + O(h_n^{2l})] \\
&\times [\sum_{\mu=1}^{l-1} \frac{h_m^{2\mu}B_{2\mu}(2^{1-2\mu}-1)}{(2\mu)!} u(s,h_n)] \\
= &\ln|\frac{d-t}{t-c}| \sum_{\mu=1}^{l-1} \frac{h_m^{2\mu}B_{2\mu}(2^{1-2\mu}-1)}{(2\mu)!} u(s,h_n) - \sum_{\mu=1}^{l-1} \frac{h_m^{2\mu}B_{2\mu}(2^{1-2\mu}-1)}{(2\mu)!} u(s,h_n) \\
&\times \sum_{\mu=1}^{l-1} \frac{h_n^{2\mu}(2^{1-2\mu}-1)B_{2\mu}(-1)^{2\mu-1}(2\mu-1)!}{(2\mu)!} w_{2\mu}(t) \\
&+ \sum_{\mu=1}^{l-1} \frac{h_m^{2\mu}B_{2\mu}(2^{1-2\mu}-1)}{(2\mu)!} u(s,h_n) O(h_n^{2l}), \qquad (3.15)
\end{aligned}
$$

where

$$
w_{2\mu}(t) = \frac{1}{(c-t)^{2\mu}} - \frac{1}{(d-t)^{2\mu}}, \qquad (3.16)
$$

and

$$
u(s,h_n) = [\frac{\partial^{2\mu-1}}{\partial x^{2\mu-1}} \frac{f(x, c+\frac{2j-1}{2}h_n)}{(x-s)}|_{x=a} - \frac{\partial^{2\mu-1}}{\partial x^{2\mu-1}} \frac{f(x, c+\frac{2j-1}{2}h_n)}{(x-s)}|_{x=b}]. \qquad (3.17)
$$

Next,

$$\begin{aligned}
I_2(s,t) =&\sum_{\mu=1}^{l-1} \frac{h_n^{2\mu} B_{2\mu}(2^{1-2\mu}-1)}{(2\mu)!} \int_a^b \frac{dx}{(x-s)} [\frac{\partial^{2\mu-1}}{\partial y^{2\mu-1}} \frac{f(x,y)}{(y-t)}|_{y=c} \\
&- \frac{\partial^{2\mu-1}}{\partial y^{2\mu-1}} \frac{f(x,y)}{(y-t)}|_{y=d}] \\
=&\sum_{\mu=1}^{l-1} \frac{h_n^{2\mu} B_{2\mu}(2^{1-2\mu}-1)}{(2\mu)!} b_\mu(s,t),
\end{aligned} \tag{3.18}$$

where

$$b_\mu(s,t) = \frac{\partial^{2\mu-1}}{\partial y^{2\mu-1}} \int_a^b F(x,y)dx|_{y=c} - \frac{\partial^{2\mu-1}}{\partial y^{2\mu-1}} \int_a^b F(x,y)dx||_{y=d}]. \tag{3.19}$$

In addition,

$$\begin{aligned}
I_3(x,y) =&O(h_n^{2l}) \int_a^b \frac{dx}{(x-s)} \\
=&O(h_n^{2l}) \ln |\frac{b-s}{s-a}| + O(h_m^2) O(h_n^{2l}) \\
=&O(h_n^{2l}).
\end{aligned} \tag{3.20}$$

According to formula (3.11-13, 3.18, 3.20), this completes the proof. □

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Some differential superordinations using Ruscheweyh derivative and generalized Sălăgean operator

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Abstract

In the present paper we establish several differential superordinations regarding the operator $RD^n_{\lambda,\alpha}$ defined by using Ruscheweyh derivative $R^n f(z)$ and the generalized Sălăgean operator $D^n_\lambda f(z)$, $RD^n_{\lambda,\alpha}: \mathcal{A} \to \mathcal{A}$, $RD^n_{\lambda,\alpha} f(z) = (1-\alpha)R^n f(z) + \alpha D^n_\lambda f(z)$, $z \in U$, where $n \in \mathbb{N}$, $\lambda, \alpha \geq 0$ and $f \in \mathcal{A}$, $\mathcal{A} = \{f \in \mathcal{H}(U) : f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \quad z \in U\}$. A number of interesting consequences of some of these superordination results are discussed. Relevant connections of some of the new results obtained in this paper with those in earlier works are also provided.

Keywords: differential superordination, convex function, best subordinant, differential operator.
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1 Introduction

Denote by U the unit disc of the complex plane, $U = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}(U)$ the space of holomorphic functions in U.

Let $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + \dots, \ z \in U\}$ with $\mathcal{A}_1 = \mathcal{A}$ and $\mathcal{H}[a,n] = \{f \in \mathcal{H}(U) : f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \dots, \ z \in U\}$ for $a \in \mathbb{C}$ and $n \in \mathbb{N}$.

Denote by $K = \left\{ f \in \mathcal{A} : \operatorname{Re} \frac{zf''(z)}{f'(z)} + 1 > 0, \ z \in U \right\}$, the class of normalized convex functions in U.

If f and g are analytic functions in U, we say that f is superordinate to g, written $g \prec f$, if there is a function w analytic in U, with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$ such that $g(z) = f(w(z))$ for all $z \in U$. If f is univalent, then $g \prec f$ if and only if $f(0) = g(0)$ and $g(U) \subseteq f(U)$.

Let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$ and h analytic in U. If p and $\psi(p(z), zp'(z); z)$ are univalent in U and satisfies the (first-order) differential superordination

$$h(z) \prec \psi(p(z), zp'(z); z), \qquad z \in U, \tag{1}$$

then p is called a solution of the differential superordination. The analytic function q is called a subordinant of the solutions of the differential superordination, or more simply a subordinant, if $q \prec p$ for all p satisfying (1).

An univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (1) is said to be the best subordinant of (1). The best subordinant is unique up to a rotation of U.

Definition 1.1 *(Al Oboudi [9]) For $f \in \mathcal{A}$, $\lambda \geq 0$ and $n \in \mathbb{N}$, the operator D^n_λ is defined by $D^n_\lambda : \mathcal{A} \to \mathcal{A}$,*

$$\begin{aligned} D^0_\lambda f(z) &= f(z) \\ D^1_\lambda f(z) &= (1-\lambda) f(z) + \lambda z f'(z) = D_\lambda f(z) \ , \dots, \\ D^{n+1}_\lambda f(z) &= (1-\lambda) D^n_\lambda f(z) + \lambda z \left(D^n_\lambda f(z)\right)' = D_\lambda \left(D^n_\lambda f(z)\right), \quad z \in U. \end{aligned}$$

Remark 1.2 *If $f \in \mathcal{A}$ and $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $D^n_\lambda f(z) = z + \sum_{j=2}^{\infty} [1 + (j-1)\lambda]^n a_j z^j$, $z \in U$.*

For $\lambda = 1$ in the above definition we obtain the Sălăgean differential operator [14].

Definition 1.3 *(Ruscheweyh [13]) For $f \in \mathcal{A}$, $n \in \mathbb{N}$, the operator R^n is defined by $R^n : \mathcal{A} \to \mathcal{A}$,*

$$\begin{aligned} R^0 f(z) &= f(z) \\ R^1 f(z) &= z f'(z) \ , ..., \\ (n+1) R^{n+1} f(z) &= z (R^n f(z))' + n R^n f(z), \quad z \in U. \end{aligned}$$

Remark 1.4 *If $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $R^n f(z) = z + \sum_{j=2}^{\infty} C_{n+j-1}^n a_j z^j$, $z \in U$.*

Definition 1.5 *[3] Let $\alpha, \lambda \geq 0$, $n \in \mathbb{N}$. Denote by $RD_{\lambda,\alpha}^n$ the operator given by $RD_{\lambda,\alpha}^n : \mathcal{A} \to \mathcal{A}$,*

$$RD_{\lambda,\alpha}^n f(z) = (1-\alpha) R^n f(z) + \alpha D_\lambda^n f(z), \qquad z \in U.$$

Remark 1.6 *If $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then*
$RD_{\lambda,\alpha}^n f(z) = z + \sum_{j=2}^{\infty} \left\{ \alpha [1 + (j-1)\lambda]^n + (1-\alpha) C_{n+j-1}^n \right\} a_j z^j$, $z \in U$.

This operator was studied also in [6], [7], [4], [10], [11].

Remark 1.7 *For $\alpha = 0$, $RD_{\lambda,0}^n f(z) = R^n f(z)$, where $z \in U$ and for $\alpha = 1$, $RD_{\lambda,1}^n f(z) = D_\lambda^n f(z)$, where $z \in U$.*

For $\lambda = 1$, we obtain $RD_{1,\alpha}^n f(z) = L_\alpha^n f(z)$ which was studied in [1], [2], [5].

For $n = 0$, $RD_{\lambda,\alpha}^0 f(z) = (1-\alpha) R^0 f(z) + \alpha D_\lambda^0 f(z) = f(z) = R^0 f(z) = D_\lambda^0 f(z)$, where $z \in U$.

Definition 1.8 *We denote by Q the set of functions that are analytic and injective on $\overline{U} \backslash E(f)$, where $E(f) = \{\zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty\}$, and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \backslash E(f)$. The subclass of Q for which $f(0) = a$ is denoted by $Q(a)$.*

We will use the following lemmas.

Lemma 1.9 *(Miller and Mocanu [12, Th. 3.1.6, p. 71]) Let h be a convex function with $h(0) = a$ and let $\gamma \in \mathbb{C} \backslash \{0\}$ be a complex number with Re $\gamma \geq 0$. If $p \in \mathcal{H}[a,n] \cap Q$, $p(z) + \frac{1}{\gamma} z p'(z)$ is univalent in U and $h(z) \prec p(z) + \frac{1}{\gamma} z p'(z)$, $z \in U$, then $q(z) \prec p(z)$, $z \in U$, where $q(z) = \frac{\gamma}{n z^{\gamma/n}} \int_0^z h(t) t^{\gamma/n - 1} dt$, $z \in U$. The function q is convex and is the best subordinant.*

Lemma 1.10 *(Miller and Mocanu [12]) Let q be a convex function in U and let $h(z) = q(z) + \frac{1}{\gamma} z q'(z)$, $z \in U$, where Re $\gamma \geq 0$. If $p \in \mathcal{H}[a,n] \cap Q$, $p(z) + \frac{1}{\gamma} z p'(z)$ is univalent in U and $q(z) + \frac{1}{\gamma} z q'(z) \prec p(z) + \frac{1}{\gamma} z p'(z)$, $z \in U$, then $q(z) \prec p(z)$, $z \in U$, where $q(z) = \frac{\gamma}{n z^{\gamma/n}} \int_0^z h(t) t^{\gamma/n - 1} dt$, $z \in U$. The function q is the best subordinant.*

2 Main results

Theorem 2.1 *Let h be a convex function, $h(0) = 1$. Let $n \in \mathbb{N}$, $\lambda, \alpha, \delta \geq 0$, $f \in \mathcal{A}$ and suppose that $\left(\frac{RD_{\lambda,\alpha}^n f(z)}{z} \right)^{\delta - 1} \left(RD_{\lambda,\alpha}^n f(z) \right)'$ is univalent and $\left(\frac{RD_{\lambda,\alpha}^n f(z)}{z} \right)^{\delta} \in \mathcal{H}[1, \delta] \cap Q$. If*

$$h(z) \prec \left(\frac{RD_{\lambda,\alpha}^n f(z)}{z} \right)^{\delta - 1} \left(RD_{\lambda,\alpha}^n f(z) \right)', \qquad z \in U, \tag{2}$$

then $q(z) \prec \left(\frac{RD_{\lambda,\alpha}^n f(z)}{z} \right)^{\delta}$, $z \in U$, where $q(z) = \frac{\delta}{z^\delta} \int_0^z h(t) t^{\delta - 1} dt$. The function q is convex and it is the best subordinant.

Proof. Consider $p(z) = \left(\frac{RD_{\lambda,\alpha}^n f(z)}{z} \right)^{\delta} = \left(\frac{z + \sum_{j=2}^{\infty} \left\{ \alpha [1 + (j-1)\lambda]^n + (1-\alpha) C_{n+j-1}^n \right\} a_j z^j}{z} \right)^{\delta} = 1 + p_\delta z^\delta + p_{\delta+1} z^{\delta+1} + ...$, $z \in U$. Differentiating we obtain $\left(\frac{RD_{\lambda,\alpha}^n f(z)}{z} \right)^{\delta - 1} \left(RD_{\lambda,\alpha}^n f(z) \right)' = p(z) + \frac{1}{\delta} z p'(z)$, $z \in U$.

Then (2) becomes $h(z) \prec p(z) + \frac{1}{\delta} z p'(z)$, $z \in U$. By using Lemma 1.9 for $n = 1$ and $\gamma = \delta$, we have $q(z) \prec p(z)$, $z \in U$, i.e. $q(z) \prec \left(\frac{RD_{\lambda,\alpha}^n f(z)}{z} \right)^{\delta}$, $z \in U$, where $q(z) = \frac{\delta}{z^\delta} \int_0^z h(t) t^{\delta - 1} dt$. The function q is convex and it is the best subordinant. ■

Corollary 2.2 *Let $h(z)=\frac{1+(2\beta-1)z}{1+z}$ be a convex function in U, where $0\le\beta<1$. Let $n\in\mathbb{N}$, $\lambda,\alpha,\delta\ge 0$, $f\in\mathcal{A}$ and suppose that $\left(\frac{RD^n_{\lambda,\alpha}f(z)}{z}\right)^{\delta-1}\left(RD^n_{\lambda,\alpha}f(z)\right)'$ is univalent and $\left(\frac{RD^n_{\lambda,\alpha}f(z)}{z}\right)^{\delta}\in\mathcal{H}[1,\delta]\cap Q$. If*

$$h(z)\prec\left(\frac{RD^n_{\lambda,\alpha}f(z)}{z}\right)^{\delta-1}\left(RD^n_{\lambda,\alpha}f(z)\right)',\qquad z\in U,\tag{3}$$

then $q(z)\prec\left(\frac{RD^n_{\lambda,\alpha}f(z)}{z}\right)^{\delta}$, $z\in U$, where q is given by $q(z)=2\beta-1+\frac{2(1-\beta)\delta}{z^\delta}\int_0^z\frac{t^{\delta-1}}{1+t}dt$, $z\in U$. The function q is convex and it is the best subordinant.

Proof. Following the same steps as in the proof of Theorem 2.1 and considering $p(z)=\left(\frac{RD^m_{\lambda,\alpha}f(z)}{z}\right)^{\delta}$, the differential superordination (3) becomes $h(z)=\frac{1+(2\beta-1)z}{1+z}\prec p(z)+\frac{z}{\delta}p'(z)$, $z\in U$.

By using Lemma 1.9 for $\gamma=\delta$ and $n=1$, we have $q(z)\prec p(z)$, i.e., $q(z)=\frac{\delta}{z^\delta}\int_0^z h(t)t^{\delta-1}dt=\frac{\delta}{z^\delta}\int_0^z t^{\delta-1}\frac{1+(2\beta-1)t}{1+t}dt=\frac{\delta}{z^\delta}\int_0^z\left[(2\beta-1)t^{\delta-1}+2(1-\beta)\frac{t^{\delta-1}}{1+t}\right]dt=(2\beta-1)+\frac{2(1-\beta)\delta}{z^\delta}\int_0^z\frac{t^{\delta-1}}{1+t}dt\prec\left(\frac{RD^m_{\lambda,\alpha}f(z)}{z}\right)^{\delta}$, $z\in U$. The function q is convex and it is the best subordinant. ■

Theorem 2.3 *Let q be convex in U and let h be defined by $h(z)=q(z)+\frac{z}{\delta}q'(z)$. If $n\in\mathbb{N}$, $\lambda,\alpha,\delta\ge 0$, $f\in\mathcal{A}$, suppose that $\left(\frac{RD^n_{\lambda,\alpha}f(z)}{z}\right)^{\delta-1}\left(RD^n_{\lambda,\alpha}f(z)\right)'$ is univalent and $\left(\frac{RD^n_{\lambda,\alpha}f(z)}{z}\right)^{\delta}\in\mathcal{H}[1,\delta]\cap Q$ and satisfies the differential superordination*

$$h(z)=q(z)+\frac{z}{\delta}q'(z)\prec\left(\frac{RD^n_{\lambda,\alpha}f(z)}{z}\right)^{\delta-1}\left(RD^n_{\lambda,\alpha}f(z)\right)',\qquad z\in U,\tag{4}$$

then $q(z)\prec\left(\frac{RD^n_{\lambda,\alpha}f(z)}{z}\right)^{\delta}$, $z\in U$, where $q(z)=\frac{\delta}{z^\delta}\int_0^z h(t)t^{\delta-1}dt$. The function q is the best subordinant.

Proof. Following the same steps as in the proof of Theorem 2.1 and considering $p(z)=\left(\frac{RD^n_{\lambda,\alpha}f(z)}{z}\right)^{\delta}$, the differential superordination (4) becomes $q(z)+\frac{z}{\delta}q'(z)\prec p(z)+\frac{z}{\delta}p'(z)$, $z\in U$.

Using Lemma 1.10 for $n=1$ and $\gamma=\delta$, we have $q(z)\prec p(z)$, $z\in U$, i.e. $q(z)=\frac{\delta}{z^\delta}\int_0^z h(t)t^{\delta-1}dt\prec\frac{RD^n_{\lambda,\alpha}f(z)}{z}$, $z\in U$, and q is the best subordinant. ■

Remark 2.4 *For $n=1$, $\lambda=\frac{1}{2}$, $\alpha=2$, $\delta=1$ we obtain the same example as in [8, Example 4.3.2, p. 136].*

Theorem 2.5 *Let h be a convex function, $h(0)=1$. Let $\lambda,\alpha,\delta\ge 0$, $n\in\mathbb{N}$, $f\in\mathcal{A}$ and suppose that $z\frac{\delta+1}{\delta}\frac{RD^n_{\lambda,\alpha}f(z)}{\left(RD^{n+1}_{\lambda,\alpha}f(z)\right)^2}+\frac{z^2}{\delta}\frac{RD^n_{\lambda,\alpha}f(z)}{\left(RD^{n+1}_{\lambda,\alpha}f(z)\right)^2}\left[\frac{\left(RD^n_{\lambda,\alpha}f(z)\right)'}{RD^n_{\lambda,\alpha}f(z)}-2\frac{\left(RD^{n+1}_{\lambda,\alpha}f(z)\right)'}{RD^{n+1}_{\lambda,\alpha}f(z)}\right]$ is univalent and $z\frac{RD^n_{\lambda,\alpha}f(z)}{\left(RD^{n+1}_{\lambda,\alpha}f(z)\right)^2}\in\mathcal{H}[0,1]\cap Q$. If*

$$h(z)\prec z\frac{\delta+1}{\delta}\frac{RD^n_{\lambda,\alpha}f(z)}{\left(RD^{n+1}_{\lambda,\alpha}f(z)\right)^2}+\frac{z^2}{\delta}\frac{RD^n_{\lambda,\alpha}f(z)}{\left(RD^{n+1}_{\lambda,\alpha}f(z)\right)^2}\left[\frac{\left(RD^n_{\lambda,\alpha}f(z)\right)'}{RD^n_{\lambda,\alpha}f(z)}-2\frac{\left(RD^{n+1}_{\lambda,\alpha}f(z)\right)'}{RD^{n+1}_{\lambda,\alpha}f(z)}\right],\qquad z\in U,\tag{5}$$

then $q(z)\prec z\frac{RD^n_{\lambda,\alpha}f(z)}{\left(RD^{n+1}_{\lambda,\alpha}f(z)\right)^2}$, $z\in U$, where $q(z)=\frac{\delta}{z^\delta}\int_0^z h(t)t^{\delta-1}dt$. The function q is convex and it is the best subordinant.

Proof. Consider $p(z)=z\frac{RD^n_{\lambda,\alpha}f(z)}{\left(RD^{n+1}_{\lambda,\alpha}f(z)\right)^2}$ and we obtain $p(z)+\frac{z}{\delta}p'(z)=z\frac{\delta+1}{\delta}\frac{RD^n_{\lambda,\alpha}f(z)}{\left(RD^{n+1}_{\lambda,\alpha}f(z)\right)^2}+\frac{z^2}{\delta}\frac{RD^n_{\lambda,\alpha}f(z)}{\left(RD^{n+1}_{\lambda,\alpha}f(z)\right)^2}\cdot\left[\frac{\left(RD^n_{\lambda,\alpha}f(z)\right)'}{RD^n_{\lambda,\alpha}f(z)}-2\frac{\left(RD^{n+1}_{\lambda,\alpha}f(z)\right)'}{RD^{n+1}_{\lambda,\alpha}f(z)}\right]$. Relation (5) becomes $h(z)\prec p(z)+\frac{z}{\delta}p'(z)$, $z\in U$.

By using Lemma 1.10 for $n=1$ and $\gamma=\delta$, we have $q(z)\prec p(z)$, $z\in U$, i.e. $q(z)=\frac{\delta}{z^\delta}\int_0^z h(t)t^{\delta-1}dt\prec z\frac{RD^n_{\lambda,\alpha}f(z)}{\left(RD^{n+1}_{\lambda,\alpha}f(z)\right)^2}$, $z\in U$. The function q is convex and it is the best subordinant. ■

Theorem 2.6 *Let q be convex in U and let h be defined by $h(z) = q(z)+\frac{z}{\delta}q'(z)$. If $n \in \mathbb{N}$, $\lambda, \alpha, \delta \geq 0$, $f \in \mathcal{A}$, suppose that $z\frac{\delta+1}{\delta}\frac{RD^n_{\lambda,\alpha}f(z)}{\left(RD^{n+1}_{\lambda,\alpha}f(z)\right)^2} + \frac{z^2}{\delta}\frac{RD^n_{\lambda,\alpha}f(z)}{\left(RD^{n+1}_{\lambda,\alpha}f(z)\right)^2}\left[\frac{\left(RD^n_{\lambda,\alpha}f(z)\right)'}{RD^n_{\lambda,\alpha}f(z)} - 2\frac{\left(RD^{n+1}_{\lambda,\alpha}f(z)\right)'}{RD^{n+1}_{\lambda,\alpha}f(z)}\right]$ is univalent and $z\frac{RD^n_{\lambda,\alpha}f(z)}{\left(RD^{n+1}_{\lambda,\alpha}f(z)\right)^2} \in \mathcal{H}[0,1] \cap Q$ and satisfies the differential superordination*

$$h(z) \prec z\frac{\delta+1}{\delta}\frac{RD^n_{\lambda,\alpha}f(z)}{\left(RD^{n+1}_{\lambda,\alpha}f(z)\right)^2} + \frac{z^2}{\delta}\frac{RD^n_{\lambda,\alpha}f(z)}{\left(RD^{n+1}_{\lambda,\alpha}f(z)\right)^2}\left[\frac{\left(RD^n_{\lambda,\alpha}f(z)\right)'}{RD^n_{\lambda,\alpha}f(z)} - 2\frac{\left(RD^{n+1}_{\lambda,\alpha}f(z)\right)'}{RD^{n+1}_{\lambda,\alpha}f(z)}\right], \quad z \in U, \tag{6}$$

then $q(z) \prec z\frac{RD^n_{\lambda,\alpha}f(z)}{\left(RD^{n+1}_{\lambda,\alpha}f(z)\right)^2}$, $z \in U$, where $q(z) = \frac{\delta}{z^\delta}\int_0^z h(t)t^{\delta-1}dt$. The function q is the best subordinant.

Proof. Let $p(z) = z\frac{RD^n_{\lambda,\alpha}f(z)}{\left(RD^{n+1}_{\lambda,\alpha}f(z)\right)^2}$, $z \in U$. Differentiating, we obtain $p(z) + \frac{z}{\delta}p'(z) = z\frac{\delta+1}{\delta}\frac{RD^n_{\lambda,\alpha}f(z)}{\left(RD^{n+1}_{\lambda,\alpha}f(z)\right)^2} + \frac{z^2}{\delta}\frac{RD^n_{\lambda,\alpha}f(z)}{\left(RD^{n+1}_{\lambda,\alpha}f(z)\right)^2}\left[\frac{\left(RD^n_{\lambda,\alpha}f(z)\right)'}{RD^n_{\lambda,\alpha}f(z)} - 2\frac{\left(RD^{n+1}_{\lambda,\alpha}f(z)\right)'}{RD^{n+1}_{\lambda,\alpha}f(z)}\right]$, $z \in U$, and (6) becomes $h(z) = q(z) + \frac{z}{\delta}q'(z) \prec p(z) + \frac{z}{\delta}p'(z)$, $z \in U$.

By using Lemma 1.10 for $n = 1$ and $\gamma = \delta$, we have $q(z) \prec p(z)$, $z \in U$, i.e. $q(z) = \frac{\delta}{z^\delta}\int_0^z h(t)t^{\delta-1}dt \prec z\frac{RD^n_{\lambda,\alpha}f(z)}{\left(RD^{n+1}_{\lambda,\alpha}f(z)\right)^2}$, $z \in U$, and q is the best subordinant. ■

Theorem 2.7 *Let h be a convex function in U with $h(0) = 1$ and let $\lambda, \alpha, \delta \geq 0$, $n \in \mathbb{N}$, $f \in \mathcal{A}$, $z^2\frac{\delta+2}{\delta}\frac{\left(RD^n_{\lambda,\alpha}f(z)\right)'}{RD^n_{\lambda,\alpha}f(z)} + \frac{z^3}{\delta}\left[\frac{\left(RD^n_{\lambda,\alpha}f(z)\right)''}{RD^n_{\lambda,\alpha}f(z)} - \left(\frac{\left(RD^n_{\lambda,\alpha}f(z)\right)'}{RD^n_{\lambda,\alpha}f(z)}\right)^2\right]$ is univalent and $z^2\frac{\left(RD^n_{\lambda,\alpha}f(z)\right)'}{RD^n_{\lambda,\alpha}f(z)} \in \mathcal{H}[0,1] \cap Q$. If*

$$h(z) \prec z^2\frac{\delta+2}{\delta}\frac{\left(RD^n_{\lambda,\alpha}f(z)\right)'}{RD^n_{\lambda,\alpha}f(z)} + \frac{z^3}{\delta}\left[\frac{\left(RD^n_{\lambda,\alpha}f(z)\right)''}{RD^n_{\lambda,\alpha}f(z)} - \left(\frac{\left(RD^n_{\lambda,\alpha}f(z)\right)'}{RD^n_{\lambda,\alpha}f(z)}\right)^2\right], \quad z \in U, \tag{7}$$

then $q(z) \prec z^2\frac{\left(RD^n_{\lambda,\alpha}f(z)\right)'}{RD^n_{\lambda,\alpha}f(z)}$, $z \in U$, where $q(z) = \frac{\delta}{z^\delta}\int_0^z h(t)t^{\delta-1}dt$. The function q is convex and it is the best subordinant.

Proof. Let $p(z) = z^2\frac{\left(RD^n_{\lambda,\alpha}f(z)\right)'}{RD^n_{\lambda,\alpha}f(z)}$, $z \in U$. Differentiating, we obtain $p(z) + \frac{z}{\delta}p'(z) = z^2\frac{\delta+2}{\delta}\frac{\left(RD^n_{\lambda,\alpha}f(z)\right)'}{RD^n_{\lambda,\alpha}f(z)} + \frac{z^3}{\delta}\left[\frac{\left(RD^n_{\lambda,\alpha}f(z)\right)''}{RD^n_{\lambda,\alpha}f(z)} - \left(\frac{\left(RD^n_{\lambda,\alpha}f(z)\right)'}{RD^n_{\lambda,\alpha}f(z)}\right)^2\right]$, $z \in U$. Using the notation in (7), the differential superordination becomes $h(z) \prec p(z) + \frac{1}{\delta}zp'(z)$, $z \in U$.

By using Lemma 1.9 for $n = 1$ and $\gamma = \delta$, we have $q(z) \prec p(z)$, $z \in U$, i.e. $q(z) = \frac{\delta}{z^\delta}\int_0^z h(t)t^{\delta-1}dt \prec z^2\frac{\left(RD^n_{\lambda,\alpha}f(z)\right)'}{RD^n_{\lambda,\alpha}f(z)}$, $z \in U$. The function q is convex and it is the best subordinant. ■

Theorem 2.8 *Let q be a convex function in U and $h(z) = q(z)+\frac{z}{\delta}q'(z)$. Let $\lambda, \alpha, \delta \geq 0$, $n \in \mathbb{N}$, $f \in \mathcal{A}$, suppose that $z^2\frac{\delta+2}{\delta}\frac{\left(RD^n_{\lambda,\alpha}f(z)\right)'}{RD^n_{\lambda,\alpha}f(z)} + \frac{z^3}{\delta}\left[\frac{\left(RD^n_{\lambda,\alpha}f(z)\right)''}{RD^n_{\lambda,\alpha}f(z)} - \left(\frac{\left(RD^n_{\lambda,\alpha}f(z)\right)'}{RD^n_{\lambda,\alpha}f(z)}\right)^2\right]$ is univalent in U and $z^2\frac{\left(RD^n_{\lambda,\alpha}f(z)\right)'}{RD^n_{\lambda,\alpha}f(z)} \in \mathcal{H}[0,\delta] \cap Q$ and satisfies the differential superordination*

$$h(z) \prec z^2\frac{\delta+2}{\delta}\frac{\left(RD^n_{\lambda,\alpha}f(z)\right)'}{RD^n_{\lambda,\alpha}f(z)} + \frac{z^3}{\delta}\left[\frac{\left(RD^n_{\lambda,\alpha}f(z)\right)''}{RD^n_{\lambda,\alpha}f(z)} - \left(\frac{\left(RD^n_{\lambda,\alpha}f(z)\right)'}{RD^n_{\lambda,\alpha}f(z)}\right)^2\right], \quad z \in U, \tag{8}$$

then $q(z) \prec z^2\frac{\left(RD^n_{\lambda,\alpha}f(z)\right)'}{RD^n_{\lambda,\alpha}f(z)}$, $z \in U$, where $q(z) = \frac{\delta}{z^\delta}\int_0^z h(t)t^{\delta-1}dt$. The function q is the best subordinant.

Proof. Let $p(z) = z^2\frac{\left(RD^n_{\lambda,\alpha}f(z)\right)'}{RD^n_{\lambda,\alpha}f(z)}$. Differentiating, we obtain $p(z) + \frac{z}{\delta}p'(z) = z^2\frac{\delta+2}{\delta}\frac{\left(RD^n_{\lambda,\alpha}f(z)\right)'}{RD^n_{\lambda,\alpha}f(z)} +$

$\frac{z^3}{\delta}\left[\frac{\left(RD^n_{\lambda,\alpha}f(z)\right)''}{RD^n_{\lambda,\alpha}f(z)}-\left(\frac{\left(RD^n_{\lambda,\alpha}f(z)\right)'}{RD^n_{\lambda,\alpha}f(z)}\right)^2\right]$, $z\in U$. Using the notation in (8), the differential superordination becomes $h(z)=q(z)+\frac{z}{\delta}q'(z)\prec p(z)+\frac{1}{\delta}zp'(z)$.

By using Lemma 1.10 for $\gamma=\delta$ and $n=1$, we have $q(z)\prec p(z)$, i.e., $q(z)=\frac{\delta}{z^\delta}\int_0^z h(t)t^{\delta-1}dt\prec z^2\frac{\left(RD^n_{\lambda,\alpha}f(z)\right)'}{RD^n_{\lambda,\alpha}f(z)}$, $z\in U$. The function q is the best subordinant. ■

Theorem 2.9 *Let h be a convex function, $h(0)=1$. Let $n\in\mathbb{N}$, $\lambda,\alpha,\delta\geq 0$, $f\in\mathcal{A}$ and suppose that $1-\frac{RD^n_{\lambda,\alpha}f(z)\cdot\left(RD^n_{\lambda,\alpha}f(z)\right)''}{\left[\left(RD^n_{\lambda,\alpha}f(z)\right)'\right]^2}$ is univalent and $\frac{RD^n_{\lambda,\alpha}f(z)}{z\left(RD^n_{\lambda,\alpha}f(z)\right)'}\in\mathcal{H}[1,1]\cap Q$. If*

$$h(z)\prec 1-\frac{RD^n_{\lambda,\alpha}f(z)\cdot\left(RD^n_{\lambda,\alpha}f(z)\right)''}{\left[\left(RD^n_{\lambda,\alpha}f(z)\right)'\right]^2},\qquad z\in U, \tag{9}$$

then $q(z)\prec\frac{RD^n_{\lambda,\alpha}f(z)}{z\left(RD^n_{\lambda,\alpha}f(z)\right)'}$, $z\in U$, where q is given by $q(z)=\frac{1}{z}\int_0^z h(t)dt$, $z\in U$. The function q is convex and it is the best subordinant.

Proof. Let $p(z)=\frac{RD^n_{\lambda,\alpha}f(z)}{z\left(RD^n_{\lambda,\alpha}f(z)\right)'}$, $z\in U$. Differentiating, we obtain $1-\frac{RD^n_{\lambda,\alpha}f(z)\cdot\left(RD^n_{\lambda,\alpha}f(z)\right)''}{\left[\left(RD^n_{\lambda,\alpha}f(z)\right)'\right]^2}=p(z)+zp'(z)$, $z\in U$, and (9) becomes $h(z)\prec p(z)+zp'(z)$, $z\in U$.

Using Lemma 1.9 for $\gamma=1$ and $n=1$, we have $q(z)\prec p(z)$, $z\in U$, i.e. $q(z)=\frac{1}{z}\int_0^z h(t)dt\prec\frac{RD^n_{\lambda,\alpha}f(z)}{z\left(RD^n_{\lambda,\alpha}f(z)\right)'}$, $z\in U$. The function q is convex and it is the best subordinant. ■

Corollary 2.10 *Let $h(z)=\frac{1+(2\beta-1)z}{1+z}$ be a convex function in U, where $0\leq\beta<1$. Let $n\in\mathbb{N}$, $\lambda,\alpha,\delta\geq 0$, $f\in\mathcal{A}$ and suppose that $1-\frac{RD^n_{\lambda,\alpha}f(z)\cdot\left(RD^n_{\lambda,\alpha}f(z)\right)''}{\left[\left(RD^n_{\lambda,\alpha}f(z)\right)'\right]^2}$ is univalent and $\frac{RD^n_{\lambda,\alpha}f(z)}{z\left(RD^n_{\lambda,\alpha}f(z)\right)'}\in\mathcal{H}[1,1]\cap Q$. If*

$$h(z)\prec 1-\frac{RD^n_{\lambda,\alpha}f(z)\cdot\left(RD^n_{\lambda,\alpha}f(z)\right)''}{\left[\left(RD^n_{\lambda,\alpha}f(z)\right)'\right]^2},\qquad z\in U, \tag{10}$$

then $q(z)\prec\frac{RD^n_{\lambda,\alpha}f(z)}{z\left(RD^n_{\lambda,\alpha}f(z)\right)'}$, $z\in U$, where q is given by $q(z)=(2\beta-1)+2(1-\beta)\frac{\ln(1+z)}{z}$, $z\in U$. The function q is convex and it is the best subordinant .

Proof. Following the same steps as in the proof of Theorem 2.9 and considering $p(z)=\frac{RD^n_{\lambda,\alpha}f(z)}{z\left(RD^n_{\lambda,\alpha}f(z)\right)'}$, the differential subordination (10) becomes $h(z)=\frac{1+(2\beta-1)z}{1+z}\prec p(z)+zp'(z)$, $z\in U$.

By using Lemma 1.9 for $\gamma=1$, we have $q(z)\prec p(z)$, i.e. $q(z)=\frac{1}{z}\int_0^z h(t)dt=\frac{1}{z}\int_0^z\frac{1+(2\beta-1)t}{1+t}dt=\frac{1}{z}\int_0^z\left[(2\beta-1)+\frac{2(1-\beta)}{1+t}\right]dt=(2\beta-1)+2(1-\beta)\frac{\ln(1+z)}{z}\prec\frac{RD^n_{\lambda,\alpha}f(z)}{z\left(RD^n_{\lambda,\alpha}f(z)\right)'}$, $z\in U$. ■

Theorem 2.11 *Let q be convex in U and let h be defined by $h(z)=q(z)+zq'(z)$. If $n\in\mathbb{N}$, $\lambda,\alpha,\delta\geq 0$, $f\in\mathcal{A}$, suppose that $1-\frac{RD^n_{\lambda,\alpha}f(z)\cdot\left(RD^n_{\lambda,\alpha}f(z)\right)''}{\left[\left(RD^n_{\lambda,\alpha}f(z)\right)'\right]^2}$ is univalent and $\frac{RD^n_{\lambda,\alpha}f(z)}{z\left(RD^n_{\lambda,\alpha}f(z)\right)'}\in\mathcal{H}[1,1]\cap Q$ and satisfies the differential superordination*

$$h(z)\prec 1-\frac{RD^n_{\lambda,\alpha}f(z)\cdot\left(RD^n_{\lambda,\alpha}f(z)\right)''}{\left[\left(RD^n_{\lambda,\alpha}f(z)\right)'\right]^2},\qquad z\in U, \tag{11}$$

then $q(z)\prec\frac{RD^n_{\lambda,\alpha}f(z)}{z\left(RD^n_{\lambda,\alpha}f(z)\right)'}$, $z\in U$, where q is given by $q(z)=\frac{1}{z}\int_0^z h(t)dt$, $z\in U$. The function q is the best subordinant.

Proof. Let $p(z)=\frac{RD^n_{\lambda,\alpha}f(z)}{z\left(RD^n_{\lambda,\alpha}f(z)\right)'}$. Differentiating, we obtain $1-\frac{RD^n_{\lambda,\alpha}f(z)\cdot\left(RD^n_{\lambda,\alpha}f(z)\right)''}{\left[\left(RD^n_{\lambda,\alpha}f(z)\right)'\right]^2}=p(z)+zp'(z)$, $z\in U$, and (11) becomes $h(z)=q(z)+zq'(z)\prec p(z)+zp'(z)$, $z\in U$.

Using Lemma 1.9 for $\gamma=1$ and $n=1$, we have $q(z)\prec p(z)$, $z\in U$, i.e. $q(z)=\frac{1}{z}\int_0^z h(t)dt\prec\frac{RD^n_{\lambda,\alpha}f(z)}{z\left(RD^n_{\lambda,\alpha}f(z)\right)'}$, $z\in U$. The function q is the best subordinant. ■

Example 2.12 *Let $h(z)=\frac{1-z}{1+z}$ a convex function in U with $h(0)=1$. Let $f(z)=z+z^2$, $z\in U$. For $n=1$, $\lambda=\frac{1}{2}$, $\alpha=2$, we obtain $RD^1_{\frac{1}{2},2}f(z)=-R^1f(z)+2D^1_{\frac{1}{2}}f(z)=-zf'(z)+2\left(\frac{1}{2}f(z)+\frac{1}{2}zf'(z)\right)=f(z)=z+z^2$, $z\in U$.*

Then $\left(RD^1_{\frac{1}{2},2}f(z)\right)'=f'(z)=1+2z$, $\frac{RD^1_{\frac{1}{2},2}f(z)}{z\left(RD^1_{\frac{1}{2},2}f(z)\right)'}=\frac{z+z^2}{z(1+2z)}=\frac{1+z}{1+2z}$, $1-\frac{RD^1_{\frac{1}{2},2}f(z)\cdot\left(RD^1_{\frac{1}{2},2}f(z)\right)''}{\left[\left(RD^1_{\frac{1}{2},2}f(z)\right)'\right]^2}=1-\frac{(z+z^2)\cdot 2}{(1+2z)^2}=\frac{2z^2+2z+1}{(1+2z)^2}$. We have $q(z)=\frac{1}{z}\int_0^z\frac{1-t}{1+t}dt=-1+\frac{2\ln(1+z)}{z}$.

Using Theorem 2.9 we obtain$\frac{1-z}{1+z}\prec\frac{2z^2+2z+1}{(1+2z)^2}$, $z\in U$, induce $-1+\frac{2\ln(1+z)}{z}\prec\frac{1+z}{1+2z}$, $z\in U$.

Theorem 2.13 *Let h be a convex function, $h(0)=1$ and let $\lambda,\alpha\geq 0$, $n\in\mathbb{N}$, $f\in\mathcal{A}$, suppose that $\left[\left(RD^n_{\lambda,\alpha}f(z)\right)'\right]^2+RD^n_{\lambda,\alpha}f(z)\cdot\left(RD^n_{\lambda,\alpha}f(z)\right)''$ is univalent and $\frac{RD^n_{\lambda,\alpha}f(z)\cdot\left(RD^n_{\lambda,\alpha}f(z)\right)'}{z}\in\mathcal{H}[0,1]\cap Q$. If*

$$h(z)\prec\left[\left(RD^n_{\lambda,\alpha}f(z)\right)'\right]^2+RD^n_{\lambda,\alpha}f(z)\cdot\left(RD^n_{\lambda,\alpha}f(z)\right)'',\ z\in U, \tag{12}$$

then $q(z)\prec\frac{RD^n_{\lambda,\alpha}f(z)\cdot\left(RD^n_{\lambda,\alpha}f(z)\right)'}{z}$, $z\in U$, where $q(z)=\frac{1}{z}\int_0^z h(t)dt$. The function q is convex and it is the best subordinant.

Proof. Let $p(z)=\frac{RD^n_{\lambda,\alpha}f(z)\cdot\left(RD^n_{\lambda,\alpha}f(z)\right)'}{z}$, $z\in U$. Differentiating, we obtain $\left[\left(RD^n_{\lambda,\alpha}f(z)\right)'\right]^2+RD^n_{\lambda,\alpha}f(z)\cdot\left(RD^n_{\lambda,\alpha}f(z)\right)''=p(z)+zp'(z)$, $z\in U$, and (15) becomes $h(z)\prec p(z)+zp'(z)$, $z\in U$.

Using Lemma 1.9 for $n=1$ and $\gamma=1$, we have $q(z)\prec p(z)$, $z\in U$, i.e. $q(z)=\frac{1}{z}\int_0^z h(t)dt\prec\frac{RD^n_{\lambda,\alpha}f(z)\cdot\left(RD^n_{\lambda,\alpha}f(z)\right)'}{z}$, $z\in U$. The function q is convex and it is the best subordinant. ■

Corollary 2.14 *Let $h(z)=\frac{1+(2\beta-1)z}{1+z}$ be a convex function in U, where $0\leq\beta<1$. Let $\lambda,\alpha\geq 0$, $n\in\mathbb{N}$, $f\in\mathcal{A}$, suppose that $\left[\left(RD^n_{\lambda,\alpha}f(z)\right)'\right]^2+RD^n_{\lambda,\alpha}f(z)\cdot\left(RD^n_{\lambda,\alpha}f(z)\right)''$ is univalent and $\frac{RD^n_{\lambda,\alpha}f(z)\cdot\left(RD^n_{\lambda,\alpha}f(z)\right)'}{z}\in\mathcal{H}[0,1]\cap Q$. If*

$$h(z)\prec\left[\left(RD^n_{\lambda,\alpha}f(z)\right)'\right]^2+RD^n_{\lambda,\alpha}f(z)\cdot\left(RD^n_{\lambda,\alpha}f(z)\right)'',\ z\in U, \tag{13}$$

then $q(z)\prec\frac{RD^n_{\lambda,\alpha}f(z)\cdot\left(RD^n_{\lambda,\alpha}f(z)\right)'}{z}$, $z\in U$, where q is given by $q(z)=2\beta-1+2(1-\beta)\frac{\ln(1+z)}{z}$, $z\in U$. The function q is convex and it is the best subordinant.

Proof. Following the same steps as in the proof of Theorem 2.13 and considering $p(z)=\frac{RD^n_{\lambda,\alpha}f(z)\cdot\left(RD^n_{\lambda,\alpha}f(z)\right)'}{z}$, the differential superordination (13) becomes $h(z)=\frac{1+(2\beta-1)z}{1+z}\prec p(z)+zp'(z)$, $z\in U$.

By using Lemma 1.9 for $\gamma=1$ and $n=1$, we have $q(z)\prec p(z)$, i.e., $q(z)=\frac{1}{z}\int_0^z h(t)dt=\frac{1}{z}\int_0^z\frac{1+(2\beta-1)t}{1+t}dt=2\beta-1+2(1-\beta)\frac{1}{z}\ln(z+1)\prec\frac{RD^n_{\lambda,\alpha}f(z)\cdot\left(RD^n_{\lambda,\alpha}f(z)\right)'}{z}$, $z\in U$. The function q is convex and it is the best subordinant. ■

Theorem 2.15 *Let q be a convex function in U and h be defined by $h(z)=q(z)+zq'(z)$. Let $\lambda,\alpha\geq 0$, $n\in\mathbb{N}$, $f\in\mathcal{A}$, suppose that $\left[\left(RD^n_{\lambda,\alpha}f(z)\right)'\right]^2+RD^n_{\lambda,\alpha}f(z)\cdot\left(RD^n_{\lambda,\alpha}f(z)\right)''$ is univalent and $\frac{RD^n_{\lambda,\alpha}f(z)\cdot\left(RD^n_{\lambda,\alpha}f(z)\right)'}{z}\in\mathcal{H}[0,1]\cap Q$ and satisfies the differential superordination*

$$h(z)=q(z)+zq'(z)\prec\left[\left(RD^n_{\lambda,\alpha}f(z)\right)'\right]^2+RD^n_{\lambda,\alpha}f(z)\cdot\left(RD^n_{\lambda,\alpha}f(z)\right)'',\ z\in U, \tag{14}$$

then $q(z)\prec\frac{RD^n_{\lambda,\alpha}f(z)\cdot\left(RD^n_{\lambda,\alpha}f(z)\right)'}{z}$, $z\in U$, where $q(z)=\frac{1}{z}\int_0^z h(t)dt$. The function q is the best subordinant.

Proof. Following the same steps as in the proof of Theorem 2.13 and considering $p(z)=\frac{RD^n_{\lambda,\alpha}f(z)\cdot\left(RD^n_{\lambda,\alpha}f(z)\right)'}{z}$, the differential superordination (14) becomes $h(z)=q(z)+zq'(z)\prec p(z)+zp'(z)$, $z\in U$.

By using Lemma 1.10 for $\gamma=1$ and $n=1$, we have $q(z)\prec p(z)$, i.e., $q(z)=\frac{1}{z}\int_0^z h(t)dt\prec\frac{RD^n_{\lambda,\alpha}f(z)\cdot\left(RD^n_{\lambda,\alpha}f(z)\right)'}{z}$, $z\in U$. The function q is the best subordinant. ■

Example 2.16 *Let $h(z)=\frac{1-z}{1+z}$ a convex function in U with $h(0)=1$. Let $f(z)=z+z^2$, $z\in U$. For $n=1$, $\lambda=\frac{1}{2}$, $\alpha=2$, we obtain $RD^1_{\frac{1}{2},2}f(z)=-R^1f(z)+2D^1_{\frac{1}{2}}f(z)=-zf'(z)+2\left(\frac{1}{2}f(z)+\frac{1}{2}zf'(z)\right)=f(z)=z+z^2$, $z\in U$.*

Then $\left(RD^1_{\frac{1}{2},2}f(z)\right)'=f'(z)=1+2z$, $\frac{RD^1_{\frac{1}{2},2}f(z)\cdot\left(RD^1_{\frac{1}{2},2}f(z)\right)'}{z}=\frac{(z+z^2)(1+2z)}{z}=2z^2+3z+1$, $\left[\left(RD^1_{\frac{1}{2},2}f(z)\right)'\right]^2+RD^1_{\frac{1}{2},2}f(z)\cdot\left(RD^1_{\frac{1}{2},2}f(z)\right)''=(1+2z)^2+(z+z^2)\cdot 2=6z^2+6z+1$. We have $q(z)=\frac{1}{z}\int_0^z\frac{1-t}{1+t}dt=-1+\frac{2\ln(1+z)}{z}$.

Using Theorem 2.13 we obtain $\frac{1-z}{1+z}\prec 6z^2+6z+1$, $z\in U$, induce $-1+\frac{2\ln(1+z)}{z}\prec 2z^2+3z+1$, $z\in U$.

Theorem 2.17 *Let h be a convex function, $h(0)=1$. Let $\lambda,\alpha\geq 0$, $\delta\in(0,1)$, $n\in\mathbb{N}$, $f\in\mathcal{A}$, and suppose that $\left(\frac{z}{RD^n_{\lambda,\alpha}f(z)}\right)^\delta\frac{RD^{n+1}_{\lambda,\alpha}f(z)}{1-\delta}\left(\frac{\left(RD^{n+1}_{\lambda,\alpha}f(z)\right)'}{RD^{n+1}_{\lambda,\alpha}f(z)}-\delta\frac{\left(RD^n_{\lambda,\alpha}f(z)\right)'}{RD^n_{\lambda,\alpha}f(z)}\right)$ is univalent and $\frac{RD^{n+1}_{\lambda,\alpha}f(z)}{z}\cdot\left(\frac{z}{RD^n_{\lambda,\alpha}f(z)}\right)^\delta\in\mathcal{H}[0,1]\cap Q$. If*

$$h(z)\prec\left(\frac{z}{RD^n_{\lambda,\alpha}f(z)}\right)^\delta\frac{RD^{n+1}_{\lambda,\alpha}f(z)}{1-\delta}\left(\frac{\left(RD^{n+1}_{\lambda,\alpha}f(z)\right)'}{RD^{n+1}_{\lambda,\alpha}f(z)}-\delta\frac{\left(RD^n_{\lambda,\alpha}f(z)\right)'}{RD^n_{\lambda,\alpha}f(z)}\right),\ z\in U,\tag{15}$$

then $q(z)\prec\frac{RD^{n+1}_{\lambda,\alpha}f(z)}{z}\cdot\left(\frac{z}{RD^n_{\lambda,\alpha}f(z)}\right)^\delta$, $z\in U$, where $q(z)=\frac{1-\delta}{z^{1-\delta}}\int_0^z h(t)t^{-\delta}dt$. The function q is convex and it is the best subordinant.

Proof. Let $p(z)=\frac{RD^{n+1}_{\lambda,\alpha}f(z)}{z}\cdot\left(\frac{z}{RD^n_{\lambda,\alpha}f(z)}\right)^\delta$, $z\in U$. Differentiating, we obtain $\left(\frac{z}{RD^n_{\lambda,\alpha}f(z)}\right)^\delta\frac{RD^{n+1}_{\lambda,\alpha}f(z)}{1-\delta}\cdot\left(\frac{\left(RD^{n+1}_{\lambda,\alpha}f(z)\right)'}{RD^{n+1}_{\lambda,\alpha}f(z)}-\delta\frac{\left(RD^n_{\lambda,\alpha}f(z)\right)'}{RD^n_{\lambda,\alpha}f(z)}\right)=p(z)+\frac{1}{1-\delta}zp'(z)$, $z\in U$, and (15) becomes $h(z)\prec p(z)+\frac{1}{1-\delta}zp'(z)$, $z\in U$.

Using Lemma 1.9, we have $q(z)\prec p(z)$, $z\in U$, i.e. $q(z)=\frac{1-\delta}{z^{1-\delta}}\int_0^z h(t)t^{-\delta}dt\prec\frac{RD^{n+1}_{\lambda,\alpha}f(z)}{z}\cdot\left(\frac{z}{RD^n_{\lambda,\alpha}f(z)}\right)^\delta$, $z\in U$. The function q is convex and it is the best subordinant. ■

Theorem 2.18 *Let q be a convex function in U and $h(z)=q(z)+\frac{z}{1-\delta}q'(z)$. If $\lambda,\alpha\geq 0$, $\delta\in(0,1)$, $n\in\mathbb{N}$, $f\in\mathcal{A}$, suppose that $\left(\frac{z}{RD^n_{\lambda,\alpha}f(z)}\right)^\delta\frac{RD^{n+1}_{\lambda,\alpha}f(z)}{1-\delta}\left(\frac{\left(RD^{n+1}_{\lambda,\alpha}f(z)\right)'}{RD^{n+1}_{\lambda,\alpha}f(z)}-\delta\frac{\left(RD^n_{\lambda,\alpha}f(z)\right)'}{RD^n_{\lambda,\alpha}f(z)}\right)$ is univalent and $\frac{RD^{n+1}_{\lambda,\alpha}f(z)}{z}\cdot\left(\frac{z}{RD^n_{\lambda,\alpha}f(z)}\right)^\delta\in\mathcal{H}[0,1]\cap Q$ satisfies the differential superordination*

$$h(z)\prec\left(\frac{z}{RD^n_{\lambda,\alpha}f(z)}\right)^\delta\frac{RD^{n+1}_{\lambda,\alpha}f(z)}{1-\delta}\left(\frac{\left(RD^{n+1}_{\lambda,\alpha}f(z)\right)'}{RD^{n+1}_{\lambda,\alpha}f(z)}-\delta\frac{\left(RD^n_{\lambda,\alpha}f(z)\right)'}{RD^n_{\lambda,\alpha}f(z)}\right),\ z\in U,\tag{16}$$

then $q(z)\prec\frac{RD^{n+1}_{\lambda,\alpha}f(z)}{z}\cdot\left(\frac{z}{RD^n_{\lambda,\alpha}f(z)}\right)^\delta$, $z\in U$, where $q(z)=\frac{1-\delta}{z^{1-\delta}}\int_0^z h(t)t^{-\delta}dt$. The function q is the best subordinant.

Proof. Let $p(z)=\frac{RD^{n+1}_{\lambda,\alpha}f(z)}{z}\cdot\left(\frac{z}{RD^n_{\lambda,\alpha}f(z)}\right)^\delta$. Differentiating, we obtain
$\left(\frac{z}{RD^n_{\lambda,\alpha}f(z)}\right)^\delta\frac{RD^{n+1}_{\lambda,\alpha}f(z)}{1-\delta}\left(\frac{\left(RD^{n+1}_{\lambda,\alpha}f(z)\right)'}{RD^{n+1}_{\lambda,\alpha}f(z)}-\delta\frac{\left(RD^n_{\lambda,\alpha}f(z)\right)'}{RD^n_{\lambda,\alpha}f(z)}\right)=p(z)+\frac{1}{1-\delta}zp'(z)$, $z\in U$.

Using the notation in (16), the differential superordination becomes $h(z)=q(z)+\frac{z}{1-\delta}q'(z)\prec p(z)+\frac{1}{1-\delta}zp'(z)$. By using Lemma 1.10, we have $q(z)\prec p(z)$, $z\in U$, i.e. $q(z)=\frac{1-\delta}{z^{1-\delta}}\int_0^z h(t)t^{-\delta}dt\prec\frac{RD^{n+1}_{\lambda,\alpha}f(z)}{z}\cdot\left(\frac{z}{RD^n_{\lambda,\alpha}f(z)}\right)^\delta$, $z\in U$, and q is the best subordinant. ■

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Generalized block diagonal and block triangular preconditioners for non-symmetric indefinite linear systems *

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Abstract

In this paper, we study the block diagonal and block triangular preconditioners containing a parameter for general two-by-two block linear systems with zero (2,2)-block. The boundary estimates of eigenvalue of preconditioned system are given. If the parameter is chosen appropriately, the preconditioned system will have a more clustered spectrum, which results in faster convergence for Krylov subspace methods in many applications. Finally, numerical experiments are presented to illustrate the performance of these preconditioners.

Keywords: Saddle point problem; Indefinite systems; Eigenvalue bounds; Preconditioners; Parameter.

AMS classification: 65F10, 65F15, 65N22.

1 Introduction

We consider the following 2×2 block linear systems of the form

$$\mathcal{A}\ v \equiv \begin{pmatrix} A & B^T \\ C & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} = b, \tag{1}$$

where $A \in R^{n\times n}$ is non-singular, and B and $C \in R^{m\times n}$ (m < n) are of full rank. Such problems are referred to as generalized saddle point problems, which appear in many applications and have attracted a lot of research [6, 7, 11, 14–16, 23, 25]; especially, see [5] for a comprehensive survey and related references.

As is known, there exist two kinds of methods to solve the linear systems: direct methods and iterative methods. Direct methods are widely employed when the size of the coefficient matrix is not too large, and are usually regarded as robust methods. The memory and the computational requirements for solving the large linear systems may seriously challenge the most efficient direct solution method available today. Naturally, it is necessary that we make use of iterative methods instead of direct methods to solve the large sparse linear systems. Meanwhile, iterative methods are easier to implement efficiently on high performance computers than direct methods. Currently, Krylov subspace methods [18] are considered as one kind of the important and efficient iterative techniques available for solving large linear systems, because they are cheap to be implemented and are able to exploit the sparsity of the coefficient matrix. However, in fact, the Krylov subspace methods are not competitive without a good preconditioner. To speed up the convergence, it is profitable to use a good preconditioner. A lot of preconditioners

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are presented for solving systems (1), such as block-diagonal preconditioners (with exact Schur complement and approximate Schur complement) [4, 6, 11, 16, 23], augmentation block preconditioners [7] and constraint preconditioners [14, 25]. For the case $C = B$, there also exist lots of iterative methods and preconditioners for solving systems (1), see [8, 17, 19, 20, 22, 24, 27] for more details. Especially, the spectral analysis of the preconditioned matrix are given on block diagonal and triangular preconditioners in [1–3, 12, 13].

Recently, De Sturler and Liesen [23] provided a detailed analysis on the eigenvalues of the preconditioned matrix with block diagonal preconditioner. Let

$$A = G - E \tag{2}$$

be a splitting of A, where $G \in R^{n\times n}$ is non-singular, then the block diagonal preconditioner

$$\mathcal{G} = \begin{pmatrix} G & 0 \\ 0 & CG^{-1}B^T \end{pmatrix} \tag{3}$$

was shown in [23]. And the preconditioner

$$\mathcal{P} = \begin{pmatrix} G & 0 \\ 0 & -CG^{-1}B^T \end{pmatrix} \tag{4}$$

was considered in [6] by Cao. When $G = A$, the preconditioners $\mathcal{G}$ and $\mathcal{P}$ were analyzed in [16] and [8], respectively. By the combination ideas of [22], we introduce a combination parameter β, and present the following preconditioner

$$\mathcal{P}(\beta) = \beta\mathcal{G} + (1-\beta)\mathcal{P} = \begin{pmatrix} G & 0 \\ 0 & (2\beta-1)CG^{-1}B^T \end{pmatrix}. \tag{5}$$

For the simple of the notations, we define the preconditioner (5) as

$$P_\alpha(G) = \begin{pmatrix} G & 0 \\ 0 & \alpha CG^{-1}B^T \end{pmatrix}. \tag{6}$$

Moreover, we also study the following two block triangular preconditioners containing a parameter in different place:

$$F_\alpha(G) = \begin{pmatrix} G & B^T \\ 0 & \alpha CG^{-1}B^T \end{pmatrix}, \tag{7}$$

and

$$H_\alpha(G) = \begin{pmatrix} G & \alpha B^T \\ 0 & CG^{-1}B^T \end{pmatrix}. \tag{8}$$

We will analyse the properties of preconditioned matrix $P_\alpha(G)^{-1}\mathcal{A}$, $F_\alpha(G)^{-1}\mathcal{A}$ and $H_\alpha(G)^{-1}\mathcal{A}$, and provide their bounds of eigenvalues.

The remainder of this paper is organized as follows. In Sections 2 and 3, the block diagonal and block triangular preconditioners are presented, and bounds of eigenvalues of the preconditioned system are analysed. Section 4 and 5 give numerical experiments and conclusions, respectively.

2 Block Diagonal Preconditioner

Throughout this paper, $\|\cdot\|$ indicates the matrix 2-norm, $|\cdot|$ stands for the modulus. Now, we discuss the properties of the block diagonal preconditioned matrix

$$P_\alpha(G)^{-1}\mathcal{A} = \begin{pmatrix} I_n - G^{-1}E & G^{-1}B^T \\ \frac{1}{\alpha}\left(CG^{-1}B^T\right)^{-1}C & 0 \end{pmatrix}.$$

Denote $M = \left(CG^{-1}B^T\right)^{-1} C,\ N = G^{-1}B^T,\ S = G^{-1}E$, then we have

$$T(S) = T_\alpha(S) = P_\alpha(G)^{-1}\mathcal{A} = \begin{pmatrix} I_n - S & N \\ \frac{1}{\alpha}M & 0 \end{pmatrix}, \tag{9}$$

where $M \in R^{m\times n},\ N \in R^{n\times m}$ and $MN = I_m, (NM)^2 = NM$. Note that NM is a projection matrix of rank m. Let $U_1 = [u_1, ..., u_{n-m}] \in R^{n\times(n-m)}$ form a basis of $\mathcal{N}(NM)$, the nullspace of NM, and let $U_2 = [u_{n-m+1}, ..., u_n] \in R^{n\times m}$ form a basis of $\mathcal{R}(NM)$, the range of NM. Thus, we have that $[U_1, U_2] \in R^{n\times n}$ is non-singular and

$$NM[U_1, U_2] = [U_1, U_2]\begin{pmatrix} 0 & 0 \\ 0 & I_m \end{pmatrix}, \tag{10}$$

i.e., $[U_1, U_2]$ is an eigenvector matrix of the projector matrix NM. Next, we give properties of eigenvalues and eigenvectors of $T(0)$ which is the preconditioned matrix $P_\alpha(A)^{-1}\mathcal{A}$ (for short $\mathcal{T}$).

Theorem 1 *The block diagonal preconditioned matrix $\mathcal{T} = P_\alpha(A)^{-1}\mathcal{A}$ is diagonalizable and has three distinct eigenvalues: 1, $\lambda^\pm = \frac{1\pm\sqrt{1+\frac{4}{\alpha}}}{2}$ $(\alpha \neq -4)$. The eigenvector matrix is given by $Y(0)$:*

$$Y(0) = \begin{pmatrix} U_1 & U_2 & U_2 \\ 0 & (\lambda^+)^{-1} MU_2 & (\lambda^-)^{-1} MU_2 \end{pmatrix}. \tag{11}$$

Proof. The proof is anaglous to that of Theorem 3.3 of [23] or Theorem 2.1 of [6], hence, it is omitted. □

remark 1 *$\mathcal{T}$ satisfies $(\mathcal{T} - I)(\mathcal{T}^2 - \mathcal{T} - \frac{1}{\alpha}I) = 0$. When $\alpha = -4$, the preconditioned matrix $\mathcal{T}$ has only two distinct eigenvalues, but it is not diagonalizable; when $\alpha < 0$, the preconditioned matrix $\mathcal{T}$ is a positive stable matrix, i.e., the eigenvalues have positive real parts; when $\alpha > 0$, the preconditioned matrix $\mathcal{T}$ is indefinite and all the eigenvalues are real number.*

Now, we derive bounds of the eigenvalues of each matrix $T(S)$ in terms of the corresponding matrix $\mathcal{T}$.

Theorem 2 *Let $Y(0)$ be the eigenvector matrix of $\mathcal{T}$, for each matrix S, and each eigenvalue λ_S of $T(S)$, there is an eigenvalue $\lambda(\lambda = 1$ or $\frac{1\pm\sqrt{1+\frac{4}{\alpha}}}{2})$ $(\alpha \neq -4)$ of $\mathcal{T}$ such that*

$$|\lambda_S - \lambda| \leq \|Y(0)^{-1}\begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} Y(0)\|, \tag{12}$$

$$\leq c_\alpha \|[U_1, U_2]^{-1} S [U_1, U_2]\|, \tag{13}$$

where positive number c_α satisfies $c_\alpha^2 = \begin{cases} \frac{3\alpha+10}{\alpha+4} - \sqrt{\frac{\alpha}{\alpha+4}}, & 0 < \alpha; \\ \frac{2\alpha+10}{\alpha+4}, & -3 \leq \alpha < 0; \\ \frac{4}{\alpha+4}, & -4 < \alpha < -3; \\ \frac{2\alpha+4}{\alpha+4}, & \alpha < -4. \end{cases}$

Proof. The proof of the Theorem 2 is similar to that of Theorem 3.5 of [23] or Theorem 2.1 of [6], but for the completeness, we also give the proof. Matrix $T(S)$ can be splitting into

$$T(S) = T(0) - \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}. \tag{14}$$

Since $T(0)$ is diagonalizable and has eigenvector matrix $Y(0)$, inequality (12) follows from a well-known result in matrix perturbation theory [21] or Bauer-Fike theorem [26].

To prove (13), we consider the following two-by-two block decomposition

$$Y(0) = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix},$$

where $Y_{11} = [U_1, U_2] \in R^{n\times n}$ and $Y_{22} = (\lambda^-)^{-1}MU_2 \in R^{m\times m}$ are both invertible. Then $Y(0)^{-1}$ can be expressed as

$$Y(0)^{-1} = \begin{pmatrix} \left(Y_{11} - Y_{12}Y_{22}^{-1}Y_{21}\right)^{-1} & -Y_{11}^{-1}Y_{12}\left(Y_{22} - Y_{21}Y_{11}^{-1}Y_{12}\right)^{-1} \\ -Y_{22}^{-1}Y_{21}\left(Y_{11} - Y_{12}Y_{22}^{-1}Y_{21}\right)^{-1} & \left(Y_{22} - Y_{21}Y_{11}^{-1}Y_{12}\right)^{-1} \end{pmatrix}.$$

By simple calculation, we have

$$\begin{aligned}(Y_{11} - Y_{12}Y_{22}^{-1}Y_{21})^{-1} =&([U_1, U_2] - U_2(\lambda^-)(MU_2)^{-1}[0, (\lambda^+)^{-1}MU_2])^{-1} \\ =&([U_1, U_2] - [0, (\lambda^-/\lambda^+)U_2])^{-1} \\ =&\begin{pmatrix} I_{n-m} & 0 \\ 0 & \frac{\lambda^+}{\lambda^+-\lambda^-}I_m \end{pmatrix}[U_1, U_2]^{-1} \\ \equiv&\hat{I}_n[U_1, U_2]^{-1}\end{aligned}$$

and

$$\begin{aligned}-Y_{22}^{-1}Y_{21}(Y_{11} - Y_{12}Y_{22}^{-1}Y_{21})^{-1} &= -[0, (\lambda^-/\lambda^+)I_m]\hat{I}_n[U_1, U_2]^{-1} \\ &= -[0, \frac{\lambda^-}{\lambda^+ - \lambda^-}I_m][U_1, U_2]^{-1} \\ &\equiv \hat{I}_m[U_1, U_2]^{-1}.\end{aligned}$$

Using these relations, we obtain

$$\begin{aligned}\left\|Y(0)^{-1}\begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}Y(0)\right\|^2 &= \left\|\begin{pmatrix} \hat{I}_n[U_1, U_2]^{-1}S[U_1, U_2] & \hat{I}_n[U_1, U_2]^{-1}SU_2 \\ \hat{I}_m[U_1, U_2]^{-1}S[U_1, U_2] & \hat{I}_m[U_1, U_2]^{-1}SU_2 \end{pmatrix}\right\|^2 \\ &\le \max_{\|c\|\le\sqrt{2}}\left\|\begin{pmatrix} \hat{I}_n \\ \hat{I}_m \end{pmatrix}[U_1, U_2]^{-1}S[U_1, U_2]c\right\|^2 \\ &\le 2(\left\|\hat{I}_n[U_1,U_2]^{-1}S[U_1,U_2]\right\|^2 + \left\|\hat{I}_m[U_1,U_2]^{-1}S[U_1,U_2]\right\|^2) \\ &\le c_\alpha^2\left\|[U_1,U_2]^{-1}S[U_1,U_2]\right\|^2.\end{aligned}$$

The remainder of the proof concerns the expression of c_α in (13) for each particular case. It is easy to know that

$$\hat{I}_n = \begin{pmatrix} I_{n-m} & 0 \\ 0 & \left(\frac{1}{2} + \frac{1}{2\sqrt{1+\frac{4}{\alpha}}}\right)I_m \end{pmatrix}, \quad \hat{I}_m = \left[0, \left(\frac{1}{2} - \frac{1}{2\sqrt{1+\frac{4}{\alpha}}}\right)I_m\right].$$

Let $s = \frac{1}{2} + \frac{1}{2\sqrt{1+\frac{4}{\alpha}}}$, $t = \frac{1}{2} - \frac{1}{2\sqrt{1+\frac{4}{\alpha}}}$. There are several cases which should be considered.

Case 1: If $\alpha > 0$, obviously, $0 < s < 1$, then $c_\alpha^2 = 2[1 + t^2] = \frac{3\alpha+10}{\alpha+4} - \sqrt{\frac{\alpha}{\alpha+4}}$.

Case 2: If $\alpha < -4$, obviously, $s > 1$, then

$$c_\alpha^2 = 2[s^2 + t^2] = 2[\frac{1}{2} + \frac{1}{2(1+\frac{4}{\alpha})}] = \frac{2\alpha+4}{\alpha+4}.$$

Case 3: If $-4 < \alpha < 0$, then s and t are complex numbers, and

$$|s|^2 = |t|^2 = \frac{1}{4} - \frac{1}{4(1+\frac{4}{\alpha})}.$$

If $|s| \le 1$, i.e., $-3 \le \alpha < 0$, then $c_\alpha^2 = 2[1 + |t|^2] = \frac{2\alpha+10}{\alpha+4}$.
If $|s| > 1$, i.e.,$-4 < \alpha < -3$, then $c_\alpha^2 = 2[|s|^2 + |t|^2] = \frac{4}{\alpha+4}$.

With the combination of cases 1, 2 and 3, the proof of the theorem is completed.

Using the general result (Lemma 3.6 of [23]), the condition number of $[U_1, U_2]$ is given by

$$\kappa([U_1, U_2]) = \left(\frac{1+\omega_1}{1-\omega_1}\right)^{\frac{1}{2}}, \tag{15}$$

where ω_1 is the largest singular value of ${U_1}^T U_2$, i.e., $\omega_1 = \|{U_1}^T U_2\|$. □

Using (15) and Theorem 2, we have the following corollary.

Corollary 1 *In the notation of Theorem 2, for each eigenvalue λ_S of $T(S)$, there is an eigenvalue λ of $T(0)$ such that*

$$|\lambda_S - \lambda| \le c_\alpha \left(\frac{1+\omega_1}{1-\omega_1}\right)^{\frac{1}{2}} \|S\| \ . \tag{16}$$

In the next section, the block triangular preconditioners will be considered, and the corresponding bounds of eigenvalues of the preconditioned matrix are given.

3 Block Triangular Preconditioners

In this section, we consider two block triangular preconditioners (7) and (8), which contain a parameter α in different location.

If the triangular preconditioner is given by $F_\alpha(G)$ in (7), then the preconditioned matrix becomes

$$F(S) = F_\alpha(G)^{-1}\mathcal{A} = \begin{pmatrix} I_n - S - \frac{1}{\alpha}NM & N \\ \frac{1}{\alpha}M & 0 \end{pmatrix}, \tag{17}$$

specially,

$$F(0) = \begin{pmatrix} I_n - \frac{1}{\alpha}NM & N \\ \frac{1}{\alpha}M & 0 \end{pmatrix}. \tag{18}$$

Theorem 3 *The block triangular preconditioned matrix $\mathcal{F} = F(0)$ is diagonalizable, and it has following eigenvalues and eigenvectors:*

• *n eigenvalue of 1, the corresponding eigenvectors are $[{u_j}^T, 0]^T$, $j = 1, 2, ..., n-m$ and $[{u_j}^T, (\frac{1}{\alpha}Mu_j)^T]^T$, $j = n-m+1, ..., n$.*

• *m eigenvalue of $-\frac{1}{\alpha}(\alpha \ne -1)$, the corresponding eigenvectors are $[{u_j}^T, (-Mu_j)^T]^T$, $j = n-m+1, ..., n$.*

Furthermore, the eigenvector matrix of $F(0)$ is given by

$$Y = \begin{pmatrix} U_1 & U_2 & U_2 \\ 0 & \frac{1}{\alpha}MU_2 & -MU_2 \end{pmatrix}. \tag{19}$$

Proof. Let $(\theta, [u^T, v^T])$ be the eigenpairs of $F(0)$, then $F(0)[u^T, v^T]^T = \theta[u^T, v^T]^T$, which is equivalent to the following two equations

$$(i)\ (I_n - \frac{1}{\alpha}NM)u + Nv = \theta u, \quad (ii)\ \frac{1}{\alpha}Mu = \theta v.$$

If $\theta = 1$, we insert (ii) into (i). By simple computation, we know that (i) is an identical equation. So the corresponding eigenvectors are $[{u_j}^T, 0]^T,\ \ j = 1, 2, ..., n-m$ and $[{u_j}^T, (\frac{1}{\alpha}Mu_j)^T]^T,\ \ j = n-m+1, ..., n$.
If $\theta \neq 1$, inserting (ii) into (i) and using some basic computation, we obtain

$$NMu = -\theta\alpha u.$$

Hence, $\theta = -\frac{1}{\alpha}$, and the corresponding eigenvectors are $[{u_j}^T, (-Mu_j)^T]^T$, $j = n-m+1, ..., n$. □

remark 2 *$\mathcal{F}$ satisfies $(\mathcal{F} - I)(\mathcal{F} + \frac{1}{\alpha}I) = 0$, and when $\alpha = -1$, the preconditioned matrix $\mathcal{F}$ is not diagonalizable. It is interesting to find that if parameter $\alpha < 0$, then the preconditioner $F_\alpha(G)$ is indefinite and the preconditioned matrix $\mathcal{F}$ is positive stable. Nevertheless, if parameter $\alpha > 0$, then the preconditioner $F_\alpha(G)$ becomes positive stable definite and the preconditioned matrix $\mathcal{F}$ is indefinite.*

The next theorem will give the relation of the eigenvalues of each matrix $F(S)$ in terms of the corresponding matrix $F(0)$.

Theorem 4 *With eigenvector matrix Y given in (19), for each matrix S, and each eigenvalue θ_S of $F(S)$, there is an eigenvalue θ ($\theta = 1$ or $-\frac{1}{\alpha}$) of $F(0)$ such that*

$$|\theta_S - \theta| \leq \|Y^{-1}\begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} Y\| \tag{20}$$

$$\leq c_\alpha \|[U_1, U_2]^{-1} S [U_1, U_2]\|, \tag{21}$$

where $c_\alpha^2 = \begin{cases} 2 + \frac{2}{(\alpha+1)^2}, & \alpha \geq -\frac{1}{2}; \\ \frac{2+2\alpha^2}{(\alpha+1)^2}, & \alpha < -\frac{1}{2}. \end{cases}$

Proof. As the proof of Theorem 2, it is easy to know that there exists the following two-by-two block decomposition

$$Y^{-1} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix},$$

where

$$Z_{11} = \begin{pmatrix} I_{n-m} & 0 \\ 0 & \frac{\alpha}{\alpha+1} I_m \end{pmatrix} [U_1, U_2]^{-1} \equiv \tilde{I}_n [U_1, U_2]^{-1}$$

and

$$Z_{21} = [0, \frac{1}{\alpha+1} I_m][U_1, U_2]^{-1} \equiv \tilde{I}_m [U_1, U_2]^{-1}.$$

Hence,

$$\begin{aligned} \left\|Y^{-1}\begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} Y\right\|^2 &\leq 2(\left\|\tilde{I}_n[U_1, U_2]^{-1}S[U_1, U_2]\right\|^2 + \left\|\tilde{I}_m[U_1, U_2]^{-1}S[U_1, U_2]\right\|^2) \\ &\leq c_\alpha^2 \left\|[U_1, U_2]^{-1}S[U_1, U_2]\right\|^2. \end{aligned}$$

Now, we give the expression of c_α. There are two cases which should be considered.

Case 1: If $-\frac{1}{2} \leq \alpha$, clearly, $|\frac{\alpha}{\alpha+1}| \leq 1$, then, $c_\alpha^2 = 2[1 + (\frac{1}{\alpha+1})^2] = 2 + \frac{2}{(\alpha+1)^2}$.

Case 2: If $\alpha < -\frac{1}{2}$, obviously, $|\frac{\alpha}{\alpha+1}| > 1$, then,

$$c_\alpha^2 = 2[(\frac{\alpha}{\alpha+1})^2 + (\frac{1}{\alpha+1})^2] = \frac{2+2\alpha^2}{(\alpha+1)^2}.$$

From what has been discussed above, the proof of the theorem is completed. □

Corollary 2 *In the notation of Theorem 4, for each eigenvalue θ_S of $F(S)$, there is an eigenvalue θ of $F(0)$ such that*

$$|\theta_S - \theta| \leq c_\alpha \left(\frac{1+\omega_1}{1-\omega_1}\right)^{\frac{1}{2}} \|S\| \ . \tag{22}$$

Next, we consider another block triangular preconditioner $H_\alpha(G)$. Then the preconditioned matrix becomes

$$H(S) = H_\alpha(G)^{-1}\mathcal{A} = \begin{pmatrix} I_n - S - \alpha NM & N \\ M & 0 \end{pmatrix}, \tag{23}$$

particularly,

$$H(0) = \begin{pmatrix} I_n - \alpha NM & N \\ M & 0 \end{pmatrix}. \tag{24}$$

Theorem 5 *The block triangular preconditioned matrix $\mathcal{H} = H(0)$ is diagonalizable, and it has the following eigenvalues and eigenvectors:*

- *$n-m$ eigenpairs of the form $(1, [{u_j}^T, 0]^T)$, where $j = 1, 2, ..., n-m$.*
- *$2m$ eigenpairs of the form $(\eta^\pm, [{u_j}^T, (\eta^\pm)^{-1}(Mu_j)^T]^T)$, where*

$\eta^\pm \equiv \frac{(1-\alpha)\pm\sqrt{4+(1-\alpha)^2}}{2}$, $j = n-m+1, ..., n$.
Furthermore, the eigenvector matrix $\mathcal{Y}$ of $H(0)$ is given as

$$\mathcal{Y} = \begin{pmatrix} U_1 & U_2 & U_2 \\ 0 & (\eta^+)^{-1} MU_2 & (\eta^-)^{-1} MU_2 \end{pmatrix}. \tag{25}$$

Proof. The proof is anaglous to that of Theorem 3, therefore, it is omitted. □

remark 3 *$\mathcal{H}$ satisfies $(\mathcal{H} - I)(\mathcal{H}^2 - (1-\alpha)\mathcal{H} - I) = 0$. It is obvious that the eigenvalues of $\mathcal{H}$ are real and the preconditioned matrix $\mathcal{H}$ is indefinite for all $\alpha \in R$.*

Theorem 6 *With eigenvector matrix $\mathcal{Y}$ given in (25), for each matrix S, and each eigenvalue η_S of $H(S)$, there is an eigenvalue η ($\eta = 1$ or $\eta^\pm = \frac{(1-\alpha)\pm\sqrt{4+(1-\alpha)^2}}{2}$) of $\mathcal{H}$ such that*

$$|\eta_S - \eta| \leq \|\mathcal{Y}^{-1} \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} \mathcal{Y}\| \tag{26}$$

$$\leq c_\alpha \|[U_1, U_2]^{-1} S [U_1, U_2]\|, \tag{27}$$

where $c_\alpha^2 = \frac{3(1-\alpha)^2+10}{4+(1-\alpha)^2} - \frac{(1-\alpha)}{\sqrt{4+(1-\alpha)^2}}$.

Proof. The proofs of (26) and (27) are similar to that of Theorem 2. Considering the following two-by-two block decomposition

$$\mathcal{Y}^{-1} = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix},$$

where

$$
\begin{aligned}
W_{11} &= \begin{pmatrix} I_{n-m} & 0 \\ 0 & \frac{\eta^+}{\eta^+-\eta^-} I_m \end{pmatrix} [U_1, U_2]^{-1} \\
&= \begin{pmatrix} I_{n-m} & 0 \\ 0 & \left(\frac{1}{2} + \frac{(1-\alpha)}{2\sqrt{4+(1-\alpha)^2}}\right) I_m \end{pmatrix} [U_1, U_2]^{-1} \\
&\equiv \bar{I}_n [U_1, U_2]^{-1}
\end{aligned}
$$

and

$$
\begin{aligned}
W_{21} &= -[0, (\eta^-/\eta^+) I_m] \bar{I}_n [U_1, U_2]^{-1} \\
&= -[0, \frac{\eta^-}{\eta^+ - \eta^-} I_m][U_1, U_2]^{-1} \\
&= \left[0, \left(\frac{1}{2} - \frac{(1-\alpha)}{2\sqrt{4+(1-\alpha)^2}}\right) I_m\right] [U_1, U_2]^{-1} \\
&\equiv \bar{I}_m [U_1, U_2]^{-1}.
\end{aligned}
$$

Denote $s = \frac{1}{2} + \frac{(1-\alpha)}{2\sqrt{4+(1-\alpha)^2}}$, $t = \frac{1}{2} - \frac{(1-\alpha)}{2\sqrt{4+(1-\alpha)^2}}$. For all $\alpha \in R$, it can be concluded that $|\frac{(1-\alpha)}{\sqrt{4+(1-\alpha)^2}}| < 1$, so $0 < s < 1$, $0 < t < 1$, and then

$$
c_\alpha^2 = 2(1+t^2) = \frac{3(1-\alpha)^2 + 10}{4 + (1-\alpha)^2} - \frac{(1-\alpha)}{\sqrt{4+(1-\alpha)^2}}.
$$

Based on the above discussion and analysis, we complete the proof of the theorem. □

Corollary 3 *In the notation of Theorem 6, for each eigenvalue η_S of $H(S)$, there is an eigenvalue η of $H(0)$ such that*

$$
|\eta_S - \eta| \leq c_\alpha \left(\frac{1+\omega_1}{1-\omega_1}\right)^{\frac{1}{2}} \|S\| \ . \tag{28}
$$

There is no zero eigenvalue of the preconditioned matrix, which means that λ_S, θ_S and $\eta_S \neq 0$. By (16), (22) and (28), the smaller c_α is, the closer λ_S, θ_S and η_{S} are to λ, θ and η, respectively. It means that the eigenvalues of the preconditioned matrix may be more clustered, and the preconditioners may significantly improve the clustering properties of the spectrum of the original systems by choosing appropriate parameter α. Therefore, a Krylov subspace method such as GMRES(l) method [18] may converge more quickly. However, these inequations (16), (22) and (28) only give a rough estimate, so the distribution of the spectrum may not describe the rate of convergence completely.

4 Numerical experiments

The problem under consideration is the Oseen problem [10]

$$
\begin{cases} -\nu \triangle u + w \cdot \nabla u + \nabla p = f, \text{in } \Omega \\ \text{div} u = 0, \text{in } \Omega \end{cases} \tag{29}
$$

with suitable boundary conditions on $\partial\Omega$, where $\Omega \subset R^2$ is a bounded domain and w is a given divergence free field. The scalar ν is the viscosity, the vector u represents the velocity, and p denotes the pressure.

The test problem is a leaky two-dimensional lid-driven cavity problem. We discretize equation (29) with the IFISS software written by Howard Elman, Alison Ramage and David Silvester [9]. The mixed finite element used is the bilinear-constant velocity-pressure $Q_1 - P_0$ pair. The finite element subdivision is based on $n \times n$ uniform grids of square elements. Since the matrix B produced by the software is rank deficient, we drop the first two rows of B to get a full rank matrix.

In our numerical experiments, the resulting saddle point-type matrix

$$\mathcal{A} = \begin{pmatrix} A & B^T \\ C & 0 \end{pmatrix}$$

satisfies $C = B$, A is positive real, i.e., $A + A^T$ is symmetric positive definite. We take G as an incomplete LU factorization of A:

$$A = LU + R, \; G = LU,$$

with drop tolerance τ [18], where L and U are the lower and upper triangular matrices, respectively. We give some results to show the convergence behaviors of preconditioned GMRES(10) method. The initial guess is taken to be $v_{(0)} = 0$ and the stopping criteria is $\|r_k\|_2 \leq 10^{-6}\|r_0\|_2$, where $r_k = b - \mathcal{A}v_{(k)}$ is the residual vector after kth iteration. The right-hand side vectors b are taken such that the exact solutions x and y are both vectors with all components being 1. We take some values of ν: $\nu = 1$, $\nu = 0.1$ and $\nu = 0.01$ and two mesh grids h: 1/16 and 1/32. We give the iteration numbers and CPU time, though the codes are not optimally tuned. In these tables, $m(n)$ means number of outer (inner) iterations, CPU time lie inside [ ]. All computations are performed in Matlab 7.0 with intel(R) Pentium CPU 3.00GHz and 1 GB memory. Denote the preconditioners $P_\alpha(G)$, $F_\alpha(G)$ and $H_\alpha(G)$ as $P(\alpha)$, $F(\alpha)$ and $H(\alpha)$, respectively.

In Tables 1– 4, we show the iteration numbers and CPU time of preconditioned GMRES(10) applied to solve the saddle point linear systems with different preconditioners $P(\alpha)$, $F(\alpha)$ and $H(\alpha)$. At the same time, we give, in Figures 1– 4, the convergence curve for these preconditioners with an appropriate parameter α. From these tables and figures we can see that:

• For the block diagonal preconditioner $P_\alpha(G)$, there is no apparent difference for different parameter α in computational performance.

• If we choose an appropriate value α, the block triangular preconditioners $F_\alpha(G)$ and $H_\alpha(G)$ are significantly more efficient than the block diagonal preconditioners $P_\alpha(G)$. Moreover, in most case, the block triangular preconditioners $F_\alpha(G)$ is better than $H_\alpha(G)$.

• Tables 1– 4 show that the smaller the value of $|\alpha|$ is, the smaller the iteration numbers of GMRES(10) preconditioned by $F_\alpha(G)$ are. A great number of numerical experiments show that $\alpha = -0.1$ and $\alpha = 1$ are relative 'optimal' choices for the block triangular preconditioners $F_\alpha(G)$ and $H_\alpha(G)$, respectively.

5 Conclusion

In this paper, the generalized block diagonal and block triangular preconditioners containing a parameter α are presented for the non-symmetric saddle point problems. The spectral properties of the preconditioned matrix are investigated and the bounds of eigenvalues of the preconditioned matrix are given. The numerical experiments indicate that the performance of

Table 1: Preconditioned GMRES(10) iteration numbers and CPU time for 16×16 grid with drop tolerance $\tau = 0.01$ and different parameter α.

α	$\nu = 1$			$\nu = 0.01$		
	$P(\alpha)$	$F(\alpha)$	$G(\alpha)$	$P(\alpha)$	$F(\alpha)$	$G(\alpha)$
-4	3(3) [0.875]	2(8) [1.125]	3(4) [1.516]	8(10) [4.969]	6(10) [5.234]	7(9) [6.109]
-3	3(3) [0.875]	2(8) [1.078]	3(4) [1.437]	8(4) [4.610]	6(9) [5.328]	9(4) [7.547]
-2	3(4) [1.031]	2(7) [1.125]	3(4) [1.641]	7(10) [4.984]	5(7) [4.250]	8(3) [7.016]
-1	3(2) [0.844]	2(6) [0.969]	3(4) [1.469]	6(9) [3.562]	4(10) [3.532]	8(3) [6.562]
-0.5	3(3) [0.875]	2(5) [0.922]	2(10)[1.187]	6(7) [3.500]	4(8) [3.453]	7(9) [6.125]
-0.25	3(2) [0.860]	2(4) [0.828]	3(2) [1.375]	6(5) [3.437]	4(5) [3.203]	8(5) [6.516]
-0.1	3(2) [0.844]	2(4) [0.875]	3(2) [1.312]	5(5) [2.797]	3(9) [2.656]	7(8) [6.016]
0.1	3(2) [1.656]	2(4) [0.953]	2(10)[1.312]	6(7) [3.500]	3(10) [2.687]	7(5) [5.672]
0.25	3(2) [0.906]	2(4) [1.009]	2(8) [1.157]	6(10) [3.657]	5(3) [3.843]	8(3) [6.360]
0.5	3(2) [0.906]	2(6) [1.016]	2(9) [1.250]	6(10) [3.765]	5(7) [4.375]	6(9) [5.266]
1	3(2) [0.922]	2(6) [1.109]	2(6) [1.031]	7(7) [4.156]	6(6) [4.969]	6(6) [5.047]
2	2(9) [0.782]	2(9) [1.203]	2(10)[1.265]	8(8) [4.844]	7(7) [5.843]	9(2) [7.141]
3	2(9) [0.781]	2(9) [1.219]	3(4) [1.546]	8(9) [4.813]	7(10) [6.250]	9(9) [8.203]
4	2(9) [0.782]	2(8) [1.203]	3(6) [1.687]	8(7) [4.609]	8(8) [6.829]	9(7) [7.843]
5	2(9) [0.782]	2(9) [1.265]	3(6) [1.672]	9(9) [5.438]	8(6) [6.750]	9(8) [7.828]

Table 2: Preconditioned GMRES(10) iteration numbers and CPU time for 32×32 grid with drop tolerance $\tau = 0.01$ and different parameter α.

α	$\nu = 1$			$\nu = 0.01$		
	$P(\alpha)$	$F(\alpha)$	$G(\alpha)$	$P(\alpha)$	$F(\alpha)$	$G(\alpha)$
-5	4(8) [30.625]	4(3) [37.719]	5(8) [55.640]	3(3) [22.640]	2(8) [25.375]	3(6) [36.468]
-4	4(8) [35.204]	4(1) [41.797]	5(7) [64.094]	3(3) [29.156]	2(8) [33.109]	3(6) [46.141]
-3	4(8) [31.766]	3(9) [34.172]	4(10)[48.297]	3(3) [32.313]	2(8) [37.063]	3(5) [49.297]
-2	4(8) [37.172]	3(7) [37.203]	4(10)[56.656]	3(6) [28.765]	2(7) [29.047]	3(7) [42.547]
-1	4(8) [33.484]	3(3) [27.563]	5(7) [57.718]	3(6) [30.094]	2(6) [27.078]	3(7) [44.797]
-0.5	4(8) [32.860]	3(3) [28.281]	4(6) [43.453]	3(5) [32.031]	2(6) [29.907]	3(6) [48.156]
-0.25	4(9) [33.359]	3(3) [27.750]	4(8) [47.859]	3(6) [35.485]	2(5) [30.906]	3(5) [49.531]
-0.1	4(8) [30.453]	3(2) [25.407]	4(8) [42.843]	3(6) [33.875]	2(4) [28.641]	3(4) [45.875]
0.1	4(9) [31.453]	3(3) [27.953]	4(6) [42.203]	3(5) [37.515]	2(6) [32.719]	3(4) [44.390]
0.25	4(9) [31.312]	3(5) [27.969]	4(5) [38.516]	3(6) [30.406]	2(6) [28.469]	3(5) [44.015]
0.5	4(8) [30.562]	3(6) [29.829]	3(10)[32.937]	3(5) [29.578]	2(6) [28.625]	2(10)[34.281]
1	4(8) [30.328]	3(9) [31.875]	3(9) [31.891]	3(5) [29.391]	2(6) [28.672]	2(6) [28.281]
2	4(8) [30.203]	4(4) [37.593]	3(10)[33.032]	3(3) [27.313]	2(9) [33.031]	3(4) [42.219]
2.5	4(8) [32.296]	4(2) [37.313]	4(3) [37.625]	3(3) [27.297]	2(9) [33.453]	3(5) [44.609]
4	4(9) [31.859]	4(5) [39.375]	4(7) [41.969]	3(2) [27.891]	2(9) [32.187]	3(6) [45.516]

Table 3: Preconditioned GMRES(10) iteration numbers and CPU time for 16×16 grid with drop tolerance $\tau = 0.1$ and different parameter α.

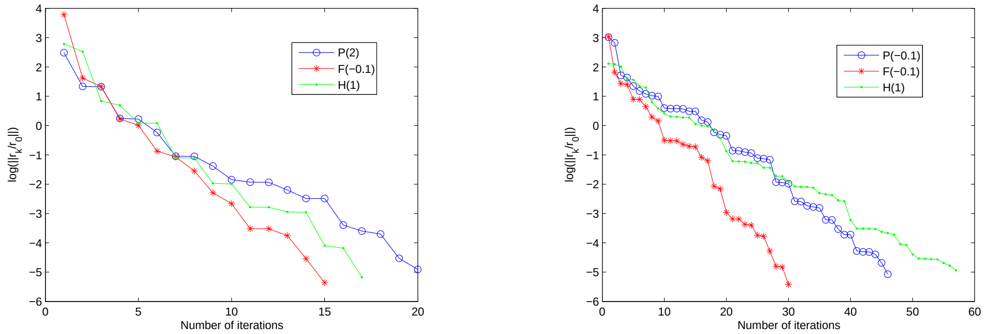
α	$\nu = 1$			$\nu = 0.1$		
	$P(\alpha)$	$F(\alpha)$	$G(\alpha)$	$P(\alpha)$	$F(\alpha)$	$G(\alpha)$
-5	4(7) [1.891]	3(10) [1.890]	5(6) [2.922]	5(1) [1.891]	4(2) [1.969]	5(4) [2.734]
-3	4(6) [1.500]	3(7) [1.735]	5(2) [2.843]	4(6) [1.390]	3(9) [1.782]	5(6) [2.922]
-2	4(6) [1.407]	3(6) [1.750]	4(8) [2.484]	4(8) [1.453]	3(6) [1.641]	4(8) [2.438]
-1	4(7) [1.437]	3(2) [1.469]	4(7) [2.437]	4(6) [1.438]	3(3) [1.516]	5(2) [2.750]
-0.5	4(5) [1.375]	2(10) [1.282]	4(5) [2.343]	4(8) [2.516]	3(2) [1.422]	4(3) [2.078]
-0.25	4(6) [1.469]	2(9) [1.187]	4(6) [2.281]	4(9) [1.562]	2(10) [1.266]	4(7) [2.328]
-0.1	4(6) [1.421]	2(10) [1.297]	4(6) [2.391]	4(8) [2.640]	2(10) [1.297]	4(6) [2.328]
0.1	4(6) [1.484]	3(2) [1.500]	4(5) [2.250]	4(6) [1.484]	2(10) [1.328]	4(3) [2.141]
0.25	4(6) [1.484]	3(3) [1.578]	3(8) [1.797]	4(6) [1.438]	3(2) [1.468]	4(2) [2.110]
0.5	4(5) [1.469]	3(4) [1.625]	3(9) [1.875]	4(5) [1.390]	3(5) [1.656]	4(1) [2.032]
1	4(5) [1.453]	3(7) [1.843]	3(7) [1.797]	4(4) [1.344]	3(7) [1.734]	3(7) [1.719]
2	4(6) [1.438]	3(9) [1.828]	3(8) [1.875]	4(6) [1.406]	4(2) [2.032]	3(8) [1.718]
3	4(5) [1.438]	3(9) [1.875]	4(6) [2.390]	4(6) [1.500]	4(2) [2.063]	3(10)[1.859]
5	4(5) [1.500]	4(6) [2.375]	4(6) [2.344]	4(6) [1.515]	4(8) [2.375]	4(6) [2.235]

Table 4: Preconditioned GMRES(10) iteration numbers and CPU time for 32×32 grid with drop tolerance $\tau = 0.1$ and different parameter α.

α	$\nu = 1$			$\nu = 0.1$		
	$P(\alpha)$	$F(\alpha)$	$G(\alpha)$	$P(\alpha)$	$F(\alpha)$	$G(\alpha)$
-5	6(9) [47.719]	5(5) [54.703]	8(5) [91.719]	6(3) [40.375]	5(3) [51.766]	7(8) [83.266]
-4	6(8) [47.109]	5(3) [53.313]	9(7) [104.984]	6(3) [40.953]	4(10) [48.250]	8(6) [93.484]
-3	6(6) [45.125]	4(8) [46.531]	9(10)[107.422]	6(4) [44.031]	4(7) [45.781]	8(3) [89.047]
-2	6(8) [54.343]	4(7) [47.454]	7(6) [79.062]	6(3) [41.125]	4(6) [44.063]	7(6) [81.859]
-1	6(9) [53.797]	4(6) [49.922]	7(7) [92.672]	6(6) [43.203]	4(5) [44.360]	8(1) [89.343]
-0.5	7(4) [64.218]	4(5) [55.172]	6(7) [87.250]	7(3) [48.235]	4(4) [41.625]	7(2) [75.109]
-0.25	7(3) [64.453]	4(5) [53.625]	7(5) [98.407]	7(3) [48.234]	4(4) [42.141]	6(9) [74.593]
-0.1	7(6) [55.328]	4(4) [44.453]	6(8) [73.219]	7(2) [47.297]	4(3) [42.234]	6(7) [71.672]
0.1	6(9) [56.062]	4(6) [53.079]	6(10)[84.609]	6(9) [46.672]	4(6) [45.012]	6(8) [70.844]
0.25	7(3) [67.750]	4(9) [64.532]	6(8) [90.672]	6(9) [47.485]	4(8) [47.640]	6(5) [68.328]
0.5	6(8) [60.515]	4(10)[64.625]	6(6) [89.219]	6(7) [45.031]	4(10) [48.922]	5(9) [60.532]
1	6(9) [62.157]	5(9) [75.250]	5(9) [73.921]	6(8) [45.437]	5(7) [57.203]	5(7) [60.250]
2	7(6) [78.828]	6(8) [104.516]	5(5) [83.297]	6(8) [44.523]	6(2) [65.656]	5(2) [54.344]
3	7(8) [81.012]	6(10)[109.797]	5(4) [76.968]	6(4) [56.203]	5(10) [80.922]	5(4) [73.359]
4	7(6) [73.359]	7(8) [111.031]	5(4) [78.625]	6(9) [61.375]	6(9) [95.172]	5(4) [72.203]
5	7(9) [76.015]	8(5) [122.110]	5(4) [71.984]	6(3) [55.437]	5(3) [72.328]	7(8) [114.032]

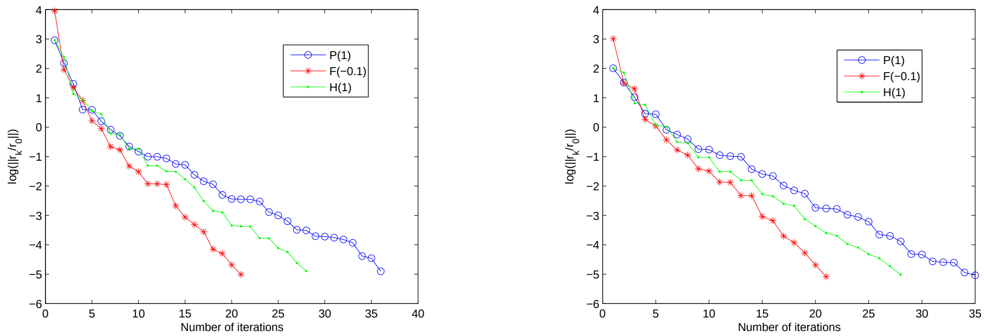
Table 5: Values of n and m, and order of $\mathcal{A}$

Grid	n	m	Order of $\mathcal{A}$
16×16	578	254	832
32×32	2178	1022	3200



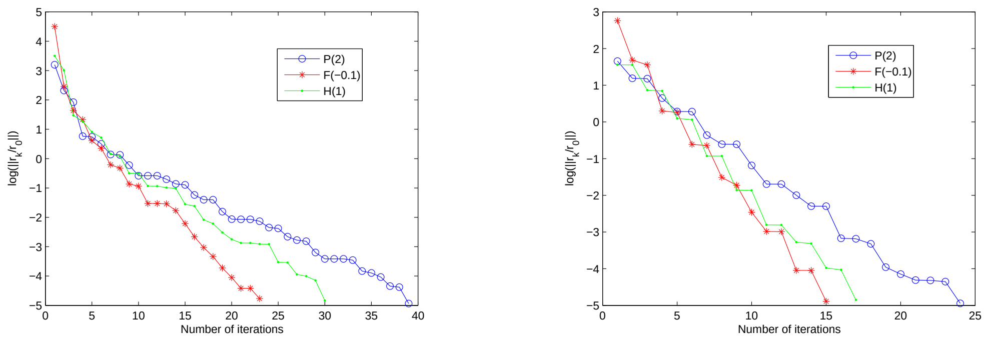
(a) $\nu = 1$ (b) $\nu = 0.01$

Figure 1: *Convergence curve and total numbers of preconditioned GMRES(10) method for* 16×16 *grid when drop tolerance* $\tau = 0.01$.



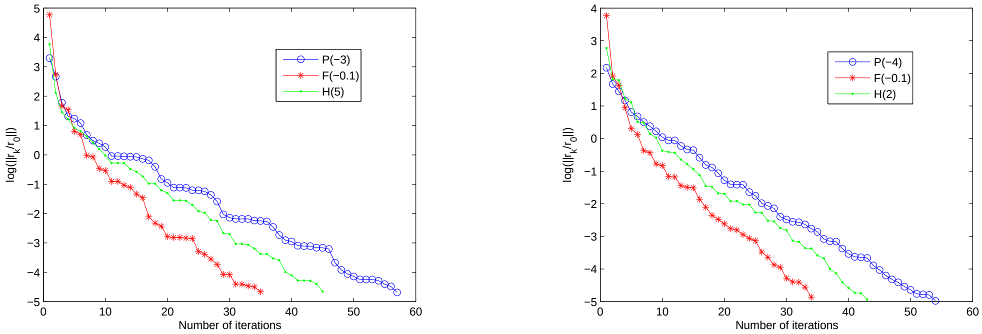
(a) $\nu = 1$ (b) $\nu = 0.1$

Figure 2: *Convergence curve and total numbers of preconditioned GMRES(10) method for* 16×16 *grid when drop tolerance* $\tau = 0.1$.



(a) $\nu = 1$ (b) $\nu = 0.01$

Figure 3: *Convergence curve and total numbers of preconditioned GMRES(10) method for* 32×32 *grid when drop tolerance* $\tau = 0.01$.



(a) $\nu = 1$ (b) $\nu = 0.1$

Figure 4: *Convergence curve and total numbers of preconditioned GMRES(10) method for* 32×32 *grid when drop tolerance* $\tau = 0.1$.

the preconditioners $P_\alpha(G)$ has no apparent difference along with the diversification of parameter α, however, the block triangular $F_\alpha(G)$ is more effective than the preconditioners $H_\alpha(G)$ for appropriate parameter α. Furthermore, the numerical experiments demonstrate the block triangular preconditioners are more effective than the block diagonal preconditioners. The determination of the optimum value of parameter α needs further to be studied in theory.

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Topological structures of soft fuzzy rough sets *

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Abstract: Soft set theory and rough set theory are mathematical tools for dealing with uncertainties. In this paper, we investigate soft fuzzy rough sets and give their topological structures.

Keywords: Soft sets; Rough sets; Soft fuzzy rough sets; Topological structures.

1 Introduction

Most of traditional methods for formal modeling, reasoning, and computing are crisp, deterministic, and precise in character. However, many practical problems within fields such as economics, engineering, environmental science, medical science and social sciences involve data that contain uncertainties. We can not use traditional methods because of various types of uncertainties present in these problems.

There are several theories: probability theory, fuzzy set theory[16], theory of interval mathematics and rough set theory[15], which can be considered as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties (see [12]). For example, probability theory can deal only with stochastically stable phenomena. To overcome these kinds of difficulties, Molodtsov [12] proposed a completely new approach, which is called soft set theory, for modeling uncertainty.

Presently, works on soft set theory are rapidly progressing. Maji et al. [9, 10] further studied soft set theory and used this theory to solve some decision making problems. Aktas and Cağman [1] defined soft groups. Jiang et al. [7] extended soft sets with description logics. Feng et al. [3, 4] investigated the relationship among soft sets, rough sets and fuzzy sets. Ge et al. [5] discussed the relationship between soft sets and topological spaces. Shabir et al. [13] introduced soft topological spaces over the universe with a fixed set of parameters. Cağman and Karatas [2] defined soft topologies on soft sets.

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Rough set theory was proposed by Pawlak [15]. It is an extension of set theory for the study of intelligent systems characterized by insufficient and incomplete information. The foundation of its object classification is an equivalence relation. The upper and lower approximation operations are two core notions in rough set theory. They can also be seen as a closure operator and an interior operator of the topology induced by an equivalence relation on a universe.

The purpose of this paper is to investigate soft fuzzy rough sets and their topological structures.

2 Preliminaries

In this paper, U denotes a nonempty finite set called the universe of discourse. 2^U denotes the family of all subsets of U. I denotes $[0,1]$. I^U denotes the family of all fuzzy sets in U. $\bar{a}$ represents the fuzzy set which satisfies $\bar{a}(x) = a$ for each $x \in U$.

Definition 2.1 ([15]). *Let R be an equivalence relation on U. The pair (U, R) is called a Pawlak approximation space. Based on (U, R), one can define the following two rough approximations:*

$$R_*(X) = \{x \in U : [x]_R \subseteq X\},$$

$$R^*(X) = \{x \in U : [x]_R \cap X \neq \emptyset\}.$$

$R_*(X)$ and $R^*(X)$ are called the Pawlak lower approximation and the Pawlak upper approximation of X, respectively. In general, we refer to $R_*(X)$ and $R^*(X)$ as Pawlak rough approximations of X.

X is Pawlak rough if $R_*(X) \neq R^*(X)$; otherwise, it is Pawlak definable.

Definition 2.2 ([12]). *A pair (f, E) is called a soft set over U, if f is a mapping given by $f : E \to 2^U$, denoted by f_E.*

In other words, a soft set over U is the parameterized family of subsets of the universe U. For $\varepsilon \in A$, $f(\varepsilon)$ may be considered as the set of ε-approximate elements of the soft set f_A. Obviously, a soft set is not a set.

Denote

$$S(U, A) = \{f_A : f_A \text{ is a soft set over } U\}.$$

Definition 2.3 ([9]). *Let $f_A \in S(U, A)$ and $g_B \in S(U, B)$.*

(1) f_A is called a soft subset of g_B, if $A \subseteq B$ and for any $\varepsilon \in A$, $f(\varepsilon) \subseteq g(\varepsilon)$. We write $f_A \widetilde{\subseteq} g_B$.

(2) f_A is called a soft super set of g_B, if $g_B \widetilde{\subseteq} f_A$. We write $f_A \widetilde{\supseteq} g_B$.

(3) f_A and g_B are called soft equal, if $A = B$ and $f(\varepsilon) = g(\varepsilon)$ for any $\varepsilon \in A$. We write $f_A = g_B$.

Obviously, $f_A = g_B$ if and only if $f_A \widetilde{\subseteq} g_B$ and $f_A \widetilde{\supseteq} g_B$.

2

Definition 2.4 ([9]). *Let $f_A \in S(U, A)$ and $g_B \in S(U, B)$.*
(1) *$h_{A\cup B}$ is called the union of f_A and g_B, if*

$$h(\varepsilon) = \begin{cases} f(\varepsilon), & \text{if } \varepsilon \in A - B, \\ g(\varepsilon), & \text{if } \varepsilon \in B - A, \\ f(\varepsilon) \cup g(\varepsilon), & \text{if } \varepsilon \in A \cap B. \end{cases}$$

We write $f_A \widetilde{\cup}\, g_B = h_{A\cup B}$.
(2) *$h_{A\cap B}$ is called the soft intersection of f_A and g_B, if $h(\varepsilon) = f(\varepsilon) \cap g(\varepsilon)$ for any $\varepsilon \in A \cap B$. We write $f_A \widetilde{\cap}\, g_B = h_{A\cap B}$.*

Definition 2.5. *Let f_E be a soft set over X.*
(1) *f_E is called full, if $\bigcup\limits_{e\in E} f(e) = X$.*
(2) *f_E is called keeping intersection, if for any $e_1, e_2 \in E$, there exists $e_3 \in E$ such that $f(e_1) \cap f(e_2) = f(e_3)$.*
(3) *f_E is called keeping union, if for any $e_1, e_2 \in E$, there exists $e_3 \in E$ such that $f(e_1) \cup f(e_2) = f(e_3)$.*
(4) *f_E is called partition if $\{f(e) : e \in E\}$ is a partition of U.*

3 Soft fuzzy rough sets

Definition 3.1 ([3]). *Let f_E be a soft set over U. The pair $P = (U, f_E)$ is called a soft approximation space. Based on P, for any $X \in 2^U$, we define:*

$$\underline{apr}_P(X) = \{x \in U : \text{ there exists } e \in E \text{ such that } x \in f(e) \subseteq X\},$$

$$\overline{apr}_P(X) = \{x \in U : \text{ there exists } e \in E \text{ such that } x \in f(e) \text{ and } f(e) \cap X \neq \emptyset\},$$

$\underline{apr}_P(X)$ and $\overline{apr}_P(X)$ called the lower and upper soft rough approximation of X in P.

X is soft rough if $\underline{apr}_P(X) \neq \overline{apr}_P(X)$; otherwise, it is soft definable.

Definition 3.2 ([3]). *Let f_E be a soft set over U. The pair $P = (U, f_E)$ is called a soft approximation space. Based on P, the lower and upper soft rough approximations of $A \in I^U$ respect to P are denoted by $\underline{sap}_P(A)$ and $\overline{sap}_P(A)$, respectively, which are defined by*

$$\underline{sap}_P(A)(x) = \bigwedge\{A(y) : \text{ there exists } e \in E \text{ such that } \{x, y\} \subseteq f(e)\},$$

$$\overline{sap}_P(A)(x) = \bigvee\{A(y) : \text{ there exists } e \in E \text{ such that } \{x, y\} \subseteq f(e)\},$$

for any $A \in U$.

$\underline{sap}_P(A)$ and $\overline{sap}_P(A)$ are called lower and upper soft rough fuzzy approximation operators on fuzzy sets.

X is soft fuzzy rough if $\underline{sap}_P(A) \neq \overline{sap}_P(A)$; otherwise, it is soft fuzzy definable.

Remark 3.3. *In [3], a soft fuzzy rough set is called a soft rough fuzzy set.*

Proposition 3.4 ([3])**.** *Let f_E be a full soft set over U and let $P = (U, f_E)$. Then for any $A, B \in I^U$,*

(1) $\underline{sap}_P(\bar{1}) = \overline{sap}_P(\bar{1}) = \bar{1}$, $\underline{sap}_P(\bar{0}) = \overline{sap}_P(\bar{0}) = \bar{0}$,

(2) $\underline{sap}_P(A) \subseteq A \subseteq \overline{sap}_P(A)$,

(3) $A \subseteq B \Rightarrow \underline{sap}_P(A) \subseteq \underline{sap}_P(B)$ *and* $\overline{sap}_P(A) \subseteq \overline{sap}_P(B)$,

(4) $\underline{sap}_P(A^c) = ((\overline{sap}_P(A))^c$, $\overline{sap}_P(A^c) = ((\underline{sap}_P(A))^c$,

(5) $\underline{sap}_P(A \cap B) = \underline{sap}_P(A) \cap \underline{sap}_P(B)$,

(6) $\overline{sap}_P(A \cup B) = \overline{sap}_P(A) \cup \overline{sap}_P(B)$,

(6) $\underline{sap}_P(A \cup B) \supseteq \underline{sap}_P(A) \cup \underline{sap}_P(B)$,

(7) $\overline{sap}_P(A \cap B) \subseteq \overline{sap}_P(A) \cap \overline{sap}_P(B)$.

Obviously, if $A = \bar{a}$, then $\underline{sap}_P(A) = \overline{sap}_P(A) = A$.

Proposition 3.5. *Let f_E be a full soft set over U and let $P = (U, f_E)$. Then for any $A \in I^U$,*

(1) $\underline{sap}_P(\underline{sap}_P(A)) \subseteq \underline{sap}_P(A)$.

(2) $\underline{sap}_P(\overline{sap}_P(A)) \subseteq \overline{sap}_P(A)$.

(3) $\overline{sap}_P(\underline{sap}_P(A)) \supseteq \underline{sap}_P(A)$.

(4) $\overline{sap}_P(\overline{sap}_P(A)) \supseteq \overline{sap}_P(A)$.

Proof. This holds by Proposition 3.4.

Example 3.6. *Let $U = \{x_1, x_2, x_3, x_4, x_5\}$ and $E = \{e_1, e_2, e_3, e_4\}$. Let f_E be a soft set over U, defined as follows*

$f(e_1) = \{x_1, x_2\}$, $f(e_2) = \{x_2, x_4, x_5\}$,
$f(e_3) = \{x_3, x_4, x_5\}$, $f(e_4) = \{x_1, x_3, x_5\}$.
Put $P = (U, f_E)$. Let

$$A = \frac{0.3}{x_1} + \frac{0.7}{x_2} + \frac{0.4}{x_3} + \frac{0.1}{x_4} + \frac{0.9}{x_5}.$$

We have

$$\underline{sap}_P(A) = \frac{0.3}{x_1} + \frac{0.1}{x_2} + \frac{0.1}{x_3} + \frac{0.1}{x_4} + \frac{0.1}{x_5},$$

$$\underline{sap}_P(\underline{sap}_P(A)) = \frac{0.1}{x_1} + \frac{0.1}{x_2} + \frac{0.1}{x_3} + \frac{0.1}{x_4} + \frac{0.1}{x_5},$$

$$\overline{sap}_P(\underline{sap}_P(A)) = \frac{0.3}{x_1} + \frac{0.3}{x_2} + \frac{0.3}{x_3} + \frac{0.1}{x_4} + \frac{0.3}{x_5}.$$

Then $\underline{sap}_P(\underline{sap}_P(A)) \not\supseteq \underline{sap}_P(A)$ *and* $\overline{sap}_P(\underline{sap}_P(A)) \not\subseteq \underline{sap}_P(A)$.

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Let

$$B = \frac{0.3}{x_1} + \frac{0.7}{x_2} + \frac{0.4}{x_3} + \frac{0.9}{x_4} + \frac{0.1}{x_5}$$

Then

$$\overline{sap}_P(B) = \frac{0.7}{x_1} + \frac{0.9}{x_2} + \frac{0.9}{x_3} + \frac{0.9}{x_4} + \frac{0.9}{x_5},$$

$$\overline{sap}_P(\overline{sap}_P(B)) = \frac{0.9}{x_1} + \frac{0.9}{x_2} + \frac{0.9}{x_3} + \frac{0.9}{x_4} + \frac{0.9}{x_5},$$

$$\underline{sap}_P(\overline{sap}_P(B)) = \frac{0.7}{x_1} + \frac{0.7}{x_2} + \frac{0.7}{x_3} + \frac{0.9}{x_4} + \frac{0.7}{x_5}.$$

Thus $\overline{sap}_P(\overline{sap}_P(B)) \not\subseteq \overline{sap}_P(B)$ *and* $\underline{sap}_P(\overline{sap}_P(B)) \not\supseteq \overline{sap}_P(B)$.

□

Proposition 3.7. *Let f_E be a full soft set over U and let $P = (U, f_E)$. For any $x \in U$, denote*

$$O(x) = \{y \in U : \textit{there exists } e \in E \textit{ such that } \{x, y\} \subseteq f(e)\}.$$

Then

(1) $\underline{sap}_P(A)(x) = \bigwedge_{y \in O(x)} A(y)$, $\overline{sap}_P(A)(x) = \bigvee_{y \in O(x)} A(y)$.

(2) $O(x) \neq \emptyset$.

(3) *If f_E is partition, then $O(x) = O(y)$ whenever $y \in O(x)$.*

(4) *If f_E is full and keeping union, then for any $x \in U$, $O(x) = U$.*

Proof. (1) This is obvious.

(2) Since f_E is full, $U = \bigcup_{e \in E} f(e)$. Then there exists $e \in E$ such that $x \in f(e)$, so $\{x, x\} \subseteq f(e)$. This is to say $x \in O(x)$. Thus $O(x) \neq \emptyset$.

(3) Let $z \in O(x)$. Then there exists $e \in E$ such that $\{x, z\} \subseteq f(e)$. Since $y \in O(x)$, there exists $e' \in E$ such that $\{x, y\} \subseteq f(e')$. Then $x \in f(e) \cap f(e') \neq \emptyset$. By f_E is partition, $e = e'$. So $\{y, z\} \subseteq f(e)$. Thus $z \in O(y)$. Hence $O(x) \subseteq O(y)$.

Similarly, $O(y) \subseteq O(x)$.

Thus $O(x) = O(y)$.

(4) For any $x, y \in U$, by f_E is full, then there exists $e, e' \in E$ such that $x \in f(e)$ and $y \in f(e')$. Since f_E keeping union, then there exists $e'' \in E$ such that $f(e) \cup f(e') = f(e'')$. Thus $\{x, y\} \subseteq f(e'')$. So $y \in O(x)$. Hence $O(x) = U$. □

Theorem 3.8. *Let f_E be a full and keeping union soft set and let $P = (U, f_E)$. Then for any $A \in I^U$,*

$$\underline{sap}_P(A) = \bar{a}_A, \quad \overline{sap}_P(A) = \bar{b}_A$$

where $\bar{a}_A = \bigwedge_{x \in U} A(x)$ *and* $\bar{b}_A = \bigvee_{x \in U} A(x)$.

Proof. For any $x \in U$ and $A \in I^U$, since f_E is full and keeping union, by Proposition 3.7, $O(x) = U$. Thus

$$\underline{sap}_P(A)(x) = \bigwedge_{y \in O(x)} A(y) = \bigwedge_{y \in U} A(y),$$

$$\overline{sap}_P(A) = \bigvee_{y \in O(x)} A(y) = \bigvee_{y \in U} A(y).$$

□

Theorem 3.9. *Let f_E be a partition soft set over U and let $P = (U, f_E)$. Then for $A \in I^U$, we have*

(1) $\underline{sap}_P(\underline{sap}_P(A)) = \underline{sap}_P(A)$.

(2) $\underline{sap}_P(\overline{sap}_P(A)) = \overline{sap}_P(A)$.

(3) $\overline{sap}_P(\underline{sap}_P(A)) = \underline{sap}_P(A)$.

(4) $\overline{sap}_P(\overline{sap}_P(A)) = \overline{sap}_P(A)$.

Proof. (1) By Proposition 3.7, for any $x \in U$,

$$\begin{aligned} \underline{sap}_P(\underline{sap}_P(A))(x) &= \bigwedge_{y \in O(x)} \underline{sap}_P(A)(y) \\ &= \bigwedge_{y \in O(x)} \bigwedge_{z \in O(y)} A(z) \\ &= \bigwedge_{z \in O(x)} A(z) \\ &= \underline{sap}_P(A)(x) \end{aligned}$$

Similarly, (2), (3) and (4) can be proved. □

Theorem 3.10. *Let f_E be a full and keeping union soft set over U and let $P = (U, f_E)$. Then for $A \in I^U$, we have*

(1) $\underline{sap}_P(\underline{sap}_P(A)) = \underline{sap}_P(A)$.

(2) $\underline{sap}_P(\overline{sap}_P(A)) = \overline{sap}_P(A)$.

(3) $\overline{sap}_P(\underline{sap}_P(A)) = \underline{sap}_P(A)$.

(4) $\overline{sap}_P(\overline{sap}_P(A)) = \overline{sap}_P(A)$.

Proof. (1) By Proposition 3.7, for any $x \in U$,

$$\begin{aligned} \underline{sap}_P(\underline{sap}_P(A))(x) &= \bigwedge_{y \in O(x)} \underline{sap}_P(A)(y) \\ &= \bigwedge_{y \in U} \bigwedge_{z \in U} A(z) \\ &= \bigwedge_{z \in U} A(z) \\ &= \underline{sap}_P(A)(x) \end{aligned}$$

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Similarly, $(2), (3)$ and (4) can be proved. □

4 Topological structures of soft fuzzy rough sets

Denote

$$\tau_P = \{A \in I^U : \underline{sap}_P(A) = A\},\ \sigma_P = \{A \in I^U : \overline{sap}_P(A) = A\}.$$

Theorem 4.1. *Let f_E be a full soft set over U and let $P = (U, f_E)$. Then τ_P is a fuzzy topology.*

Proof. (1) Obviously, $\bar{0}, \bar{1} \in \tau_P$.

(2) Let $A, B \in \tau_P$. Then we have $\underline{sap}_P(A) = A$ and $\underline{sap}_P(B) = B$. By Theorem 3.4, $\underline{sap}_P(A \cap B) = \underline{sap}_P(A) \cap \underline{sap}_P(B) = A \cap B$. Thus $A \cap B \in \tau_P$.

(3) Let $\{A_\alpha : \alpha \in \Lambda\} \subseteq \tau_P$. By Theorem 3.4, we have

$\underline{sap}_P(\bigcup\limits_{\alpha \in \Lambda} A_\alpha) \subseteq \bigcup\limits_{\alpha \in \Lambda} A_\alpha$ and $\underline{sap}_P(\bigcup\limits_{\alpha \in \Lambda} A_\alpha) \supseteq \bigcup\limits_{\alpha \in \Lambda} \underline{sap}_P(A_\alpha) = \bigcup\limits_{\alpha \in \Lambda} A_\alpha$.

Thus $\underline{sap}_P(\bigcup\limits_{\alpha \in \Lambda} A_\alpha) = \bigcup\limits_{\alpha \in \Lambda} A_\alpha$. So $\bigcup\limits_{\alpha \in \Lambda} A_\alpha \in \tau_P$.

Hence τ_P is a fuzzy topology. □

Theorem 4.2 below gives topological structures of soft fuzzy rough sets.

Theorem 4.2. *Let f_E be a partition (resp. full and keeping union) soft set over U and let $P = (U, f_E)$. Then for any $A \in I^U$.*

(1) $\{\overline{sap}_P(A) : A \in I^U\} = \tau_P = \sigma_P = \{\underline{sap}_P(A) : A \in I^U\}$.

(2) $\underline{sap}_P$ *is an interior operator of* τ_P.

Proof. (1) (i) Let f_E be partition. By Theorem 3.9, we have $\{\overline{sap}_P(A) : A \in I^U\} \subseteq \tau_P$.

For any $A \in \tau_P$, then $\underline{sap}_P(A) = A$. By Theorem 3.9,

$$\overline{sap}_P(A) = \overline{sap}_P(\underline{sap}_P(A)) = \underline{sap}_P(A) = A.$$

This implies that$A \in \{\overline{sap}_P(A) : A \in I^U\}$.

Thus $\{\overline{sap}_P(A) : A \in I^U\} = \tau_P$.

Similarly, $\{\underline{sap}_P(A) : A \in I^U\} = \sigma_P$.

(ii) Obviously,

$$\tau_P \subseteq \{\underline{sap}_P(A) : A \in I^U\}.$$

Let $B \in \{\underline{sap}_P(A) : A \in I^U\}$. Then $B = \underline{sap}_P(A)$ for some $A \in I^U$. By Theorem 3.9, $\underline{sap}_P(\underline{sap}_P(A)) = \underline{sap}_P(A)$. This implies that $B \in \tau_P$.

Thus $\tau_P \supseteq \{\underline{sap}_P(A) : A \in I^U\}$.

Hence $\{\overline{sap}_P(A) : A \in I^U\} = \tau_P = \{\underline{sap}_P(A) : A \in I^U\}$.

Therefore, $\{\overline{sap}_P(A) : A \in I^U\} = \tau_P = \sigma_P = \{\underline{sap}_P(A) : A \in I^U\}$.

(2) It suffices to show that

$$\underline{sap}_P(A) = int(A) \text{ for any } A \in I^U.$$

By (1), $\underline{sap}_P(A) \in \tau_P$. By Theorem 3.4, $\underline{sap}_P(A) \subseteq A$. Thus

$$\underline{sap}_P(A) \subseteq int(A).$$

Conversely, for any $B \in \tau_P$ with $B \subseteq A$, we have $B = \underline{sap}_P(B) \subseteq \underline{sap}_P(A)$. Thus

$$int(A) = \bigcup\{B : B \in \tau_P \text{ and } B \subseteq A\} \subseteq \underline{sap}_P(A).$$

Hence

$$\underline{sap}_P(A) = int(A).$$

Similarly, if f_E is full and keeping union, the properties hold. □

Denote

$$\mathcal{R} = \{A \in I^U : A \text{ is a soft fuzzy rough set}\},$$

$$\mathcal{D} = \{A \in I^U : A \text{ is a soft fuzzy definable set}\}.$$

Proposition 4.3. *Let f_E be a partition (resp. full and keeping union) soft set over U and let $P = (U, f_E)$. Then for any $A \in I^U$,*

$$A \in \mathcal{D} \Longleftrightarrow \underline{sap}_P(A) = A \Longleftrightarrow \overline{sap}_P(A) = A.$$

Proof. Obviously, $A \in \mathcal{D}$, we have $\underline{sap}_P(A) = A$ and $\overline{sap}_P(A) = A$.

Suppose $\underline{sap}_P(A) = A$. We only need to show that $\overline{sap}_P(A) = A$. Since f_E is partition (resp. full and keeping union), then by Theorem 3.9 (resp. Theorem 3.10), we have

$$\overline{sap}_P(A) = \overline{sap}_P(\underline{sap}_P(A)) = \underline{sap}_P(A) = A.$$

Thus $A \in \mathcal{D}$.

Suppose $\overline{sap}_P(A) = A$. Similarly, we have

$$\underline{sap}_P(A) = \underline{sap}_P(\overline{sap}_P(A)) = \overline{sap}_P(A) = A.$$

Thus $A \in \mathcal{D}$. □

Theorem 4.4 below gives structures of soft fuzzy rough sets.

Theorem 4.4. *Let f_E be a full soft set over U and let $P = (U, f_E)$. Then*

(1) $\mathcal{R} \cup \mathcal{D} = I^U$, $\mathcal{R} \cap \mathcal{D} = \emptyset$.

(2) *If f_E is partition (resp. full and keeping union), then*

$$\mathcal{R} = I^U \setminus \tau_P \text{ and } \mathcal{D} = \tau_P = \sigma_P.$$

Proof. This holds by Theorem 3.9, Theorem 3.10 and Proposition 4.3. □

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Approximate controllability of impulsive fractional neutral evolution equations with Riemann-Liouville fractional derivatives *

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Abstract

In this paper, we study the control systems of impulsive fractional neutral evolution differential equations involving Riemann-Liouville fractional derivatives in Banach spaces. Firstly, we establish the $PC_{1-\alpha}$-mild solution for the impulsive fractional neutral evolution differential equations . Secondly, some assumptions is made to guarantee the existence and uniqueness results of mild solutions. And under this condition, the approximate controllability of the associated impulsive fractional neutral evolution systems are formulated and proved. An example is provided to illustrate the application of the obtained theory.

Key words: *Impulsive fractional evolution equations; Riemann-Liouville fractional derivatives; $PC_{1-\alpha}$-mild solutions; Approximate controllability;*

1 Introduction

The objective of this paper is to investigate the approximate controllability of the following impulsive fractional neutral control systems involving Riemann-Liouville fractional

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derivatives:

$$\begin{cases} D_t^\alpha(x(t)-g(t,x(t))) = Ax(t)+f(t,x(t))+Bu(t), t\in(0,b], t\neq t_k, k=1,2,\cdots,m,\\ \Delta I_{0^+}^{1-\alpha}x(t)|_{t=t_k} = I_k(x(t_k)), \qquad k=1,2,\cdots,m,\\ I_{0^+}^{1-\alpha}x(t)|_{t=0}=x_0\in X, I_{0^+}^{1-\alpha}g(t,x(t))|_{t=0}=g_0\in X, \end{cases} \tag{1.1}$$

where $0<\alpha\leq 1, D_t^\alpha$ denotes the Riemann−Liouville fractional derivative of order α with the lower limit zero. $A: D(A)\subseteq X\to X$ is the infinitesimal generator of a C_0-semigroup $T(t)(t\geq 0)$ on a Banach space X. $f:[0,b]\times X\to X$, $g:[0,b]\times X\to D(A)$ are given function that will be specified later. $0=t_0<t_1<\cdots<t_m<t_{m+1}=b, I_k: X\mapsto X, \Delta I_{0^+}^{1-\alpha}x(t_k) = I_{0^+}^{1-\alpha}x(t_k^+) - I_{0^+}^{1-\alpha}x(t_k^-), I_{0^+}^{1-\alpha}x(t_k^+)$ and $I_{0^+}^{1-\alpha}x(t_k^-)$ denote the right and the left limits of $I_{0^+}^{1-\alpha}x(t)$ at $t=t_k, k=1,2,\cdots,m$. The control function $u(t)$ takes value in $V=L^p([0,b];U)$, $p>\frac{1}{\alpha}$ and U is a Banach space, B is a linear operator from V into $L^p([0,b];X)$.

Fractional calculus is a generalization of ordinary differentiation and integration, it is also as old as ordinary differential calculus. For the last decades, fractional differential equations have been received intensive attention because they provide an excellent tool for the description of memory and hereditary properties of various materials and processes, such as physics, mechanics, chemistry, engineering, etc. For more details, one can see [2, 8, 11, 16–19, 21, 23] and the reference therein.

The definitions of Riemann-Liouville fractional derivatives or integrals initial conditions play an important role in some fractional differential problems in the real world. Heymans and Podlubny [9] had verified that it was possible to attribute physical meaning to initial conditions expressed in terms of Riemann-Liouville fractional derivatives or integrals on the field of the viscoelasticity.

The impulsive differential systems have been originated from the real world problems to describe the sudden, discontinuous jumps and other dynamics processes. Impulsive differential equations have become more important in many mathematical models of real processes and phenomena which have been studied in control, physics, chemistry, population dynamics, aeronautics and engineering, for example, one can see([13–15, 27]).

The concept of controllability plays an important part in the analysis and design of control systems, since Kalman [10] first time gave the definition of it in 1963. Controllability of the deterministic and stochastic dynamical control systems in infinite-dimensional spaces is well developed by applying different approaches, more details can be found in [1, 3, 6] and the references therein. Some authors [4, 5, 7] studied the concept of exact controllability for systems represented by nonlinear evolution equations, in which the authors effectively used the fixed point approach. Most of the controllability results in infinite-dimensional control system concern the so-called semilinear system that consists of a linear part and a nonlinear part. In recent years, several researchers ([24, 25, 32]) studied a weaker concept of the controllability which has been called approximately controllability for control systems.

In recent years, the control systems governed by Caputo fractional evolution equations

with impulsive conditions have been extensively studied (one can see [12, 22, 26, 28]). In [27], J.R.Wang, M. Fečkan and Y. Zhuo gave the mild solutions of impulsive fractional evolution equations with Caputo fractional derivative. In [30], Y. Zhou and F. Jiao discussed the existence and uniqueness of mild solutions of a class of fractional neutral evolution equations by using the operators and some fixed point theorems. However, the solvability and controllability of the impulsive fractional neutral differential evolution with Riemann-Liouville fractional derivatives are still untreated topics in the literature. For this reason, it is necessary and important to study it.

The rest of this paper is organized as follows: In section 2, we will present some preliminaries and give the $PC_{1-\alpha}$-mild solution . In section 3, some sufficient conditions are established for the existence and uniqueness of mild solutions of the system (1.1). In section 4, we will study the approximate controllability for impulsive fractional neutral evolution differential equations with Riemann-Liouville fractional derivatives. Finally, we present an example to illustrate our main results .

2 Preliminaries

In this section, we introduce some basic definitions and preliminaries which are used throughout this paper. The norm of a Banach space X will be denoted by $\|\cdot\|_X$. $L_b(X,Y)$ denotes the space of bounded linear operators from X to Y. For the uniformly bounded C_0-semigroup $T(t)(t\geq 0)$, we set $M := \sup_{t\in[0,\infty)}\|T(t)\|_{L_b(X)} < \infty$. Let $C(J,X)$ denote the Banach space of all $X-$value continuous functions from $J=[0,b]$ into X with the norm $\|x\|_C = \sup_{t\in J}\{\|x(t)\|_X\}$.

To define the mild solutions of (1.1), we also consider the Banach space $PC_{1-\alpha}(J,X) = \{x : (t-t_k)^{1-\alpha}x(t) \in PC(J,X)$ is continuous at $t\in(t_k,t_{k+1}], k=0,1,2,\cdots,m$, and $t^{1-\alpha}x(t)$ is continuous from left and has right limits at $t\in\{t_1,t_2,\cdots,t_m\}\}$ with the norm

$$\|x\|_{PC_{1-\alpha}} = \sup_{t\in J}\{(t-t_k)^{1-\alpha}\|x(t)\|\}.$$

It is easily to verify that the $PC_{1-\alpha}(J,X)$ is a Banach space.

Firstly, let us recall the following definitions of fractional calculus. For more details, one can see [11, 21]:

Definition 2.1. The integral

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds, \qquad \alpha>0,$$

is called Riemann-Liouville fractional integral of order α, where Γ is the gamma function.

Definition 2.2. For a function $f(t)$ given in the interval $[0,\infty)$, the expression

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n\int_0^t (t-s)^{n-\alpha-1}f(s)dt,$$

is called the Riemann-Liouville fractional derivative of order α,where $n=[\alpha]+1,\ [\alpha]$ denotes the integer part of number α .

Lemma 2.3.([11]) let $\alpha>0,\ m=[\alpha]+1$, and let $x_{m-\alpha}(t)=I_{0^+}^{m-\alpha}x(t)$ be the fractional integral of order $m-\alpha$. If $x(t)\in L^1(J,X)$ and $x_{m-\alpha}(t)\in AC^m(J,X)$, then we have the following equality

$$I_t^\alpha D_t^\alpha x(t)=x(t)-\sum_{k=1}^{m}\frac{x_{m-\alpha}^{(m-k)}(0)}{\Gamma(\alpha-k+1)}t^{\alpha-k}.$$

Lemma 2.4. Let $0<\alpha\le 1$, and let $x_{1-\alpha}(t)=I_{0^+}^{1-\alpha}x(t)$ be the fractional integral of order $1-\alpha$. If $x(t)\in PC_{1-\alpha}(J,X)$ and $x_{1-\alpha}(t)\in PC(J,X)$, then we have the following equality

$$I_{0^+}^\alpha D_{0^+}^\alpha x(t)=\begin{cases} x(t)-x_{1-\alpha}(t)|_{t=0}\frac{t^{\alpha-1}}{\Gamma(\alpha)}, & t\in[0,t_1],\\ x(t)-\displaystyle\sum_{i=1}^{k}\frac{\Delta x_{1-\alpha}(t_i)}{\Gamma(\alpha)}(t-t_i)^{\alpha-1}-x_{1-\alpha}(t)|_{t=0}\frac{t^{\alpha-1}}{\Gamma(\alpha)}, & t\in(t_k,t_{k+1}],\end{cases}\tag{2.1}$$

where $\Delta x_{1-\alpha}(t_k)=x_{1-\alpha}(t_k^+)-x_{1-\alpha}(t_k^-),k=1,2,\cdots,m.$

Proof. At first, we can easily get

$$\begin{aligned} I_{0^+}^\alpha\ {}^LD_{0^+}^\alpha x(t)&=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\ {}^LD_{0^+}^\alpha x(s)ds\\ &=\frac{d}{dt}\{\frac{1}{\Gamma(\alpha+1)}\int_0^t(t-s)^{\alpha}\ {}^LD_{0^+}^\alpha x(s)ds\}.\end{aligned}$$

If $t\in[0,t_1]$, by Lemma 2.3, the result hold.

If $t\in(t_1,t_2]$, then

$$\begin{aligned} &\frac{1}{\Gamma(\alpha+1)}\int_0^t(t-s)^{\alpha}\ {}^LD_{0^+}^\alpha x(s)ds\\ =&\frac{1}{\Gamma(\alpha+1)}\int_0^t(t-s)^{\alpha}\frac{d}{ds}\{I_{0^+}^{1-\alpha}x(s)\}ds\\ =&\frac{1}{\Gamma(\alpha+1)\Gamma(1-\alpha)}\int_0^t(t-s)^{\alpha}\frac{d}{ds}\int_0^s(s-\tau)^{1-\alpha-1}x(\tau)d\tau ds\\ =&\frac{1}{\Gamma(\alpha+1)\Gamma(1-\alpha)}\int_0^{t_1}(t-s)^{\alpha}\frac{d}{ds}\int_0^s(s-\tau)^{1-\alpha-1}x(\tau)d\tau ds\\ &+\frac{1}{\Gamma(\alpha+1)\Gamma(1-\alpha)}\int_{t_1}^t(t-s)^{\alpha}\frac{d}{ds}\int_0^{t_1}(s-\tau)^{1-\alpha-1}x(\tau)d\tau ds\\ &+\frac{1}{\Gamma(\alpha+1)\Gamma(1-\alpha)}\int_{t_1}^t(t-s)^{\alpha}\frac{d}{ds}\int_{t_1}^s(s-\tau)^{1-\alpha-1}x(\tau)d\tau ds\\ =&\frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^{t_1}(t-s)^{\alpha-1}\int_0^s(s-\tau)^{1-\alpha-1}x(\tau)d\tau ds\\ &+\frac{1}{\Gamma(\alpha+1)\Gamma(1-\alpha)}[(t-s)^{\alpha}\int_0^s(s-\tau)^{1-\alpha-1}x(\tau)d\tau]|_{s=0}^{s=t_1}\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_{t_1}^{t}(t-s)^{\alpha-1}\int_0^{t_1}(s-\tau)^{1-\alpha-1}x(\tau)d\tau ds\\
&+\frac{1}{\Gamma(\alpha+1)\Gamma(1-\alpha)}[(t-s)^{\alpha}\int_0^{t_1}(s-\tau)^{1-\alpha-1}x(\tau)d\tau]|_{s=t_1}^{s=t}\\
&+\frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_{t_1}^{t}(t-s)^{\alpha-1}\int_{t_1}^{s}(s-\tau)^{1-\alpha-1}x(\tau)d\tau ds\\
&+\frac{1}{\Gamma(\alpha+1)\Gamma(1-\alpha)}[(t-s)^{\alpha}\int_{t_1}^{s}(s-\tau)^{1-\alpha-1}x(\tau)d\tau]|_{s=t_1}^{s=t}\\
=&\frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^{t_1}x(\tau)d\tau\int_{\tau}^{t_1}(t-s)^{\alpha-1}(s-\tau)^{1-\alpha-1}ds\\
&+\frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^{t_1}x(\tau)d\tau\int_{t_1}^{t}(t-s)^{\alpha-1}(s-\tau)^{1-\alpha-1}ds\\
&+\frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_{t_1}^{t}x(\tau)d\tau\int_{\tau}^{t}(t-s)^{\alpha-1}(s-\tau)^{1-\alpha-1}ds\\
&-\frac{x_{1-\alpha}(0)}{\Gamma(\alpha+1)}t^{\alpha}-\frac{\Delta x_{1-\alpha}(t_1)}{\Gamma(\alpha+1)}(t-t_1)^{\alpha}\\
=&\int_0^t x(\tau)d\tau-\frac{x_{1-\alpha}(0)}{\Gamma(\alpha+1)}t^{\alpha}-\frac{\Delta x_{1-\alpha}(t_1)}{\Gamma(\alpha+1)}(t-t_1)^{\alpha},
\end{aligned}
$$

where

$$
\begin{aligned}
\int_{\tau}^{t}(t-s)^{\alpha-1}(s-\tau)^{-\alpha}ds=&\int_0^1(1-z)^{\alpha-1}z^{1-\alpha-1}dz=B(\alpha,1-\alpha)\\
&=\Gamma(\alpha)\Gamma(1-\alpha).
\end{aligned}
$$

Thus, by (2.4), we get

$$
I_{0^+}^{\alpha}\ {}^{L}D_{0^+}^{\alpha}x(t)=x(t)-\frac{x_{1-\alpha}(0)}{\Gamma(\alpha)}t^{\alpha-1}-\frac{\Delta x_{1-\alpha}(t_1)}{\Gamma(\alpha)}(t-t_1)^{\alpha-1}.
$$

If $t\in(t_k,t_{k+1}], k=2,\cdots,m$, by the same way,we can get

$$
I_{0^+}^{\alpha}\ {}^{L}D_{0^+}^{\alpha}x(t)=x(t)-\sum_{i=1}^{k}\frac{\Delta x_{1-\alpha}(t_i)}{\Gamma(\alpha)}(t-t_i)^{\alpha-1}-x_{1-\alpha}(t)|_{t=0}\frac{t^{\alpha-1}}{\Gamma(\alpha)}.
$$

The proof is completed.

Accoding to the concept of solutions of [27],[30],[31], we have the following lamma

Lemma 2.5. Let $\alpha\in(0,1]$ and $h\in L^p(J,X)$, $p>\frac{1}{\alpha}$, if $x(t)\in PC^{1-\alpha}(J,X)$, $x_{1-\alpha}(t)\in PC(J,X)$ and x is a solution of the following problem

$$
\begin{cases}
D_t^{\alpha}(x(t)-g(t,x(t))=Ax(t)+h(t), & t\in(0,b],t\neq t_k,k=1,2,\cdots,m,\\
\Delta I_{0^+}^{1-\alpha}x(t)|_{t=t_k}=I_k(x(t_k)), & k=1,2,\cdots,m,\\
I_{0^+}^{1-\alpha}x(t)|_{t=0}=x_0\in X, I_{0^+}^{1-\alpha}g(t,x(t)|_{t=0}=g_0\in X
\end{cases}
\tag{2.2}
$$

then, x satisfies the following integral equation

$$x(t)=\begin{cases} t^{\alpha-1}T_\alpha(t)(x_0-g_0)+g(t,x(t))+\int_0^t(t-s)^{\alpha-1}AT_\alpha(t-s)g(s,x(s))ds+ \\ \qquad\int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)h(s)ds, \qquad\qquad t\in[0,t_1], \\ t^{\alpha-1}T_\alpha(t)(x_0-g_0)+g(t,x(t))+\displaystyle\int_0^t(t-s)^{\alpha-1}AT_\alpha(t-s)g(s,x(s))ds+ \\ \sum_{i=1}^k T_\alpha(t-t_i)(t-t_i)^{\alpha-1}I_i(x(t_i))+\int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)h(s)ds, t\in(t_k,t_{k+1}], \end{cases} \tag{2.3}$$

where

$$T_\alpha(t)=\alpha\int_0^\infty\theta\xi_\alpha(\theta)T(t^\alpha\theta)d\theta,$$

$$\xi_\alpha(\theta)=\frac{1}{\alpha}\theta^{-1-\frac{1}{\alpha}}\varpi_\alpha(\theta^{-\frac{1}{\alpha}}),$$

$$\varpi_\alpha(\theta)=\frac{1}{\pi}\sum_{n=1}^\infty(-1)^{n-1}\theta^{-n\alpha-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha), \quad \theta\in(0,\infty),$$

ξ_α is a probability density function defined on $(0,\infty)$, that is

$$\xi_\alpha(\theta)\geq 0,\ \theta\in(0,\infty), \quad \text{and} \quad \int_0^\infty\xi_\alpha(\theta)\mathrm{d}\theta=1.$$

According to Lemma 2.5, we give the following definition:

Definition 2.5. A function $x\in PC_{1-\alpha}(J,X)$ is called a mild solution of (1.1) if it satisfies the following fractional integral equation

$$x(t)=\begin{cases} t^{\alpha-1}T_\alpha(t)(x_0-g_0)+g(t,x(t))+\int_0^t(t-s)^{\alpha-1}AT_\alpha(t-s)g(s,x(s))ds+ \\ \quad\int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)Bu(s)ds+\int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)f(s,x(s))ds, \quad t\in[0,t_1], \\ t^{\alpha-1}T_\alpha(t)(x_0-g_0)+g(t,x(t))+\displaystyle\int_0^t(t-s)^{\alpha-1}AT_\alpha(t-s)g(s,x(s))ds+ \\ \sum_{i=1}^k T_\alpha(t-t_i)(t-t_i)^{\alpha-1}I_i(x(t_i))+\int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)Bu(s)ds+ \\ \int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)f(s,x(s))ds, t\in(t_k,t_{k+1}], \end{cases} \tag{2.4}$$

Definition 2.6. Let $x(t;0,x_0,u)$ be a solution of system (1.1) at time t corresponding to the control $u(\cdot)\in V$ and the initial value $x_0\in X$. The set $K_b(f)=\{x(b;0,x_0,u):u(\cdot)\in V\}$ is called the reachable set of system (1.1) at terminal time b. If $\overline{K_b(f)}=X$, then the system (1.1) is said to be approximate controllable on J.

Due to the work of the paper ([30]), we can know the following Lemma:

Lemma 2.7. The operator $T_\alpha(t)$ has the following properties:

(i) For any fixed $t\geq 0$, $T_\alpha(t)$ is linear and bounded operators, i.e., for any $x\in X$,

$$\|T_\alpha(t)x\|\leq\frac{M}{\Gamma(\alpha)}\|x\|.$$

where $M=\sup|T(t)|<\infty, t\geq 0$.

(ii) $T_\alpha(t)(t\geq 0)$ is strongly continuous.

3 Existence and uniqueness of mild solution

This section is devoted to the study of the existence and uniqueness results for a class of impulsive fractional neutral evolution differential equations involving Riemann-Liouville fractional derivatives.

In the following, we will make the following hypotheses that will be used in our main result.

$H(1)$: $T(t)$ is a C_0-semigroup and $T(t)$ is continuous in the uniform operator topology for $t>0$.

$H(2)$: There exist functions $\phi(\cdot),\varphi(\cdot)\in L^p(J,R^+),\ p>\frac{1}{\alpha}$ and a constants $c>0$, such that

$$\|f(t,x)\|\leq\phi(t)+c\|x\|_X,\|g(t,x)\|\leq\varphi(t)+c\|x\|_X\quad\text{for a.e. } t\in J,\ \text{and all } x\in X.$$

$H(3)$: There exists a constant $L_f>0$,$H>0$, $L_{ag}>0$ such that

$$\|f(t,x)-f(t,y)\|\leq L_f\|x-y\|_X,$$
$$\|Ag(t,x)-Ag(t,y)\|\leq L_{ag}\|x-y\|_X,$$

and the inequality

$$\|Ag(t,x)\|\leq H(1+\|x\|_X)\quad\text{for a.e. } t\in J,\ \text{and all } x\in X.$$

$H(4)$: there exist constants $d_i>0,i=1,2,\cdots,m$ with $M\sum_{i=1}^{m}d_ib^{\alpha-1}<1$ such that

$$||I_i(x)-I_i(y)||\leq d_i||x-y||_X,\forall x,y\in X.$$

The key tool in our first main result is the following Banach fixed point theorem:

Theorem 3.1. Let X be a Banach space and $\mathcal{B}$ be a operator from X to itself. If there exists a positive integer n, such that $\mathcal{B}^n$ is a compact operator on X. Then $\mathcal{B}$ has a unique fixed point on X.

Now, we will present the main result of this section.

Theorem 3.2. Assume that the hypotheses $H(1),H(3),H(4)$ are satisfied and

$$\big(L_{ag}||A^{-1}||+\frac{L_{ag}Mb^\alpha}{\Gamma(\alpha+1)}+\sum_{i=1}^{k}Md_i(t-t_i)^{\alpha-1}+\frac{L_fMb^\alpha}{\Gamma(\alpha+1)}\big)<1$$

Then for each control function $u(\cdot)\in V$, the initial problem (1.1) has a unique mild solution on $PC_{1-\alpha}$.

Proof. Consider the operator $\digamma$ defined by

$$(\digamma x)(t)=\begin{cases} t^{\alpha-1}T_\alpha(t)(x_0-g_0)+g(t,x(t))+\int_0^t(t-s)^{\alpha-1}AT_\alpha(t-s)g(s,x(s))ds+\\ \quad\int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)Bu(s)ds+\int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)f(s,x(s))ds, \quad t\in[0,t_1],\\ t^{\alpha-1}T_\alpha(t)(x_0-g_0)+g(t,x(t))+\displaystyle\int_0^t(t-s)^{\alpha-1}AT_\alpha(t-s)g(s,x(s))ds+\\ \sum_{i=1}^k T_\alpha(t-t_i)(t-t_i)^{\alpha-1}I_i(x(t_i))+\int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)Bu(s)ds+\\ \int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)f(s,x(s))ds, \qquad t\in(t_k,t_{k+1}],\end{cases} \tag{3.1}$$

Firstly, under the assumptions of our theorem, it is not difficult to check that $\digamma$ maps $PC_{1-\alpha}$ into itself.

Next, we show that $\digamma^n$ is a contraction operator on $PC_{1-\alpha}$.

If $t\in[0,t_1]$, for any $x,y\in PC_{1-\alpha}(J,X)$ and $t\in J$, we have

$$\begin{aligned}(t-t_1)^{1-\alpha}\|(\digamma x)(t)-(\digamma y)(t)\| &\le (t-t_1)^{1-\alpha}||g(t,x(t))-g(t,y(t))||\\ &\quad+(t-t_1)^{1-\alpha}\int_0^t(t-s)^{\alpha-1}\|AT_\alpha(t-s)[g(s,x(s))-g(s,y(s))]\|ds\\ &\quad+(t-t_1)^{1-\alpha}\int_0^t(t-s)^{\alpha-1}\|T_\alpha(t-s)[f(s,x(s))-f(s,y(s))]\|ds\\ &\le L_{ag}||A^{-1}||(t-t_1)^{1-\alpha}\|x(t)-y(t)\|_X\\ &\quad+\frac{L_{ag}M}{\Gamma(\alpha)}(t-t_1)^{1-\alpha}\int_0^t(t-s)^{\alpha-1}\|x(s)-y(s)\|ds\\ &\quad+\frac{L_fM}{\Gamma(\alpha)}(t-t_1)^{1-\alpha}\int_0^t(t-s)^{\alpha-1}\|x(s)-y(s)\|ds\\ &\le \big(L_{ag}||A^{-1}||+\frac{L_{ag}Mb^\alpha}{\Gamma(\alpha+1)}+\frac{L_fMb^\alpha}{\Gamma(\alpha+1)}\big)\|x-y\|_{PC_{1-\alpha}}.\end{aligned} \tag{3.2}$$

Using (3.1),(3.2) repeatedly ,we can get

$$(t-t_1)^{1-\alpha}||(\digamma^n x)(t)-(\digamma^n y)(t)||\le \big(L_{ag}||A^{-1}||+\frac{L_{ag}Mb^\alpha}{\Gamma(\alpha+1)}+\frac{L_fMb^\alpha}{\Gamma(\alpha+1)}\big)^n\|x-y\|_{PC_{1-\alpha}}.$$

If $t\in(t_k,t_{k+1}]$, $k\ge1$,for any $x,y\in PC_{1-\alpha}(J,X)$ and $t\in J$, we also have

$$\begin{aligned}(t-t_k)^{1-\alpha}\|(\digamma x)(t)-(\digamma y)(t)\| &\le (t-t_k)^{1-\alpha}||g(t,x(t))-g(t,y(t))||\\ &\quad+(t-t_k)^{1-\alpha}\int_0^t(t-s)^{\alpha-1}\|AT_\alpha(t-s)[g(s,x(s))-g(s,y(s))]\|ds\\ &\quad+(t-t_k)^{1-\alpha}\sum_{i=1}^k T_\alpha(t-t_i)(t-t_i)^{\alpha-1}(I_i(x(t_i))-I_i(y(t_i)))\\ &\quad+(t-t_k)^{1-\alpha}\int_0^t(t-s)^{\alpha-1}\|T_\alpha(t-s)[f(s,x(s))-f(s,y(s))]\|ds\\ &\le L_{ag}||A^{-1}||(t-t_k)^{1-\alpha}\|x(t)-y(t)\|_X\end{aligned}$$

$$
\begin{aligned}
&+\frac{L_{ag}M}{\Gamma(\alpha)}(t-t_k)^{1-\alpha}\int_0^t (t-s)^{\alpha-1}\|x(s)-y(s)\|ds \\
&+\sum_{i=1}^{k} Md_i(t-t_i)^{\alpha-1}(t-t_k)^{1-\alpha}\|x(t)-y(t)\|_X \\
&+\frac{L_f M}{\Gamma(\alpha)}(t-t_k)^{1-\alpha}\int_0^t (t-s)^{\alpha-1}\|x(s)-y(s)\|ds \qquad (3.3)\\
\le&\ \big(L_{ag}||A^{-1}||+\frac{(L_{ag}+L_f)Mb^\alpha}{\Gamma(\alpha+1)}+\sum_{i=1}^{k} Md_i(t-t_i)^{\alpha-1}\big)\|x-y\|_{PC_{1-\alpha}}.
\end{aligned}
$$

Using (3.1),(3.3) repeatedly ,we can get

$$(t-t_k)^{1-\alpha}||(\mathcal{F}^n x)(t)-(\mathcal{F}^n y)(t)|| \le \big(L_{ag}||A^{-1}||+\frac{(L_{ag}+L_f)Mb^\alpha}{\Gamma(\alpha+1)}+\sum_{i=1}^{k} Md_i(t-t_i)^{\alpha-1}\big)^n\|x-y\|_{PC_{1-\alpha}}.$$

Hence, $\mathcal{F}^n$ is a contraction mapping operator on $PC_{1-\alpha}(J,X)$. As a consequence of Theorem 3.1, we can deduce that $\mathcal{F}$ has a unique fixed point $x(\cdot)$ on $PC_{1-\alpha}(J,X)$, and this fixed point is the desired solution of the system (1.1). The proof is completed.

Remark 3.3. In fact, from Lemma 2.3.2 and Lemma 2.4.2 of the book of Pazy [20], we know that if the C_0-semigroup $T(t)$ is compact or differentiable for $t>t_0\ge 0$, then $T(t)$ is continuous in the uniform operator topology for $t>t_0$. Therefore, if we replace the condition $H(1)$ into that the semigroup $T(t)$ is compact or differentiable, we can also deduce that the system (1.1) has a unique solution on $PC_{1-\alpha}(J,X)$ with the hypotheses $H(2)$ and $H(3)$ holding.

4 Approximate controllability results

In this section, we will consider the approximate controllability results of the fractional evolution differential systems with Riemann-Liouville fractional derivatives.

Without loss of generality,let $t\in(t_k,t_{k+1}], k\ge 1$. Let us denote the Nemytskil operator corresponding to the nonlinear functions f and g defined by:

$$\Phi_f : C(J,X)\to L^p(J,X),\quad \Phi_f(x)(t)=f(t,x(t)),$$

and

$$\Phi_g : C(J,X)\to L^p(J,X),\quad \Phi_g(x)(t)=g(t,x(t)).$$

Define the bounded and linear operator $\mathcal{G} : L^p(J,X)\to X$ by

$$\mathcal{G}h=\int_0^b (b-s)^{\alpha-1}T_\alpha(b-s)h(s)ds,\quad h(\cdot)\in L^p(J,X).$$

$$\mathscr{T}_\alpha(t_i)=T_\alpha(b-t_i)(b-t_i)^{\alpha-1}I_i(x(t_i))$$

From the definition (2.6), we know that if for any $x_0 \in X$ and $u(\cdot) \in V$, $\overline{K_b(f)} = X$, then system (1.1) is approximately controllable on J. Equivalently, if for every desired final state $\zeta \in X$ and any $\epsilon > 0$, there exists a control function $u_\epsilon(\cdot) \in V$, such that the mild solution of system (1.1) satisfies

$$\|\zeta - b^{\alpha-1}T_\alpha(b)(x_0 - g_0) - g(b, x(b)) - \sum_{i=1}^{k} \mathscr{T}_\alpha(t_i) - \mathcal{G}A\Phi_g(x_\varepsilon) - \mathcal{G}\Phi_f(x_\varepsilon) - \mathcal{G}Bu_\varepsilon\|_X < \epsilon, \quad (4.1)$$

where $x_\epsilon(t) = x(t; 0, x_0, u_\varepsilon), k \in (t_k, t_{k+1}], t \geq 1$, then system (1.1) is approximately controllable on J.

In what follows, to discuss the approximate controllability of system (1.1), we suppose:

$H(5)$: For any $\varepsilon > 0$ and $\varphi(\cdot) \in L^p(J, X)$, there exists a $u(\cdot) \in L^p(J, U)$, such that

$$\|\mathcal{G}\varphi - \mathcal{G}Bu\|_X < \varepsilon, \quad (4.2)$$

$$\|Bu(\cdot)\|_{L^p} < N\|\varphi(\cdot)\|_{L^p}, \quad (4.3)$$

where N is a constant which is independent of $\varphi(\cdot) \in L^p(J, X)$, and

$$\frac{M(L_f + L_{ag})N}{\Gamma(\alpha)c}\left(\frac{p-1}{p\alpha - 1}\right)^{\frac{p-1}{p}} b^{1-\frac{1}{p}} E_\alpha\left(\frac{M(L_f + L_{ag})b}{c}\right) < 1, \quad (4.4)$$

where $c = 1 - L_{ag}\|A^{-1}\| - \sum_{i=1}^{k} Md_i(t - t_i)^{\alpha-1}$.

In order to discuss the approximate controllability of system (1.1), we need:

Lemma 4.1. If $H(2), H(3)$ and $H(4)$ are satisfied and $c > 0$, then any mild solutions of system (1.1) satisfy the following inequality

$$\|x_1(\cdot) - x_2(\cdot)\|_{PC_{1-\alpha}} \leq \frac{\rho}{c} E_\alpha\left(\frac{M(L_f + L_{ag})b}{c}\right)\|Bu_1(\cdot) - Bu_2(\cdot)\|_{L^p}, \quad \text{for any } u_1(\cdot),\ u_2(\cdot) \in V,$$

where

$$\rho = \frac{M}{\Gamma(\alpha)}\left(\frac{p-1}{p\alpha - 1}\right)^{\frac{p-1}{p}} b^{1-\frac{1}{p}}.$$

Proof. If $x_j (j = 1, 2)$ is a mild solution of system (1.1) with respect to $u_j(\cdot) \in V$ on $PC_{1-\alpha}(J, X)$, then

$$\begin{aligned} x_j(t) &= t^{\alpha-1}T_\alpha(t)(x_0 - g_0) + g(t, x_j(t)) + \sum_{i=1}^{k} T_\alpha(t - t_i)(t - t_i)^{\alpha-1} I_i(x_j(t_i)) \\ &+ \int_0^t (t - s)^{\alpha-1} A T_\alpha(t - s) g(s, x_j(s)) ds + \int_0^t (t - s)^{\alpha-1} T_\alpha(t - s) Bu_j(s) ds \\ &+ \int_0^t (t - s)^{\alpha-1} T_\alpha(t - s) f(s, x_j(s)) ds. \end{aligned}$$

For $t \in (t_k, t_{k+1}], k \geq 1$, we obtain that

$$(t - t_k)^{1-\alpha}\|x_2(t) - x_1(t)\|_X \leq (t - t_k)^{1-\alpha}\|g(t, x_2(t)) - g(t, x_1(t))\|$$

$$
\begin{aligned}
&+(t-t_k)^{1-\alpha}\sum_{i=1}^{k}T_\alpha(t-t_i)(t-t_i)^{\alpha-1}(I_i(x_2(t_i))-I_i(x_1(t_i)))\\
&+(t-t_k)^{1-\alpha}\int_0^t(t-s)^{\alpha-1}\|AT_\alpha(t-s)(g(s,x_2(s))-g(s,x_1(s)))\|ds\\
&+(t-t_k)^{1-\alpha}\int_0^t(t-s)^{\alpha-1}\|T_\alpha(t-s)(Bu_2(s)-Bu_1(s))\|ds\\
&+(t-t_k)^{1-\alpha}\int_0^t(t-s)^{\alpha-1}\|T_\alpha(t-s)(f(s,x_2(s))-f(s,x_1(s)))\|ds\\
\leq\ & L_{ag}\|A^{-1}\|(t-t_k)^{1-\alpha}\|x_2(t)-x_1(t)\|_X\\
&+\sum_{i=1}^{k}Md_i(t-t_i)^{\alpha-1}(t-t_k)^{1-\alpha}\|x_2(t)-x_1(t)\|_X\\
&+\frac{M}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}L_{ag}(s-t_k)^{1-\alpha}\|x_2(s)-x_1(s)\|_X ds\\
&+\frac{M}{\Gamma(\alpha)}(\frac{p-1}{p\alpha-1})^{\frac{p-1}{p}}b^{1-\frac{1}{p}}\|Bu_2-Bu_1\|_{L^p}\\
&+\frac{M}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}L_f(s-t_k)^{1-\alpha}\|x_2(s)-x_1(s)\|_X ds\\
\leq\ & (L_{ag}\|A^{-1}\|+\sum_{i=1}^{k}Md_i(t-t_i)^{\alpha-1})(t-t_k)^{1-\alpha}\|x_2(t)-x_1(t)\|_X\\
&+\frac{M}{\Gamma(\alpha)}(\frac{p-1}{p\alpha-1})^{\frac{p-1}{p}}b^{1-\frac{1}{p}}\|Bu_2-Bu_1\|_{L^p}\\
&+\frac{M(L_f+L_{ag})}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}(s-t_k)^{1-\alpha}\|x_2(s)-x_1(s)\|_X ds
\end{aligned}
$$

.

Let

$$W(t)=(t-t_k)^{1-\alpha}\|x_2(t)-x_1(t)\|_X.$$

Then by (4.5), we have

$$W(t)\leq\frac{\rho}{c}\|Bu_1(\cdot)-Bu_2(\cdot)\|_{L^p}+\frac{M(L_f+L_{ag})}{\Gamma(\alpha)c}\int_0^t(t-s)^{\alpha-1}W(s)ds.$$

By the Gronwall inequality ,we get

$$W(t)\leq\frac{\rho}{c}E_\alpha(\frac{M(L_f+L_{ag})b}{c})\|Bu_1(\cdot)-Bu_2(\cdot)\|_{L^p}.$$

Therefore, we obtain

$$\|x_1(\cdot)-x_2(\cdot)\|_{PC_{1-\alpha}}=\sup_{t\in J}\{(t-t_k)^{1-\alpha}\|x_2(t)-x_1(t)\|_X\}\leq\frac{\rho}{c}E_\alpha(\frac{M(L_f+L_{ag})b}{c})\|Bu_1(\cdot)-Bu_2(\cdot)\|_{L^p}.$$

The proof is completed.

Theorem 4.2. Suppose that hypotheses of Theorem 3.2, $H(5)$, are satisfied. Then system (1.2) is approximately controllable on J, if A is a infinitesimal generator of an analytic semigroup $T(t)$ on the Banach space X.

Proof. Since the domain $D(A)$ of the operator A is dense in X, it is sufficient to show that $D(A) \subset K_b(f)$, i.e., for any $\epsilon > 0$ and $\eta \in D(A)$, there exists a $u_\epsilon(\cdot) \in V$, such that

$$\|\zeta - b^{\alpha-1}T_\alpha(b)(x_0 - g_0) - g(b, x(b)) - \sum_{i=1}^{k} \mathscr{T}_\alpha(t_i) - \mathcal{G}A\Phi_g(x_\varepsilon) - \mathcal{G}\Phi_f(x_\varepsilon) - \mathcal{G}Bu_\varepsilon\|_X < \epsilon, \quad (4.5)$$

where $x_\epsilon(t) = x(t; 0, x_0, u_\varepsilon)$ and $t \in [0, b]$.

Firstly, for any $x_0 \in X$, we know that $b^{\alpha-1}T_\alpha(b)(x_0-g_0)+g(b, x(b))+\sum_{i=1}^{k} \mathscr{T}_\alpha(t_i) \in D(A)$ lies on the fact that $T(t)$ is an analytic semigroup. Therefore, for any given $\eta \in D(A)$, it can be seen that there exists a function $\varphi(\cdot) \in L^p(J, X)$, such that

$$\mathcal{G}\varphi = \eta - b^{\alpha-1}T_\alpha(b)(x_0 - g_0) - g(b, x(b)) - \sum_{i=1}^{k} \mathscr{T}_\alpha(t_i).$$

Next, we show that one can get a control function $u_\epsilon(\cdot) \in V$ such that the inequality (4.5) holds. In fact, for any given $\epsilon > 0$ and $u_1(\cdot) \in V$, from the hypotheses of $H(5)$, there exists a $u_2(\cdot) \in V$, such that

$$\|\eta - b^{\alpha-1}T_\alpha(b)(x_0 - g_0) - g(b, x(b)) - \sum_{i=1}^{k} \mathscr{T}_\alpha(t_i) - \mathcal{G}A\Phi_g(x_1) - \mathcal{G}\Phi_f(x_1) - \mathcal{G}Bu_2\|_X < \frac{\epsilon}{2^2},$$

where $x_1(t) = x(t; 0, x_0, u_1)$, $0 \le t \le b$. Denote $x_2(t) = x(t; 0, x_0, u_2), 0 \le t \le b$. By the hypotheses of $H(5)$ again, there exists $w_2(\cdot) \in V$, such that

$$\|\mathcal{G}[\Phi_f(x_2) - \Phi_f(x_1) + A\Phi_g(x_2) - A\Phi_g(x_1)] - \mathcal{G}Bw_2\|_X < \frac{\epsilon}{2^3},$$

and

$$\begin{aligned}
\|Bw_2(\cdot)\|_{L^p} &\le N\|\Phi_f(x_2)(\cdot) - \Phi_f(x_1)(\cdot) + \Phi_g(x_2)(\cdot) - \Phi_g(x_1)(\cdot)\|_X \\
&\le N(L_f + L_{ag})(t - t_k)^{1-\alpha}\|x_2(\cdot) - x_1(\cdot)\|_X \\
&\le \frac{M(L_f + L_{ag})N}{\Gamma(\alpha)c}\left(\frac{p-1}{p\alpha - 1}\right)^{\frac{p-1}{p}} b^{1-\frac{1}{p}} E_\alpha\left(\frac{M(L_f + L_{ag})b}{c}\right)\|Bu_1(\cdot) - Bu_2(\cdot)\|_{L^p}.
\end{aligned}$$

Now, we define

$$u_3(t) = u_2(t) - w_2(t), \qquad u_3(\cdot) \in V,$$

and it follows from that

$$\|\eta - b^{\alpha-1}T_\alpha(b)(x_0 - g_0) - g(b, x(b)) - \sum_{i=1}^{k} \mathscr{T}_\alpha(t_i) - \mathcal{G}\Phi_f(x_2) - A\Phi_g(x_2) - \mathcal{G}Bu_3\|_X$$

$$
\begin{aligned}
&\leq \|\eta - b^{\alpha-1}T_\alpha(b)(x_0 - g_0) - g(b, x(b)) - \sum_{i=1}^{k} \mathscr{T}_\alpha(t_i) - \mathcal{G}\Phi_f(x_1) - A\Phi_g(x_1) - \mathcal{G}Bu_2\|_X \\
&\quad + \|\mathcal{G}Bw_2 - [\mathcal{G}\Phi_f(x_2) - \mathcal{G}\Phi_f(x_1) - A\Phi_g(x_2) - A\Phi_g(x_1)]\|_X \\
&\leq (\frac{1}{2^2} + \frac{1}{2^3})\epsilon.
\end{aligned}
$$

By inductions, we can get the sequence $\{u_n(\cdot)\} \subset V$, which follows that

$$
\begin{aligned}
&\|\eta - b^{\alpha-1}T_\alpha(b)(x_0 - g_0) - g(b, x(b)) - \sum_{i=1}^{k} \mathscr{T}_\alpha(t_i) - \mathcal{G}\Phi_f(x_n) - \mathcal{G}A\Phi_g(x_n) - \mathcal{G}Bu_{n+1}\|_X \\
&< (\frac{1}{2^2} + \cdots + \frac{1}{2^n})\epsilon
\end{aligned}
$$

where $x_n(\cdot) = x(\cdot; 0, x_0, u_n), 0 \leq t \leq b$, and

$$
\|Bu_{n+1} - Bu_n\|_{L^p} < \frac{M(L_f + L_{ag})N}{\Gamma(\alpha)c}(\frac{p-1}{p\alpha - 1})^{\frac{p-1}{p}} b^{1-\frac{1}{p}} E_\alpha(\frac{M(L_f + L_{ag})b}{c})\|Bu_n(\cdot) - Bu_{n-1}(\cdot)\|_{L^p}.
$$

From (4.3),(4.4), we know that the sequence $\{Bu_n : n = 1, 2, \cdots\}$ is a Cauchy sequences on the Banach space $L^p(J, X)$. Therefore, there exists a sequence $\psi(\cdot) \in L^p(J, X)$, such that

$$
\lim_{n\to\infty} Bu_n(\cdot) = \psi(\cdot) \quad \text{in} \quad L^p(J, X).
$$

Then, for any $\epsilon > 0$, there exists a positive integer number N, such that

$$
\|\mathcal{G}Bu_{N+1} - \mathcal{G}Bu_N\|_X < \frac{\epsilon}{2}.
$$

Therefore, we have

$$
\begin{aligned}
&\|\eta - b^{\alpha-1}T_\alpha(b)(x_0 - g_0) - g(b, x(b)) - \sum_{i=1}^{k} \mathscr{T}_\alpha(t_i) - \mathcal{G}\Phi_f(x_N) - \mathcal{G}A\Phi_g(x_N) - \mathcal{G}Bu_N\|_X \\
&\leq \|\eta - b^{\alpha-1}T_\alpha(b)(x_0 - g_0) - g(b, x(b)) - \sum_{i=1}^{k} \mathscr{T}_\alpha(t_i) - \mathcal{G}\Phi_f(x_N) - \mathcal{G}A\Phi_g(x_N) - \mathcal{G}Bu_{N+1}\|_X \\
&\quad + \|\mathcal{G}Bu_{N+1} - \mathcal{G}Bu_N\|_X \\
&\leq (\frac{1}{2^2} + \cdots + \frac{1}{2^n})\epsilon + \frac{\epsilon}{2} < \epsilon.
\end{aligned}
$$

This proves the approximate controllability of system (1.1).

5 An example

Consider the following initial-boundary value problem of fractional parabolic control system with Riemann-Liouville fractional derivatives:

$$\begin{cases} D_t^{\alpha}(x(t,y)-g(t,y(t)))=\frac{\partial^2}{\partial y^2}x(t,y)+f(t,x(t))+Bu(t), & t\in J=[0,1]\backslash\{\frac{1}{2}\},\ y\in[0,\pi],\\ \Delta I_{0+}^{1-\alpha}x(\frac{1}{2},y)=\frac{|x(y)|}{2+|x(y)|}, & y\in[0,\pi]\\ x(t,0)=x(t,\pi)=0, & t\in J=[0,1],\\ I_{0+}^{1-\alpha}x(t,y)|_{t=0}=x_0(y),\ I_{0+}^{1-\alpha}g(t,x(t))|_{t=0}=g_0(y), & t\in[0,1],\ y\in[0,\pi], \end{cases} \tag{5.1}$$

where $\alpha=\frac{5}{6}$.

Take $X=U=L^2([0,\pi])$ and the operator $A:D(A)\subset X\to X$ is defined by

$$Ax=x'',$$

where the domain $D(A)$ is given by

$$\{x\in X: x,\ x' \text{ are absolutely continuous},\ x''\in X,\ x(0)=x(\pi)=0\}.$$

then, A can be written as

$$Ax=-\sum_{n=1}^{\infty}n^2(x,x_n)x_n,\qquad x\in D(A),$$

where $e_n(x)=\sqrt{2/\pi}\sin ny(n=1,2,\cdots)$ is an orthonormal basis of X. It is well known that A is the infinitesimal generator of a differentiable semigroup $T(t)(t>0)$ in X given by

$$T(t)x=\sum_{n=1}^{\infty}\exp^{-n^2t}(x,e_n)e_n,\quad x\in X,\qquad \text{and}\quad \|T(t)\|\le e^{-t}<1=M.$$

For every $u(\cdot)\in V=L^2(J,U)$, we have

$$u(t)=\sum_{n=1}^{\infty}u_n(t)e_n,\qquad u_n(t)=\langle u(t),e_n\rangle,$$

Define the operator B as

$$Bu(t)=\sum_{n=1}^{\infty}\overline{u}_n(t)e_n,$$

where

$$\overline{u}_n(t)=\begin{cases}0, & 0\le t<1-\frac{1}{n^2},\\ u_n(t), & 1-\frac{1}{n^2}\le t\le 1,\end{cases}\quad n=1,2,\cdots,$$

then, one can easily obtain that $\|Bu(\cdot)\|\le\|u(\cdot)\|$, which implies that $B\in L(V,L^2(J,X))$.

Firstly, by the definition of the operator B, the corresponding linear system of (5.1) as following:

$$\begin{cases} D_t^{\frac{5}{6}}(x_n(t)-\overline{g}(t))+n^2x_n(t)=\widehat{u}_n(t), \quad 1-\frac{1}{n^2}<t<1, \\ \Delta I_{0^+}^{\frac{1}{6}}x_n(\frac{1}{2})=\frac{|x_n(y)|}{2+|x_n(y)|} \\ I_{0^+}^{\frac{1}{6}}(x_n(t)-\overline{g}(t))|_{t=0}=x_0-\overline{g}_0\in X. \end{cases} \tag{5.2}$$

Next, we will check that the hypotheses $H(5)$ are satisfied. To check these, let us denote

$$h=\int_0^1(1-s)^{-\frac{1}{6}}T_{\frac{5}{6}}(1-s)g(s)ds=\sum_{n=1}^{\infty}h_ne_n, \ h_n=\langle h,e_n\rangle, \quad \text{for every } g(\cdot)\in L^2(J,X).$$

In fact, we can choose $\widetilde{u}_n(t)$ which follows from

$$\widetilde{u}_n(t)=\frac{2}{1-e^{-2}}h_ne^{-n^2(1-t)}, \quad 1-\frac{1}{n^2}\leq t\leq 1,$$

and

$$h_n=\int_{1-\frac{1}{n^2}}^1\int_0^{\infty}(1-t)^{-\frac{1}{6}}\theta\xi_{\frac{5}{6}}(\theta)e^{-n^2\theta(1-t)^{\frac{5}{6}}}\widetilde{u}_n(t)d\theta dt.$$

For this, we define

$$u(t)=\sum_{n=1}^{\infty}u_n(t)e_n,$$

where

$$u_n(t)=\begin{cases} 0, & 0\leq t<1-\frac{1}{n^2}, \\ \widetilde{u}_n(t), & 1-\frac{1}{n^2}\leq t\leq 1, \end{cases} \quad n=1,2,\cdots.$$

Therefore, for any given function $g(\cdot)\in L^2([0,1],X)$, there exists $u(\cdot)\in V$, such that

$$\int_0^1(1-s)^{-\frac{1}{6}}T_{\frac{5}{6}}(1-s)Bu(s)ds=\int_0^1(1-s)^{-\frac{1}{6}}T_{\frac{5}{6}}(1-s)g(s)ds,$$

which implies the condition (4.2) of $H(5)$ is satisfied. Moreover, we can get

$$\begin{aligned}\|Bu(\cdot)\|_{L^2}^2&=\sum_{n=1}^{\infty}\int_{1-\frac{1}{n^2}}^1|\widetilde{u}_n(t)|^2dt \\ &=2(1-e^{-2})^{-1}\sum_{n=1}^{\infty}h_n^2 \\ &\leq 2(1-e^{-2})^{-1}||g(\cdot)||_{L^2}^2\end{aligned}$$

Hence, it can be seen that the conditions $H(5)$ are satisfied, then system (5.1) is approximate controllable on J, if

$$c=1-L_{ag}\|A^{-1}\|-\sum_{i=1}^{k}d_i(t-t_i)^{\alpha-1}>0,$$

$$\frac{2}{\Gamma(\frac{5}{6})c}(1-e^{-2})^{-1}(L_f+L_{ag})(\frac{3}{2})^{\frac{1}{2}}E_{\frac{5}{6}}(\frac{L_f+L_{ag}}{c})<1,$$

is satisfied.

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Newton-Kantorovich type theorem by using recurrence relations for a fifth-order method in Banach spaces*

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Abstract

The Newton-Kantorovich type theorem of a fifth order Newton's method for solving nonlinear equations in Banach spaces is established by using recurrence relations in this paper, this theorem is proved under the assumption that the second Fréchet derivative of F satisfies Lipschitz condition. Using recurrence relations, a priori error bounds are derived along with the domains of existence and uniqueness of the solutions. The R-order convergence of the method is five. Finally, a numerical example is worked out to demonstrate the efficacy of our approach.

Keywords: Nonlinear equations; Newton-Kantorovich type theorem; Recurrence relations; A priori error bounds

MSC2000: 65D10; 65D99; 47H17

1 Introduction

One of the important problems in nonlinear science is to find the numerical solutions of nonlinear equations, it is the key problem in scientific and engineering computations field. We consider the nonlinear equations

$$F(x) = 0, \tag{1}$$

where $F : \Omega \subset X \to Y$ is a nonlinear operator on an open convex subset Ω of a Banach space X with values in a Banach space Y. As we all known, Newton's method [1,2] is the best iterative method for solving nonlinear equations (1),

$$x_{n+1} = x_n - [F'(x_n)]^{-1} F(x_n), \tag{2}$$

with quadratic convergence and the low computational cost. And the efficiency index of Newton's method equal to $\sqrt{2} \approx 1.414$.

To improve the convergence order and operational efficiency, numerous variants of modified methods were developed [3–12]. And a number of Newton-Kantorovich type theorems were obtained by using recurrence relations and majorizing sequences, under Lipschitz, Hölder and $\omega-$weak conditions respectively [5, 13–26].

Some modified Newton's methods with fifth order convergence are proposed by Kou etc. in [12], with the local order of convergence of Newton's method were improved by additional evaluations of the function. In this paper, we consider that the Newton-Kantorovich theorem by using recurrence relations for the method proposed in [12, Eqn. (24)] under the assumption that the second Fréchet derivative of F satisfies the Lipschitz condition.

The paper is organized as follows. Section 1 is the introduction. In section 2, the fifth order Newton's method is described, some preliminary results are given in Section 3 which will be required by the recurrence

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relations. In Section 4, the convergence analysis is established by using recurrence relations under the assumption that the second order Fréchet derivative satisfies Lipschitz condition and the Newton-Kantorovich type theorem of this method are given. A priori error bounds for the method are also derived. In section 4, an example is worked out. Finally, conclusions are given in Section 5.

2 The fifth-order Newton's method

Kou etc. proposed a modified Newton's method in [12, Eqn. (24)]

$$u_{n+1} = x_n = \frac{f(x_n)}{f'(\frac{1}{2}(x_n + x^*_{n+1}))}$$
$$x_{n+1} = u_{n+1} - \frac{f(u_{n+1})}{2f'(\frac{1}{2}(x_n + x^*_{n+1})) - f'(x_n)},$$

where $x^*_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. This method is proved to have the order of convergence five. Per iteration of this method require two evaluations of the function and two of its first derivative, this method have efficiency index equal to $\sqrt[4]{5} \approx 1.495$.

We first extend this method to Banach spaces as

$$\begin{aligned} u_n &= x_n - \Gamma_n F(x_n), \\ z_n &= x_n - [F'(y_n)]^{-1} F(x_n), \\ x_{n+1} &= z_n - [2F'(y_n) - F'(x_n)]^{-1} F(z_n), \end{aligned} \tag{3}$$

where I is the identity operator on X, $\Gamma_n = [F'(x_n)]^{-1}$, $y_n = x_n + \frac{1}{2}(u_n - x_n)$.

We denote $B(x, r) = \{y \in X : \|y - x\| < r\}$ and $\overline{B(x, r)} = \{y \in X : \|y - x\| \le r\}$ in this paper. Let $x_0 \in \Omega$ and the nonlinear operator $F : \Omega \subset X \to Y$ be continuously second order Fréchet differentiable where Ω is an open set and X and Y are Banach spaces. We assume that

(C1) $\|\Gamma_0 F(x_0)\| \le \eta$,

(C2) $\|\Gamma_0\| \le \beta$,

(C3) $\|F''(x)\| \le M$, $x \in \Omega$,

(C4) there exists a positive real number N such that

$$\|F''(x) - F''(y)\| \le N\|x - y\|, \quad \forall x, y \in \Omega.$$

The above condition (C4) is that the second order Fréchet derivative of F satisfies Lipschitz condition.

We now denote $\eta_0 = \eta$, $\beta_0 = \beta$, $a_0 = M\beta_0\eta_0$, $b_0 = N\beta_0\eta_0^2$ and $c_0 = h(a_0, b_0)\varphi(a_0, b_0)$. Let $a_0 < \sigma(s)$ and $h(a_0, b_0)c_0 < 1$, we can define the real sequences for $n \ge 0$.

$$\eta_{n+1} = c_n\eta_n, \tag{4}$$

$$\beta_{n+1} = h(a_n, b_n)\beta_n, \tag{5}$$

$$a_{n+1} = M\beta_{n+1}\eta_{n+1}, \tag{6}$$

$$b_{n+1} = N\beta_{n+1}\eta_{n+1}^2, \tag{7}$$

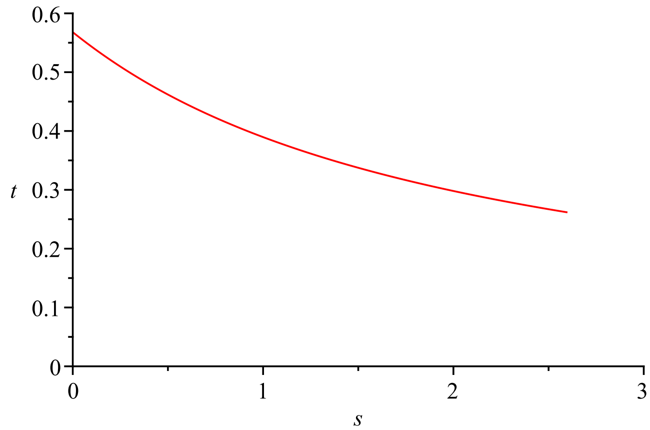
$$c_{n+1} = h(a_{n+1}, b_{n+1})\varphi(a_{n+1}, b_{n+1}). \tag{8}$$

where,

$$g(t, s) = \frac{1}{1 - t}\left[\frac{t^3}{2(2 - t)^2} + \frac{t^2}{4 - 2t} + \frac{3}{4}s\right] + \frac{2}{2 - t}, \tag{9}$$

$$h(t, s) = \frac{1}{1 - tg(t, s)}, \tag{10}$$

Figure 1: Figure of $\sigma(s)$



$$\varphi(t,s)=\frac{t}{2(1-t)^2}\left[\frac{t^3}{2(2-t)^2}+\frac{t^2}{4-2t}+\frac{3}{4}s\right]^2+\left(\frac{3}{2}s+\frac{t^2}{2-t}\right)\frac{1}{1-t}\left[\frac{t^3}{2(2-t)^2}+\frac{t^2}{4-2t}+\frac{3}{4}s\right], \quad (11)$$

$\sigma(s)$ be the smallest positive zero of the scalar function $p(t)=tg(t,s)-1$ for $s\in(0,3)$. We obtain $\sigma(s)$ is decreasing and $\sigma(s)>0$ for all $s\in(0,3)$. We can conclude that $p(t)$ has at least a real root in $(0,\frac{2}{3})$ and (See Fig. 1)

$$\max_{0<s<3}\sigma(s)\approx 0.5679591996669042<\frac{2}{3}.$$

From the definition of a_{n+1}, b_{n+1}, (4) and (5), we also have

$$a_{n+1}=h(a_n,b_n)c_na_n, \quad (12)$$

$$b_{n+1}=h(a_n,b_n)c_n^2b_n, \quad (13)$$

3 Preliminary results

Now we shall study some properties of the functions defined in (9)-(11) and the previous scalar sequences defined in (4)-(13), later developments will require the following lemma.

Lemma 1. *Let the real functions g,h and φ be given in (9)-(11). Then*

(a) $g(t,s)$ and $h(t,s)$ are increasing and $g(t,s)>1$, $h(t,s)>1$ for all $t\in(0,\sigma(s))$ and for all $s\in(0,3)$,

(b) $\varphi(t,s)$ is increasing for all $t\in(0,\sigma(s))$ and for all $s\in(0,3)$,

(c) $g(\theta t,\theta^2 s)<g(t,s)$, $h(\theta t,\theta^2 s)<h(t,s)$ and $\varphi(\theta t,\theta^2 s)<\theta^4\varphi(t,s)$ for $\theta\in(0,1)$, $t\in(0,\sigma(s))$ and $s\in(0,3)$.

Lemma 2. *Let the real functions g,h and φ be given in (9)-(11). If*

$$a_0<\sigma(s) \quad \textit{and} \quad h(a_0,b_0)c_0<1, \quad (14)$$

then we have

(a) $h(a_0,b_0)>1$ and $c_n<1$ for $n\geq 0$,

(b) the sequence $\{\eta_n\}$, $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are decreasing while $\{\beta_n\}$ is increasing,

(c) $g(a_n,b_n)a_n<1$ and $h(a_n,b_n)c_n<1$ for $n\geq 0$.

Proof. By Lemma 1 and (14), $h(a_0, b_0) > 1$ and $c_n < 1$ hold. It follows from the definitions that $\eta_1 < \eta_0$, $a_1 < a_0$, $b_1 < b_0$. Moreover, by Lemma 1, we have $1 < h(a_1, b_1) < h(a_1, b_0) < h(a_0, b_0)$ and $\varphi(a_1, b_1) < \varphi(a_1, b_0) < \varphi(a_0, b_0)$. This yields $c_1 < c_0$ and (b) holds. Based on these results we obtain $g(a_1, b_1)a_1 < g(a_0, b_0)a_0 < 1$ and $h(a_1, b_1)c_1 < h(a_0, b_0)c_0 < 1$ and (c) holds. By induction we can derive that the items (a), (b) and (c) hold. □

Lemma 3. *Under the assumptions of Lemma 2 and define $\gamma = h(a_0, b_0)c_0$, then*

$$c_n \leq \lambda\gamma^{5^n}, \quad n \geq 0, \tag{15}$$

where $\lambda = 1/h(a_0, b_0)$. Also for $n \geq 0$, we have

$$\prod_{i=0}^{n} c_i \leq \lambda^{n+1}\gamma^{\frac{5^{n+1}-1}{4}}. \tag{16}$$

Proof. By the definition of a_{n+1} and b_{n+1} given in (12)-(13), we obtain $a_1 = h(a_0, b_0)c_0a_0 = \gamma a_0$, $b_1 = h(a_0, b_0)c_0^2 b_0 < \gamma^2 b_0$, by Lemma 1 we have

$$c_1 < h(\gamma a_0, \gamma^2 b_0)\varphi(\gamma a_0, \gamma^2 b_0) < \gamma^4 h(a_0, b_0)\varphi(a_0, b_0) = \gamma^{5^1-1}c_0 = \lambda\gamma^{5^1}.$$

Suppose $c_k \leq \lambda\gamma^{5^k}$, $k \leq 1$. Then by Lemma 2, we have $a_{k+1} < a_k$, $b_{k+1} < b_k$ and $h(a_k, b_k)c_k < 1$. Thus

$$\begin{aligned} c_{k+1} &< h(a_k, b_k)\varphi(h(a_k, b_k)c_k a_k, h(a_k, b_k)c_k^2 b_k) \\ &< h(a_k, b_k)\varphi(h(a_k, b_k)c_k a_k, h^2(a_k, b_k)c_k^2 b_k) \\ &< h^5(a_k, b_k)c_k^4\varphi(a_k, b_k) = h^4(a_k, b_k)c_k^5 \\ &< \lambda\gamma^{5^{k+1}}. \end{aligned}$$

Therefore it holds that $c_n \leq \lambda\gamma^{5^n}$, $n \geq 0$.

By (15), we get

$$\prod_{i=0}^{n} c_i \leq \prod_{i=0}^{n} \lambda\gamma^{5^i} = \lambda^{n+1}\gamma^{\sum_{i=0}^{n} 5^i} = \lambda^{n+1}\gamma^{\frac{5^{n+1}-1}{4}}, \quad n \geq 0.$$

This shows (16) holds. The proof is completed. □

Lemma 4. *Under the assumptions of Lemma 2. Let $\gamma = h(a_0, b_0)c_0$ and $\lambda = 1/h(a_0, b_0)$. The sequence $\{\eta_n\}$ satisfies*

$$\eta_n \leq \eta\lambda^n\gamma^{\frac{5^n-1}{4}}, \quad n \geq 0.$$

Hence the sequence $\{\eta_n\}$ converges to 0. Moreover, for any $n \geq 0$, $m \geq 1$, it holds

$$\sum_{i=n}^{n+m} \eta_i \leq \eta\lambda^n\gamma^{\frac{5^n-1}{4}} \frac{1 - \lambda^{m+1}\gamma^{\frac{5^n(5^m+3)}{4}}}{1 - \lambda\gamma^{5^n}}.$$

Proof. From the definition of sequence $\{\eta_n\}$ given in (4) and (16), we have

$$\eta_n = c_{n-1}\eta_{n-1} = c_{n-1}c_{n-2}\eta_{n-2} = \cdots = \eta\left(\prod_{i=0}^{n-1} c_i\right) \leq \eta\lambda^n\gamma^{\frac{5^n-1}{4}}.$$

Because $\lambda < 1$ and $\gamma < 1$, it follows that $\eta_n \to 0$ as $n \to \infty$, hence the sequence $\{\eta_n\}$ converges to 0.

Since

$$\begin{aligned} \sum_{i=n}^{n+m} \lambda^i\gamma^{\frac{5^i}{4}} &\leq \lambda^n\gamma^{\frac{5^n}{4}} + \gamma^{5^n}\left(\sum_{i=n+1}^{n+m} \lambda^i\gamma^{\frac{5^{i-1}}{4}}\right) = \lambda^n\gamma^{\frac{5^n}{4}} + \lambda\gamma^{5^n}\left(\sum_{i=n}^{n+m-1} \lambda^i\gamma^{\frac{5^i}{4}}\right) \\ &= \lambda^n\gamma^{\frac{5^n}{4}} + \lambda\gamma^{5^n}\left(\sum_{i=n}^{n+m} \lambda^i\gamma^{\frac{5^i}{4}} - \lambda^{n+m}\gamma^{\frac{5^{n+m}}{4}}\right), \end{aligned}$$

where $n \geq 0$, $m \geq 1$. we can obtain

$$\sum_{i=n}^{n+m} \lambda^i \gamma^{\frac{5^i}{4}} \leq \lambda^n \gamma^{\frac{5^n}{4}} \frac{1-\lambda^{m+1}\gamma^{\frac{5^n(5^m+3)}{4}}}{1-\lambda\gamma^{5^n}},$$

Furthermore,

$$\sum_{i=n}^{n+m} \eta_i \leq \eta \sum_{i=n}^{n+m} \lambda^i \gamma^{\frac{5^i-1}{4}} = \eta\gamma^{-\frac{1}{4}} \sum_{i=n}^{n+m} \lambda^i \gamma^{\frac{5^i}{4}} \leq \eta\lambda^n \gamma^{\frac{5^n-1}{4}} \frac{1-\lambda^{m+1}\gamma^{\frac{5^n(5^m+3)}{4}}}{1-\lambda\gamma^{5^n}}.$$

Therefore $\sum_{n=0}^{\infty} \eta_n$ exists. The proof is completed. □

Lemma 5. *Let $R = \frac{g(a_0,b_0)}{1-c_0}$. If $h(a_0,b_0)c_0 < 1$, then $R < \frac{1}{a_0}$.*

Proof. Since

$$c_0 < \frac{1}{h(a_0,b_0)} = 1 - a_0 g(a_0,b_0),$$

we obtain $R < \frac{1}{a_0}$. □

Lemma 6. *Assume that the nonlinear operator $F : \Omega \subset X \to Y$ is continuously second-order Fréchet differentiable where Ω is an open set and X and Y are Banach spaces. Then we have*

$$\begin{aligned} F(z_n) = &\int_0^1 F''(u_n + t(z_n - u_n))(1-t)dt(z_n - u_n)^2 \\ &- \int_0^1 F''(y_n + t(u_n - y_n))(u_n - y_n)dt[F'(y_n)]^{-1}[F'(y_n) - F'(x_n)](u_n - x_n) \\ &+ \int_0^1 [F''(x_n + t(u_n - x_n)) - F''(x_n)](1-t)dt(u_n - x_n)^2 \\ &+ \frac{1}{2}\int_0^1 \left[F''(x_n) - F''(x_n + \frac{1}{2}t(u_n - x_n))\right] dt(u_n - x_n)^2 \end{aligned} \tag{17}$$

and

$$\begin{aligned} F(x_{n+1}) = &\int_0^1 [F''(x_n + t(u_n - x_n)) - F''(x_n)](u_n - x_n)dt(x_{n+1} - z_n) \\ &- \int_0^1 \left[F''\left(x_n + \frac{1}{2}t(u_n - x_n)\right) - F''(x_n)\right](u_n - x_n)dt(x_{n+1} - z_n) \\ &+ \int_0^1 [F'(z_n + t(x_{n+1} - z_n)) - F'(u_n)]dt(x_{n+1} - z_n). \end{aligned} \tag{18}$$

Proof. By the Taylor Expansion, we obtain

$$F(z_n) = F(u_n) + F'(u_n)(z_n - u_n) + \int_0^1 F''(u_n + t(z_n - u_n))(1-t)dt(z_n - u_n)^2, \tag{19}$$

$$\begin{aligned} F(u_n) = &F(x_n) + F'(x_n)(u_n - x_n) + \frac{1}{2}F''(x_n)(u_n - x_n)^2 \\ &+ \int_0^1 [F''(x_n + t(u_n - x_n)) - F''(x_n)](1-t)dt(u_n - x_n)^2, \end{aligned} \tag{20}$$

$$F'(u_n) = F'(y_n) + \int_0^1 F''(y_n + t(u_n - y_n))(u_n - y_n)dt$$

and

$$F'(y_n) = F'(x_n) + \frac{1}{2}F''(x_n)(u_n - x_n) + \frac{1}{2}\int_0^1 \left[F''\left(x_n + \frac{1}{2}t(u_n - x_n)\right) - F''(x_n)\right] dt(u_n - x_n),$$

we obtain

$$F'(u_n)(z_n - u_n) = F'(y_n)(z_n - u_n) + \int_0^1 F''(y_n + t(u_n - y_n))(u_n - y_n)dt(z_n - u_n), \tag{21}$$

$$\begin{aligned}[F'(y_n) - F'(x_n)](u_n - x_n) = &\frac{1}{2}F''(x_n)(u_n - x_n)^2 \\ &+ \frac{1}{2}\int_0^1 \left[F''\left(x_n + \frac{1}{2}t(u_n - x_n)\right) - F''(x_n)\right] dt(u_n - x_n)^2.\end{aligned} \tag{22}$$

By the first two-steps of method given in (3) and (20), (21), (22), we obtain

$$\begin{aligned}F'(u_n)(z_n - u_n) = &- \frac{1}{2}F''(x_n)(u_n - x_n)^2 \\ &+ \frac{1}{2}\int_0^1 \left[F''(x_n) - F''\left(x_n + \frac{1}{2}t(u_n - x_n)\right)\right] dt(u_n - x_n)^2 \\ &+ \int_0^1 F''(y_n + t(u_n - y_n))(u_n - y_n)dt(z_n - u_n).\end{aligned} \tag{23}$$

Substituting (20) and (23) into (19), we obtain (17).

We now consider $F(x_{n+1})$. Since

$$F'(y_n)(x_{n+1} - z_n) + [F'(y_n) - F'(x_n)](x_{n+1} - z_n) + F(z_n) = 0.$$

Using Taylor's formula, we have

$$\begin{aligned}F(x_{n+1}) = &F(z_n) + F'(u_n)(x_{n+1} - z_n) + \int_0^1 [F'(z_n + t(x_{n+1} - z_n)) - F'(u_n)]dt(x_{n+1} - z_n) \\ = &[F'(u_n) - F'(y_n)](x_{n+1} - z_n) - [F'(y_n) - F'(x_n)](x_{n+1} - z_n) \\ &+ \int_0^1 [F'(z_n + t(x_{n+1} - z_n)) - F'(u_n)]dt(x_{n+1} - z_n).\end{aligned} \tag{24}$$

Similarly, we obtain

$$F'(u_n) = F'(x_n) + F''(x_n)(u_n - x_n) + \int_0^1 [F''(x_n + t(u_n - x_n)) - F''(x_n)](u_n - x_n)dt.$$

It follows that

$$\begin{aligned}F'(u_n) - F'(y_n) = &\frac{1}{2}F''(x_n)(u_n - x_n) + \int_0^1 [F''(x_n + t(u_n - x_n)) - F''(x_n)](u_n - x_n)dt \\ &- \frac{1}{2}\int_0^1 \left[F''\left(x_n + \frac{1}{2}t(u_n - x_n)\right) - F''(x_n)\right](u_n - x_n)dt\end{aligned} \tag{25}$$

and

$$F'(y_n) - F'(x_n) = \frac{1}{2}F''(x_n)(u_n - x_n) + \frac{1}{2}\int_0^1 \left[F''\left(x_n + \frac{1}{2}t(u_n - x_n)\right) - F''(x_n)\right](u_n - x_n)dt. \tag{26}$$

Substituting (25) and (26) into (24), we can obtain (18). The proof is completed. □

4 Newton-Kontorovich type theorem of the method by using recurrence relations

In the following, the recurrence relations are derived for the method given by (3) under the assumptions mentioned in the previous section.

For $n = 0$, the existence of Γ_0 implies the existence of u_0, y_0. This gives us

$$\|u_0 - x_0\| = \|\Gamma_0 F(x_0)\| \leq \eta_0, \tag{27}$$

this means that $u_0, y_0 \in B(x_0, R\eta)$ where $R = \frac{g(a_0,b_0)}{1-c_0}$. By the initial hypotheses, we have

$$\|I - \Gamma_0 F'(y_0)\| \leq \|\Gamma_0\| \, \|F'(x_0) - F'(y_0)\| \leq \frac{1}{2}M\|\Gamma_0\|\|u_0 - x_0\| \leq \frac{1}{2}a_0.$$

Because of the assumption $a_0 < \sigma(s) < 2/3$, by the Banach lemma [27] it follows that

$$\left\|[I + 2\Gamma_0 \left[F'(x_0) - F'(y_0)\right]^{-1}\right\| \leq \frac{1}{1 - a_0}$$

and $\left\|[F'(y_0)]^{-1}\right\|$ exists

$$\left\|[F'(y_0)]^{-1}\right\| \leq \frac{2}{2 - a_0}\beta_0.$$

Consequently z_0 is well defined and

$$\|z_0 - x_0\| \leq \left\|[F'(y_0)]^{-1} F(x_0)\right\| \leq \left\|[F'(y_0)]^{-1} F'(x_0)\right\| \|\Gamma_0 F(x_0)\| \leq \frac{2}{2 - a_0}\eta_0.$$

It is similar to obtain

$$\begin{aligned}\|z_0 - u_0\| &= \left\|[F'(y_0)^{-1} - \Gamma_0]F(x_0)\right\| = \left\|F'(y_0)^{-1}[F'(x_0) - F'(y_0)]\Gamma_0 F(x_0)\right\| \\ &\leq \left\|F'(y_0)^{-1}\right\| \|[F'(x_0) - F'(y_0)]\| \, \|\Gamma_0 F(x_0)\| \leq \frac{a_0}{2 - a_0}\eta_0.\end{aligned}$$

By the Lemma 6, we can get

$$\|F(z_0)\| \leq \frac{1}{2}M\|z_0 - u_0\|^2 + M^2\|F'(y_0)^{-1}\|\|u_0 - y_0\|\|y_0 - x_0\|\|u_0 - x_0\| + \frac{3}{4}N\|u_0 - x_0\|^3$$

and

$$\|F(x_1)\| \leq \frac{1}{2}M\|x_1 - z_0\|^2 + \left(M\|z_0 - u_0\| + \frac{3}{2}N\|u_0 - x_0\|^2\right)\|x_1 - z_0\|. \tag{28}$$

Therefore we have

$$\begin{aligned}\|x_1 - z_0\| \leq& \|[I + 2\Gamma_0(F'(x_0) - F'(y_0))]^{-1}\|\|\Gamma_0\|\|F(z_0)\| \\ \leq& \frac{1}{1 - a_0}\left[\frac{a_0}{2}\left(\frac{a_0}{2 - a_0}\right)^2 + \frac{a_0^2}{2(2 - a_0)} + \frac{3}{4}b_0\right]\eta_0,\end{aligned}$$

and

$$\|x_1 - x_0\| \leq \|x_1 - z_0\| + \|z_0 - x_0\| \leq g(a_0, b_0)\eta_0.$$

From the assumption $c_0 < 1/h(a_0, b_0) < 1$, it follows that $x_1 \in B(x_0, R\eta)$.

By $a_0 < \sigma(s)$ and $g(t, s)$ is increasing in $t \in (0, \sigma(s))$ and $s \in (0, 3)$, we have

$$\|I - \Gamma_0 F'(x_1)\| \leq \|\Gamma_0\|\|F'(x_0) - F'(x_1)\| \leq M\beta_0\|x_1 - x_0\| \leq a_0 g(a_0, b_0) < 1,$$

it follows by the Banach lemma [27] that $\Gamma_1 = [F'(x_1)]^{-1}$ exists and

$$\|\Gamma_1\| \leq \frac{\beta_0}{1 - a_0 g(a_0, b_0)} = h(a_0, b_0)\beta_0 = \beta_1. \tag{29}$$

Then from (28) and (29), we have

$$\|u_1 - x_1\| = \|\Gamma_1 F(x_1)\| \leq \|\Gamma_1\|\|F(x_1)\| \leq h(a_0, b_0)\varphi(a_0, b_0)\eta_0 = c_0\eta_0 = \eta_1.$$

Because of $g(a_0, b_0) > 1$, we obtain

$$\|u_1 - x_0\| \leq \|u_1 - x_1\| + \|x_1 - x_0\| \leq (g(a_0, b_0) + c_0)\eta_0 < g(a_0, b_0)(1 + c_0)\eta < R\eta,$$

which shows $u_1, y_1 \in B(x_0, R\eta)$.

In addition, we have

$$M\|\Gamma_1\|\|\Gamma_1 F(x_1)\| \leq h(a_0, b_0)c_0 a_0 = a_1,$$
$$N\|\Gamma_1\|\|\Gamma_1 F(x_1)\|^2 \leq h(a_0, b_0)c_0^2 b_0 = b_1.$$

Repeating the above derivation, we can obtain the system of recurrence relations for all $n \geq 0$:

(I) There exists $\Gamma_n = [F'(x_n)]^{-1}$ and $\|\Gamma_n\| \leq \beta_n$,

(II) $\|\Gamma_n F(x_n)\| \leq \eta_n$,

(III) $M\|\Gamma_n\|\|\Gamma_n F(x_n)\| \leq a_n$,

(IV) $N\|\Gamma_n\|\|\Gamma_n F(x_n)\|^2 \leq b_n$,

(V) $\|x_{n+1} - x_n\| \leq g(a_n, b_n)\eta_n$,

(VI) $\|x_{n+1} - x_0\| \leq \sum_{i=0}^{n} g(a_i, b_i)\eta_i \leq g(a_0, b_0)\eta \dfrac{1 - \lambda^{n+1}\gamma^{\frac{5^n+3}{4}}}{1 - c_0} \leq R\eta$, where $R = g(a_0, b_0)/(1 - c_0)$.

Now we give a Newton-Kantorovich type theorem to establish the semilocal convergence of the method (3), the existence and uniqueness of the solution, the domain in which the solution is located, along with a priori error bounds, which lead to the R-order of convergence at least five of iteration (3).

Theorem 1. *Let $F : \Omega \subset X \to Y$ be a nonlinear two times Fréchet differentiable operator in an open convex subset Ω of a Banach space X with values in a Banach space Y. Assume that $x_0 \in \Omega$ and all conditions (C1)-(C4) hold. Let $a_0 = M\beta\eta$, $b_0 = N\beta\eta^2$ and $c_0 = h(a_0, b_0)\varphi(a_0, b_0)$ satisfy $a_0 < \sigma(s)$ and $h(a_0, b_0)c_0 < 1$ where g, h, φ are defined by (9)-(11). Let $\overline{B(x_0, R\eta)} \subset \Omega$ where $R = g(a_0, b_0)/(1 - c_0)$, then starting from x_0, the sequence $\{x_n\}$ generated by the method (3) converges to a solution x^* of $F(x)$ with x_n, x^* belong to $\overline{B(x_0, R\eta)}$ and x^* is the unique solution of $F(x)$ in $B(x_0, \frac{2}{M\beta} - R\eta) \bigcap \Omega$.*

Moreover, a priori error estimate is given by

$$\|x_n - x^*\| \leq \frac{g(a_0, b_0)\eta\lambda^n\gamma^{\frac{5^n-1}{4}}}{1 - \lambda\gamma^{5^n}}, \tag{30}$$

where $\gamma = h(a_0, b_0)c_0$ and $\lambda = 1/h(a_0, b_0)$.

Proof. By the system of recurrence relations, the sequences $\{x_n\}$ is well-defined in $\overline{B(x_0, R\eta)}$. Now we prove that $\{x_n\}$ is a Cauchy sequence. Since

$$\begin{aligned}\|x_{n+m} - x_n\| &\leq \sum_{i=n}^{n+m-1} \|x_{i+1} - x_i\| \leq \sum_{i=n}^{n+m-1} g(a_i, b_i)\eta_i \leq g(a_0, b_0) \sum_{i=n}^{n+m-1} \eta_i \\ &\leq g(a_0, b_0)\eta\lambda^n\gamma^{\frac{5^n-1}{4}} \frac{1 - \lambda^m\gamma^{\frac{5^n(5^{m-1}+3)}{4}}}{1 - \lambda\gamma^{5^n}},\end{aligned} \tag{31}$$

it follow that $\{x_n\}$ is a Cauchy sequence, and hence the sequence $\{x_n\}$ is convergent. So there exists a x^* such that $\lim_{n\to\infty} x_n = x^*$.

By letting $n = 0$, $m \to \infty$ in (31), we obtain

$$\|x^* - x_0\| \leq R\eta.$$

This shows that $x^* \in \overline{B(x_0, R\eta)}$.

Now we prove that x^* is a solution of $F(x) = 0$. Since

$$\|\Gamma_0\|\|F(x_n)\| \leq \|\Gamma_n\|\|F(x_n)\| \leq \eta_n, \tag{32}$$

by letting $n \to \infty$ in (32), we obtain $\|F(x_n)\| \to 0$ since $g(a_n, b_n) < g(a_0, b_0)$ and $\eta_n \to 0$. Hence, by the continuity of F in Ω, we obtain $F(x^*) = 0$.

We prove the uniqueness of x^* in $B(x_0, \frac{2}{M\beta} - R\eta) \bigcap \Omega$. Firstly we can obtain $x^* \in B(x_0, \frac{2}{M\beta} - R\eta) \bigcap \Omega$, since it follows by the system of recurrence relations

$$\frac{2}{M\beta} - R\eta = \left(\frac{2}{a_0} - R\right)\eta > \frac{1}{a_0}\eta > R\eta,$$

and then $\overline{B(x_0, R\eta)} \subset B(x_0, \frac{2}{M\beta} - R\eta) \bigcap \Omega$. Let x^{**} be another zero of $F(x)$ in $B(x_0, \frac{2}{M\beta} - R\eta) \bigcap \Omega$. By Taylor theorem, we have

$$0 = F(x^{**}) - F(x^*) = \int_0^1 F'((1-t)x^* + tx^{**})dt(x^{**} - x^*).$$

Since

$$\begin{aligned}\|\Gamma_0\| \left\| \int_0^1 [F'((1-t)x^* + tx^{**}) - F'(x_0)]dt \right\| &\leq M\beta \int_0^1 [(1-t)\|x^* - x_0\| + t\|x^{**} - x_0\|]dt \\ &< \frac{M\beta}{2}\left[R\eta + \frac{2}{M\beta} - R\eta\right] = 1,\end{aligned}$$

it follows by the Banach lemma that $\int_0^1 F'((1-t)x^* + tx^{**})dt$ is invertible and hence $x^{**} = x^*$.

Finally, by letting $m \to \infty$ in (31), we obtain (30) and furthermore

$$\|x_n - x^*\| \leq \frac{g(a_0, b_0)\eta}{\gamma^{1/4}(1 - c_0)} \left(\gamma^{1/4}\right)^{5^n}.$$

This means that the method given by (3) is of R-order of convergence at least five. This ends the proof. □

5 Numerical Example

Let $X = C[0, 1]$ be the space of continuous functions defined on the interval $[0, 1]$, with the max-norm and consider the integral equation $F(x) = 0$, where

$$F(x)(s) = x(s) - 1 - \frac{1}{2}\int_0^1 s\cos(x(t))dt,$$

with $s \in [0, 1]$, $x \in \Omega = B(0, 2) \subset X$. Integral equations of this kind (called Chandrasekhar equations) arise in elasticity or neutron transport problems.

It is easy to obtain the derivatives of F as

$$F'(x)y(s) = y(s) - \frac{1}{2}\int_0^1 s\sin(x(t))y(t)dt, \quad y \in \Omega,$$

$$F''(x)yz(s) = -\frac{1}{2}\int_0^1 s\cos(x(t))y(t)z(t)dt, \quad y, z \in \Omega.$$

The derivative F'' satisfies

$$\|F''(x)\| \leq \frac{1}{2} = M, \quad x \in \Omega,$$

and the Lipschitz condition with $N = 1/2$

$$\|F''(x) - F''(y)\| \leq \frac{1}{2}\|x - y\|, \quad x, y \in \Omega,$$

since the norm is taken as max-norm.

Starting at $x_0(t) = 4/3$, we have

$$\|F(x_0)\| = \left\| \frac{1}{2} \int_0^1 s\cos(x_0(t))dt \right\| \leq \frac{1}{2}\cos\frac{4}{3}.$$

Since

$$\|I - F'(x_0)\| = \left\| \frac{1}{2} \int_0^1 s\sin(x_0(t))dt \right\| \leq \frac{1}{2}\sin\frac{4}{3}.$$

and by the Banach lemma that Γ_0 exists and

$$\|\Gamma_0\| \leq \frac{2}{2 - \sin\frac{4}{3}} = \beta.$$

It follow that

$$\|\Gamma_0 F(x_0)\| \leq \frac{\cos\frac{4}{3}}{2 - \sin\frac{4}{3}} = \eta.$$

Consequently, we obtain

$$a_0 = M\beta\eta = \frac{\cos\frac{4}{3}}{(2 - \sin\frac{4}{3})^2}, \quad b_0 = N\beta\eta^2 = \frac{\cos^2\frac{4}{3}}{(2 - \sin\frac{4}{3})^3}, \quad c_0 = h(a_0, b_0)\varphi(a_0, b_0),$$

which satisfy

$$a_0 = 0.22257070108520 < \sigma(b_0) \approx 0.5550559412301918 < \frac{2}{3},$$

and

$$c_0 h(a_0, b_0) = 0.01439799450735 < 1.$$

This means that the hypotheses of Theorem 1 is satisfied. Hence the recurrence relations for the method given by (3) is demonstrated in Table 1. Besides, the solution x^* belongs to $\overline{B(x_0, R\eta)} = \overline{B(\frac{4}{3}, 0.27624602286429)} \subset \Omega$ and it is unique in $B(\frac{4}{3}, 1.77987817440908) \bigcap \Omega$.

Table 1: Results of recurrence relations

n	η_n	β_n	a_n	b_n	c_n
0	2.2882e-001	1.9454	2.2257e-001	5.0928e-002	1.0570e-002
1	2.4186e-003	2.6499	3.2046e-003	7.7506e-006	1.4169e-010
2	3.4268e-013	2.6585	4.5550e-013	1.5609e-025	5.7082e-050
3	1.9561e-062	2.6585	2.6001e-062	5.0860e-124	6.0602e-247
4	1.1854e-308	2.6585	1.5757e-308	0	0

6 Conclusions

A family of recurrence relations is developed for establishing the Newton-Kantorovich type convergence theorem of a modified Newton's method (3) used for solving nonlinear equations $F(x) = 0$ in Banach spaces. Based on these recurrence relations, the existence and uniqueness of the solution are proved to show the R-order convergence of the method to be five. Also a priori error bounds is given. A numerical example is worked out to demonstrate our approach and show our method can be of practical interest.

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NEW SYMMETRIC IDENTITIES INVOLVING THE EULERIAN POLYNOMIALS

YUAN HE AND CHUNPING WANG

ABSTRACT. In this paper, a further investigation for the Eulerian polynomials is performed, and some new symmetric identities involving the Eulerian polynomials are established by applying the Padé approximation technique. It turns out that some corresponding results are derived as special cases.

1. INTRODUCTION

In an attempt to describe a method of computing values of the alternating ζ-function (also called as Dirichlet eta function)

$$\eta(s)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}=1-\frac{1}{2^s}+\frac{1}{3^s}-\frac{1}{4^s}+\cdots\quad(\mathrm{Re}(s)>0)\tag{1.1}$$

at negative integers by a precursor of Abel's theorem applied to a divergent series, Leonhard Euler introduced the Eulerian polynomials

$$\sum_{k=0}^{\infty}(k+1)^n t^k=\frac{A_n(t)}{(1-t)^{n+1}}\quad(n\geq 0),\tag{1.2}$$

and determined $\eta(-n)=2^{-n-1}A_n(-1)$ for positive integer n. In fact, the Eulerian polynomials can be also defined by means of the exponential generating function

$$A(t,u)=\sum_{n=0}^{\infty}A_n(t)\frac{u^n}{n!}=\frac{t-1}{t-e^{u(t-1)}},\tag{1.3}$$

and computed by view of the recurrence

$$A_0(t)=1,\quad A_n(t)=\sum_{k=0}^{n-1}\binom{n}{k}A_k(t)(t-1)^{n-1-k}\quad(n\geq 1).\tag{1.4}$$

In particular, the coefficients $A_{n,k}$ of the Eulerian polynomials $A_n(t)=\sum_{k=0}^{n}A_{n,k}t^k$ are called as the Eulerian numbers which obey the relationship $A_{n,0}=1$ $(n\geq 0)$, $A_{n,k}=0$ $(k\geq n)$ and

$$A_{n,k}=(k+1)A_{n-1,k}+(n-k)A_{n-1,k-1}\quad(1\leq k\leq n-1),\tag{1.5}$$

and the close formula

$$A_{n,k}=\sum_{i=0}^{k}(-1)^i\binom{n+1}{i}(k+1-i)^n\quad(0\leq k\leq n-1),\tag{1.6}$$

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see [4, 5] for good introduction; also see [7] for further study.

Recently, Chung, Graham and Knuth [3] noticed that if one modify the value of $A_{0,0}$ in (1.3) by taking the convention that $A_{0,0} = 0$, then for any positive integers a and b, the following symmetrical identity holds:

$$\sum_k \binom{a+b}{k} A_{k,a-1} = \sum_k \binom{a+b}{k} A_{k,b-1} \quad (a, b \geq 1), \tag{1.7}$$

which has been extended to the q-Eulerian numbers by Han, Lin and Zeng [6]. Further, Lin [11] considered the restricted q-Eulerian polynomials and gave a similar symmetric identity on the restricted q-Eulerian numbers.

Inspired by the work of Chung, Graham and Knuth, in this paper we perform a further investigation for the Eulerian polynomials and establish some new symmetric identities between them by applying the Padé approximation technique.

This paper is organized as follows. In second section, we recall the Padé approximation to the exponential function. The third section is contributed to the statement of the new symmetric identities for the Eulerian polynomials by applying the Padé approximants to the exponential function.

2. Padé approximants

We begin by recalling here the definition of Padé approximation to general series and their expression in the case of the exponential function. Let m and n be any non-negative integers and let $\mathcal{P}_k$ be the set of all polynomials of degree $\leq k$. Given a function f with a Taylor expansion

$$f(u) = \sum_{k=0}^{\infty} c_k u^k \tag{2.1}$$

in a neighborhood of the origin, a Padé form of type (m, n) is a pair (P, Q) such that

$$P = \sum_{k=0}^{m} p_k u^k \in \mathcal{P}_m, \quad Q = \sum_{k=0}^{n} q_k u^k \in \mathcal{P}_n \quad (Q \not\equiv 0), \tag{2.2}$$

and

$$Qf - P = \mathcal{O}(u^{m+n+1}) \quad \text{as } u \to 0, \tag{2.3}$$

It is well known that every Padé form of type (m, n) for $f(u)$ always exists and obeys the same rational function. And the uniquely determined rational function P/Q is called the Padé approximant of type (m, n) for $f(u)$, and is denoted by $[m/n]_f(u)$ or $r_{m,n}[f; u]$; see for example, [1, 2].

The study of Padé approximants to the exponential function was initiated by C. Hermite [9] and continued by H. Padé [12]. Given a pair (m, n) of nonnegative integers, the Padé approximant of type (m, n) for e^t is the unique rational function

$$R_{m,n}(u) = \frac{P_{m,n}(u)}{Q_{m,n}(u)} \quad (P_{m,n} \in \mathcal{P}_m, Q_{m,n} \in \mathcal{P}_n, Q_{m,n}(0) = 1), \tag{2.4}$$

with the property that

$$e^u - R_{m,n}(u) = \mathcal{O}(u^{m+n+1}) \quad \text{as } u \to 0. \tag{2.5}$$

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Unlike Padé approximants to most other functions, it is possible to write simple explicit formulae for $P_{m,n}$ and $Q_{m,n}$ (see e.g. [13, p. 245] or [16]):

$$P_{m,n}(u)=\sum_{k=0}^{m}\frac{(m+n-k)!m!}{(m+n)!(m-k)!}\cdot\frac{u^k}{k!},\tag{2.6}$$

$$Q_{m,n}(u)=\sum_{k=0}^{n}\frac{(m+n-k)!n!}{(m+n)!(n-k)!}\cdot\frac{(-u)^k}{k!}.\tag{2.7}$$

and

$$Q_{m,n}(u)e^u-P_{m,n}(u)=(-1)^n\frac{u^{m+n+1}}{(m+n)!}\int_0^1 x^n(1-x)^m e^{xu}\mathrm{d}x.\tag{2.8}$$

We refer respectively to the polynomials $P_{m,n}(u)$ and $Q_{m,n}(u)$ as the Padé numerator and denominator of type (m,n) for e^u.

The properties of these approximants have played important roles in number theory (for example, Hermite's proof of the transcendency of e, Lindemann's proof of the transcendency of π and continued fractions, see [10, 14] for details), Orthogonal polynomials [15] and so on.

3. The restatement of results

In this section, we shall replace the exponential function e^u not by its Taylor expansion around $u=0$ but by its Padé approximant in the generating function of the Eulerian polynomials. We first rewrite the formula (1.3), as follows,

$$(t-e^{u(t-1)})\sum_{i=0}^{\infty}A_i(t)\frac{u^i}{i!}=t-1,\tag{3.1}$$

If we denote the right hand side of (2.8) by $S_{m,n}(t)$, then the Padé approximant for the exponential function e^u can be expressed in the following way:

$$e^u=\frac{P_{m,n}(u)+S_{m,n}(u)}{Q_{m,n}(u)}.\tag{3.2}$$

We now substitute $u(t-1)$ for u in (3.2) and it follows from (3.1) that

$$\begin{aligned}\big(tQ_{m,n}(u(t-1))-P_{m,n}(u(t-1))-S_{m,n}(u(t-1))\big)\sum_{i=0}^{\infty}A_i(t)\frac{u^i}{i!}\\ =(t-1)Q_{m,n}(u(t-1)).\end{aligned}\tag{3.3}$$

If applying the exponential series $e^{xu}=\sum_{k=0}^{\infty}x^ku^k/k!$ in the right hand side of (2.8), with the help of the beta function, we get

$$\begin{aligned}S_{m,n}(u)&=(-1)^n\frac{u^{m+n+1}}{(m+n)!}\sum_{k=0}^{\infty}\frac{u^k}{k!}\int_0^1 x^{n+k}(1-x)^m dx\\ &=\sum_{k=0}^{\infty}\frac{(-1)^n m!(n+k)!}{(m+n)!(m+n+k+1)!}\cdot\frac{u^{m+n+k+1}}{k!}.\end{aligned}\tag{3.4}$$

Like the definition of the Eulerian numbers, we consider $p_{m,n;k}$, $q_{m,n;k}$ and $s_{m,n;k}$ of the coefficients of the polynomials

$$P_{m,n}(u)=\sum_{k=0}^{m}p_{m,n;k}u^k,\quad Q_{m,n}(u)=\sum_{k=0}^{n}q_{m,n;k}u^k,\tag{3.5}$$

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and

$$S_{m,n}(u)=\sum_{k=0}^{\infty}s_{m,n;k}u^{m+n+k+1}. \tag{3.6}$$

Obviously, $p_{m,n;k}$, $q_{m,n;k}$ and $s_{m,n;k}$ satisfies

$$p_{m,n;k}=\frac{m!(m+n-k)!}{k!(m+n)!(m-k)!},\quad q_{m,n;k}=\frac{(-1)^k n!(m+n-k)!}{k!(m+n)!(n-k)!}, \tag{3.7}$$

and

$$s_{m,n;k}=\frac{(-1)^n m!(n+k)!}{k!(m+n)!(m+n+k+1)!}, \tag{3.8}$$

respectively. If applying (3.5) and (3.6) to (3.3) we obtain

$$\begin{aligned}\left(t\sum_{k=0}^{n}q_{m,n;k}(t-1)^k u^k\right)\sum_{i=0}^{\infty}A_i(t)\frac{u^i}{i!}-\left(\sum_{k=0}^{m}p_{m,n;k}(t-1)^k u^k\right)\sum_{i=0}^{\infty}A_i(t)\frac{u^i}{i!}\\ -\left(\sum_{k=0}^{\infty}s_{m,n;k}(t-1)^{m+n+k+1}u^{m+n+k+1}\right)\sum_{i=0}^{\infty}A_i(t)\frac{u^i}{i!}\\ =(t-1)\left(\sum_{k=0}^{n}q_{m,n;k}(t-1)^k u^k\right),\end{aligned} \tag{3.9}$$

from which and the familiar Cauchy product it follows that

$$\begin{aligned}t\sum_{l=0}^{\infty}u^l\sum_{i+k=l}q_{m,n;k}(t-1)^k\frac{A_i(t)}{i!}-\sum_{l=0}^{\infty}u^l\sum_{i+k=l}p_{m,n;k}(t-1)^k\frac{A_i(t)}{i!}\\ -\sum_{l=0}^{\infty}u^l\sum_{i+k=l-m-n-1}s_{m,n;k}(t-1)^{m+n+k+1}\frac{A_i(t)}{i!}\\ =(t-1)\left(\sum_{k=0}^{n}q_{m,n;k}(t-1)^k u^k\right).\end{aligned} \tag{3.10}$$

Comparing the coefficients of u^l in (3.10) gives for $0\le l\le m+n$

$$\begin{aligned}t\sum_{i+k=l}q_{m,n;k}(t-1)^k\frac{A_i(t)}{i!}-\sum_{i+k=l}p_{m,n;k}(t-1)^k\frac{A_i(t)}{i!}\\ =(t-1)q_{m,n;l}(t-1)^l,\end{aligned} \tag{3.11}$$

which together with (3.7) yields the following

Theorem 3.1. *Let l,m,n be any non-negative integers with $l\le m+n$,*

$$\begin{aligned}t\sum_{k=0}^{l}\binom{n}{k}(m+n-k)!(1-t)^k\frac{A_{l-k}(t)}{(l-k)!}-\sum_{k=0}^{l}\binom{m}{k}(m+n-k)!(t-1)^k\frac{A_{l-k}(t)}{(l-k)!}\\ =-(1-t)^{l+1}\binom{n}{l}(m+n-l)!.\end{aligned} \tag{3.12}$$

It follows that we show some special cases of Theorem 3.1. Setting $l = n$ in Theorem 3.1 gives that for any non-negative integers m and n,

$$t\sum_{k=0}^{n}\binom{n}{k}(m+n-k)!(1-t)^k\frac{A_{n-k}(t)}{(n-k)!}-\sum_{k=0}^{n}\binom{m}{k}(m+n-k)!(t-1)^k\frac{A_{n-k}(t)}{(n-k)!}$$
$$=-(1-t)^{n+1}m!. \quad (3.13)$$

In particuler, the case $m = 0$ in (3.13) leads to

$$A_n(t)=(1-t)^{n+1}+t\sum_{k=0}^{n}\binom{n}{k}A_k(t)(1-t)^{n-k} \quad (n\geq 0), \quad (3.14)$$

which is similar to (1.2) and can be regarded as a new recurrence formula to compute the Eulerian polynomials. On the other hand, setting $l = m + n$ in Theorem 3.1, we obtain that for any non-negative integer n and positive integer m,

$$t\sum_{k=0}^{n}\binom{n}{k}(1-t)^{n-k}A_{m+k}(t)=\sum_{k=0}^{m}\binom{m}{k}(t-1)^{m-k}A_{n+k}(t). \quad (3.15)$$

It is obvious that the case $n = 0$ in (3.15) gives the formula (1.2).

If comparing the coefficients of u^l in (3.10) for $l \geq m + n + 1$ then

$$t\sum_{i+k=l}q_{m,n;k}(t-1)^k\frac{A_i(t)}{i!}-\sum_{i+k=l}p_{m,n;k}(t-1)^k\frac{A_i(t)}{i!}$$
$$=\sum_{i+k=l-m-n-1}s_{m,n;k}(t-1)^{m+n+k+1}\frac{A_i(t)}{i!}. \quad (3.16)$$

Hence, applying (3.7) and (3.8) to (3.16) arises

$$t\sum_{k=0}^{n}\binom{n}{k}(m+n-k)!(1-t)^k\frac{A_{l-k}(t)}{(l-k)!}-\sum_{k=0}^{m}\binom{m}{k}(m+n-k)!(t-1)^k\frac{A_{l-k}(t)}{(l-k)!}$$
$$=\frac{(-1)^n m!n!}{l!}\sum_{i=0}^{l-m-n-1}\binom{l-m-i-1}{n}\binom{l}{i}(t-1)^{l-i}A_i(t). \quad (3.17)$$

If substituting $m + n + r$ for l with r positive integer in (3.17) then we state

Theorem 3.2. *Let m, n be any non-negative integers. Then for positive integer r,*

$$t\sum_{k=0}^{n}\binom{n}{k}(1-t)^{n-k}\frac{A_{m+k+r}(t)}{(m+k+1)_r}-\sum_{k=0}^{m}\binom{m}{k}(t-1)^{m-k}\frac{A_{n+k+r}(t)}{(n+k+1)_r}$$
$$=\frac{(-1)^n m!n!(t-1)^{m+n+1}}{(m+n+r)!}$$
$$\times\sum_{i=0}^{r-1}\binom{n+r-i-1}{n}\binom{m+n+r}{i}(t-1)^{r-1-i}A_i(t), \quad (3.18)$$

where $(x)_r$ is the Pochhammer symbol defined by $(x)_r = x(x + 1)\cdots(x + r - 1)$.

It is interesting to point out that the formulae (3.15) and (3.18) are analogous to the results in [8, Theorem 1.1 and Theorem 1.2] which are obtained by applying the

familiar generating function method. We now give some special cases of Theorem 3.2. Taking $r = 1$ in Theorem 3.2 leads to

$$t\sum_{k=0}^{n}\binom{n}{k}(1-t)^{n-k}\frac{A_{m+k+1}(t)}{m+k+1}-\sum_{k=0}^{m}\binom{m}{k}(t-1)^{m-k}\frac{A_{n+k+1}(t)}{n+k+1}=\frac{(-1)^n m!n!(t-1)^{m+n+1}}{(m+n+1)!}. \quad (3.19)$$

In particular, the case $m = 0$ in (3.19) yields

$$t\sum_{k=0}^{n}\binom{n}{k}(1-t)^{n-k}\frac{A_{k+1}(t)}{k+1}=\frac{A_{n+1}(t)+(-1)^n(t-1)^{n+1}}{n+1}. \quad (3.20)$$

More generally, if setting $m = 0$ in Theorem 3.2, we get that for any non-negative integer n and positive integer r,

$$t\sum_{k=0}^{n}\binom{n}{k}(1-t)^{n-k}\frac{A_{k+r}(t)}{(k+1)_r}=\frac{(-1)^n n!(t-1)^{n+1}}{(n+r)!}\sum_{i=0}^{r-1}\binom{n+r-i-1}{n}\binom{n+r}{i}(t-1)^{r-1-i}A_i(t)+\frac{A_{n+r}(t)}{(n+1)_r}. \quad (3.21)$$

And the case $n = 0$ in (3.21) gives the formula (1.2) again.

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ON THE STABILITY OF A QUADRATIC FUNCTIONAL EQUATION

SANG-BAEK LEE* AND WON-GIL PARK**

Abstract. In this paper, we investigate the generalized Hyers–Ulam stability of a new quadratic functional equation

$$\begin{aligned} &f(x+y+z)+f(x-y+z)+f(x+y-z)+f(-x+y+z) \\ &= 4[f(x)+f(y)+f(z)] \end{aligned}$$

in p-Banach spaces, where $0 < p \leq 1$. And we prove the same stability of the above functional equation in 2-Banach spaces.

1. Introduction

The problem of stability of functional equations was originally stated by S. M. Ulam [31]. In 1941, D. H. Hyers [15] proved the stability of the linear functional equation for the case when the groups $\mathcal{G}_1$ and $\mathcal{G}_2$ are Banach spaces. In 1950, T. Aoki discussed the Hyers-Ulam stability theorem in [1]. His result was further generalized and rediscovered by Th. M. Rassias in [26] in 1978. The stability problem for functional equation have been extensively investigated by a number of mathematicians ([5], [8], [12], [19], [20], [30]).

The quadratic function $f(x) = cx^2$ satisfies the functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

and therefore the equation (1.1) is called the quadratic functional equation. Every solution of the equation (1.1) is said to be a quadratic mapping. The Hyers-Ulam stability theorem for the quadratic functional equation (1.1) was by F. Skof [30] for the functions $f : \mathcal{E}_1 \to \mathcal{E}_2$ where $\mathcal{E}_1$ is a normed space and $\mathcal{E}_2$ a Banach space. The result of F. Skof is still true if the relevant domain $\mathcal{E}_1$ is replaced by an Abelian group and this was dealt with by P. W. Cholewa [6]. S. Czerwik [7] proved the Hyers-Ulam-Rassias stability of the quadratic functional equation (1.1). This result was further generalized by Th. M. Rassias [27], C. Borelli and G. L. Forti [4]. During the last three decades a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers-Ulam stability of several functional equations, and there are many interesting results concerning this problem ([2], [18], [16], [17]). In

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particular, J. M. Rassias investigated the stability of S. M. Ulam for the relative Euler-Lagrange functional equation

$$f(ax+by)+f(bx-ay)=(a^2+b^2)[f(x)+f(y)]$$

in the publications ([23], [24], [25]).

In 2008, K. Ravi, R. Murali and M. Arunkumar [28] investigated the generalized Hyers-Ulam stability of a quadratic functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+4f(x)-2f(y)$$

Definition 1.1. ([3], [29]) Let $\mathcal{X}$ be a linear space. A *quasi-norm* is a real-valued function on $\mathcal{X}$ satisfying the following:

- $\|x\| \ge 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$;
- $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in \mathcal{X}$;
- There is a constant $K \ge 1$ such that $\|x+y\| \le K(\|x\|+\|y\|)$ for all $x, y \in \mathcal{X}$.

The pair $(\mathcal{X}, \|\cdot\|)$ is called a *quasi-normed space* if $\|\cdot\|$ is a quasi-norm on $\mathcal{X}$. The smallest possible K is called the *modules of concavity* of $\|\cdot\|$. A *quasi-Banach space* is a complete quasi-normed space.

A quasi-norm $\|\cdot\|$ is called a *p-norm* $(0 < p \le 1)$ if

$$\|x+y\|^p \le \|x\|^p + \|y\|^p$$

for all $x, y \in \mathcal{X}$. In this case, a quasi-Banach space is called a *p-Banach space*.

In the 1960's, S. Gähler [9, 10, 11] introduced the concept of linear 2-normed spaces.

Definition 1.2. Let $\mathcal{X}$ be a linear space over $\mathbb{R}$ with $\dim \mathcal{X} > 1$ and let $\|\cdot,\cdot\| : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ be a function satisfying the following properties:
(a) $\|x, y\| = 0$ if and only if x and y are linearly dependent,
(b) $\|x, y\| = \|y, x\|$,
(c) $\|\alpha x, y\| = |\alpha|\|x, y\|$,
(d) $\|x, y+z\| \le \|x, y\| + \|x, z\|$
for all $x, y, z \in \mathcal{X}$ and $\alpha \in \mathbb{R}$. Then the mapping $\|\cdot,\cdot\|$ is called a 2-*norm* on $\mathcal{X}$ and the pair $(\mathcal{X}, \|\cdot,\cdot\|)$ is called a *linear* 2-*normed space*. Sometimes the condition (d) called the *triangle inequality*.

In 2011, the author [22] introduce a basic property of linear 2-normed spaces as follows.

Lemma 1.3. *Let $(\mathcal{X}, \|\cdot,\cdot\|)$ be a linear 2-normed space. If $\|x, y\| = 0$ for all $y \in \mathcal{X}$, then $x = 0$.*

For a linear 2-normed space $(\mathcal{X}, \|\cdot,\cdot\|)$, the function $x \to \|x, y\|$ is a continuous function of $\mathcal{X}$ into $\mathbb{R}$ for each fixed $y \in \mathcal{X}$ as follows.

Remark 1.4. Let $(\mathcal{X}, \|\cdot,\cdot\|)$ be a linear 2-normed space. Note that the conditions (a) and (d) implies that

$$\|x+y,z\| \le \|x,z\| + \|y,z\|$$

for all $x,y,z \in \mathcal{X}$. Putting $w := x+y$, we get $\|w,z\| \le \|x,z\| + \|w-x,z\|$ for all $x,y,z \in \mathcal{X}$. So $\|w,z\| - \|x,z\| \le \|w-x,z\|$ for all $x,z,w \in \mathcal{X}$. Replacing w by x and x by w in the above inequality, we get $\|x,z\| - \|w,z\| \le \|x-w,z\|$ for all $x,z,w \in \mathcal{X}$. Thus we have

$$|\,\|x,z\| - \|y,z\|\,| \le \|x-y,z\|$$

for all $x,y,z \in \mathcal{X}$. Hence the function $x \to \|x,y\|$ is a continuous function of $\mathcal{X}$ into $\mathbb{R}$ for each fixed $y \in \mathcal{X}$.

Let $(\mathcal{X}, \|\cdot,\cdot\|)$ be a linear 2-normed space. For $x,z \in \mathcal{X}$, let $p_z(x) := \|x,z\|$. Then, for each $z \in \mathcal{X}$, p_z is a real-valued function on $\mathcal{X}$ such that $p_z(x) = \|x,z\| \ge 0$, $p_z(\alpha x) = |\alpha|\,\|x,z\| = |\alpha| p_z(x)$ and $p_z(x+y) = \|x+y,z\| = \|z,x+y\| \le \|z,x\| + \|z,y\| = \|x,z\| + \|y,z\| = p_z(x) + p_z(y)$ for all $\alpha \in \mathbb{R}$ and all $x,y \in \mathcal{X}$. Thus p_z is a pseudo-norm (or a semi-norm) for each $z \in \mathcal{X}$.

For $x \in \mathcal{X}$, let $\|x,z\| = 0$ for all $z \in \mathcal{X}$. By Lemma 1.3, $x = 0$. Thus $0 \ne x \in \mathcal{X}$ implies that there is some $z \in \mathcal{X}$ satisfying $p_z(x) = \|x,z\| \ne 0$. Hence the family $\{p_z : z \in \mathcal{X}\}$ is a separating family of pseudo-norms.

For $\varepsilon > 0$ and $z \in \mathcal{X}$, let $U_{z,\varepsilon} := \{x \in \mathcal{X} : p_z(x) < \varepsilon\} = \{x \in \mathcal{X} : \|x,z\| < \varepsilon\}$. Let $\mathcal{S}_0 := \{U_{z,\varepsilon} : \varepsilon > 0, z \in \mathcal{X}\}$ and $\mathcal{B}_0 := \{\bigcap \mathcal{F} : \mathcal{F}$ is a finite subcollection of $\mathcal{S}_0\}$. Define a topology $\mathcal{T}$ on $\mathcal{X}$ by saying that a set U is open if and only if for every $x \in U$ there is some $N \in \mathcal{B}_0$ such that $x + N := \{x+y : y \in N\} \subset U$. That is, $\mathcal{T}$ is the topology on $\mathcal{X}$ that has as a subbase the sets $\{x \in \mathcal{X} : p_z(x - x_0) < \varepsilon\}$, $z \in \mathcal{X}$, $x_0 \in \mathcal{X}$, $\varepsilon > 0$. The topology $\mathcal{T}$ gives $\mathcal{X}$ the structure of topological vector space. Since the collection $\mathcal{B}_0$ is a local base whose members are convex, $\mathcal{X}$ is locally convex.

In the 1960's, S. Gähler and A. White [11, 32, 33] introduced the concept of 2-Banach spaces. In order to define completeness, the concepts of Cauchy sequences and convergence are required.

Definition 1.5. A sequence $\{x_n\}$ in a linear 2-normed space $\mathcal{X}$ is called a *Cauchy sequence* if $\lim_{m,n\to\infty} \|x_n - x_m, y\| = 0$ for all $y \in X$.

Definition 1.6. A sequence $\{x_n\}$ in a linear 2-normed space $\mathcal{X}$ is called a *convergent sequence* if there is an $x \in \mathcal{X}$ such that

$$\lim_{n\to\infty} \|x_n - x, y\| = 0$$

for all $y \in \mathcal{X}$. If $\{x_n\}$ converges to x, write $x_n \to x$ as $n \to \infty$ and call x the limit of $\{x_n\}$. In this case, we also write $\lim_{n\to\infty} x_n = x$.

Triangle inequality implies the following lemma (see [22]).

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Lemma 1.7. *For a convergent sequence $\{x_n\}$ in a linear 2-normed space $\mathcal{X}$,*

$$\lim_{n\to\infty} \|x_n, y\| = \left\| \lim_{n\to\infty} x_n, y \right\|$$

for all $y \in \mathcal{X}$.

Definition 1.8. A linear 2-normed space in which every Cauchy sequence is a convergent sequence is called a 2-*Banach space.*

Note that a 2-Banach space is not a p-Banach space for any p, since $0 < p \leq 1$.

In this paper, we investigate the functional equation

$$f(x+y+z)+f(x-y+z)+f(x+y-z)+f(-x+y+z) = 4[f(x)+f(y)+f(z)] \tag{1.2}$$

It is easy to see that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = ax^2$ is a solution of the functional equation (1.2). The main purpose of this paper ivestigate Hyers-Ulam stability for the functional equation (1.2).

2. Stability of (1.2) in p-Banach spaces

Theorem 2.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be vector spaces. A mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfies the functional equation (1.2) if and only if the mapping f satisfies the functional equation (1.1).*

Proof. We first assume that f is a solution of the functional equation (1.2). Set $x = y = z = 0$ in (1.2) to get $f(0) = 0$. Putting $z = 0$ in (1.2),we get $f(x + y) + f(x - y) + f(x + y) + f(-(x - y)) = 4[f(x) + f(y)]$ for all $x, y \in \mathcal{X}$. Letting $y = 0$ in the above equality, we have $f(-x) = f(x)$ for all $x \in \mathcal{X}$. From the above two equalities, we see that f satisfies (1.1).

Conversely, assume that the mapping f satisfies the functional equation (1.1). Then we have

$$\begin{aligned}
&f(x + y + z) + f(x - y + z) + f(x + y - z) + f(-x + y + z)\\
&= f\big((x + y) + z\big) + f\big((x + y) - z\big) + f\big(z + (x - y)\big) + f\big(z - (x - y)\big)\\
&= 2\big[f(x + y) + f(x - y)\big] + 4f(z)\\
&= 4\big[f(x) + f(y) + f(z)\big]
\end{aligned}$$

for all $x, y, z \in \mathcal{X}$. Thus f satisfies the equation (1.2). This completes the proof of the theorem. In this section, let $\mathcal{X}$ be a linear space and $\mathcal{Y}$ a p-Banach space $(0 < p \leq 1)$. We investigate the Hyers-Ulam stability problem for the functional equation (1.2). Define

$$\begin{aligned}
Df(x, y, z) :=&f(x + y + z) + f(x - y + z) + f(x + y - z)\\
&+ f(-x + y + z) - 4[f(x) + f(y) + f(z)]
\end{aligned}$$

for all $x, y, z \in \mathcal{X}$.

Theorem 2.2. *Let $\psi : \mathcal{X}^3 \to [0, \infty)$ be a function such that*

$$\widetilde{\psi}(x, y, z) := \sum_{i=1}^{\infty} \frac{\psi(3^{i-1}x, 3^{i-1}y, 3^{i-1}z)^p}{3^{2pi}} < \infty \tag{2.1}$$

for all $x, y, z \in \mathcal{X}$. If $f : \mathcal{X} \to \mathcal{Y}$ is a mapping satisfying $f(0) = 0$ such that

$$\|Df(x, y, z)\| \le \psi(x, y, z) \tag{2.2}$$

for all $x, y, z \in \mathcal{X}$, then there exists a unique quadratic mapping $Q : \mathcal{X} \to \mathcal{Y}$ satisfying the equation (1.2) *such that*

$$\|f(x) - Q(x)\| \le \widetilde{\psi}(x, x, x)^{\frac{1}{p}} \tag{2.3}$$

for all $x \in \mathcal{X}$.

Proof. Letting $y = z = x$ in (2.2) and dividing by 3^2, we have

$$\left\| f(x) - \frac{1}{3^2} f(3x) \right\|^p \le \frac{1}{3^{2p}}\, \psi(x, x, x)^p \tag{2.4}$$

for all $x \in \mathcal{X}$. Replacing x by $3x$ in (2.4) and then dividing by 3^{2p}, we get

$$\left\| \frac{1}{3^2} f(3x) - \frac{1}{3^4} f(3^2x) \right\|^p \le \frac{1}{3^{4p}} \psi(3x, 3x, 3x)^p$$

for all $x \in \mathcal{X}$. Adding (2.4) and the above inequality, we have

$$\left\| f(x) - \frac{1}{3^4} f(3^2x) \right\|^p \le \frac{1}{3^{2p}}\, \psi(x, x, x)^p + \frac{1}{3^{4p}} \psi(3x, 3x, 3x)^p$$

for all $x \in \mathcal{X}$. Continuing in this way, one can obtain that

$$\left\| f(x) - \frac{1}{3^{2k}} f(3^k x) \right\|^p \le \sum_{i=1}^{k} \frac{1}{3^{2pi}} \psi(3^{i-1}x, 3^{i-1}x, 3^{i-1}x)^p \tag{2.5}$$

for all $k \in \mathbb{N}$ and all $x \in \mathcal{X}$. Now, for $j \in \mathbb{N}$, dividing the preceding inequality by 3^{2pj} and then substituting x by $3^j x$, we see that

$$\left\| \frac{1}{3^{2j}} f(3^j x) - \frac{1}{3^{2(j+k)}} f(3^{j+k}x) \right\|^p \le \sum_{i=1}^{k} \frac{1}{3^{2p(i+j)}} \psi(3^{i+j-1}x, 3^{i+j-1}x, 3^{i+j-1}x)^p$$

for all $k \in \mathbb{N}$ and all $x \in \mathcal{X}$. Taking $j, k \to \infty$ in the previous inequality, by (2.1), we conclude that $\left\{\frac{1}{3^{2k}} f(3^k x)\right\}$ is a Cauchy sequence in $\mathcal{Y}$ for all $x \in \mathcal{X}$. Because of the completeness of $\mathcal{Y}$, we can define a mapping $Q : \mathcal{X} \to \mathcal{Y}$ by

$$Q(x) := \lim_{k\to\infty} \frac{1}{3^{2k}} f(3^k x)$$

for all $x \in \mathcal{X}$. By (2.1) and (2.2), we obtain that

$$\begin{aligned} \|DQ(x, y, z)\|^p &= \lim_{k\to\infty} \frac{1}{3^{2pk}} \left\| Df(3^k x, 3^k y, 3^k z) \right\|^p \\ &\le \lim_{k\to\infty} \frac{1}{3^{2pk}} \psi(3^k x, 3^k y, 3^k z)^p = 0 \end{aligned}$$

for all $x, y, z \in \mathcal{X}$. Hence the mapping $Q : \mathcal{X} \to \mathcal{Y}$ satisfies (1.2). Taking $k \to \infty$ in (2.5), we get the inequality (2.3). To prove the uniqueness of the quadratic mapping Q, let us assume that there exists a quadratic mapping $Q' : \mathcal{X} \to \mathcal{Y}$ satisfying (1.2) and (2.3). We have

$$\begin{aligned} \|Q(x) - Q'(x)\|^p &\leq \frac{1}{3^{2pk}} \big\|Q(3^k x) - Q'(3^k x)\big\|^p \\ &= \frac{2}{3^{2pk}} \widetilde{\psi}(3^k x, 3^k x, 3^k x) \to 0 \quad \text{as} \quad k \to \infty \end{aligned}$$

for all $x \in \mathcal{X}$. Therefore Q is unique.

Corollary 2.3. *Let $\mathcal{X}$ be a quasi-normed space and $\mathcal{Y}$ a p-Banach space. Let θ, q be real numbers such that $\theta \geq 0$, $0 < q < 2$. Suppose that a mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfies*

$$\|Df(x, y, z)\| \leq \theta\big(\|x\|^q + \|y\|^q + \|z\|^q\big)$$

for all $x, y, z \in \mathcal{X}$. Then there exists a unique quadratic mapping $Q : \mathcal{X} \to \mathcal{Y}$ satisfying (1.2) *such that*

$$\|f(x) - Q(x)\| \leq \frac{\theta}{3^{q-1}\big[3^{p(2-q)} - 1\big]^{\frac{1}{p}}} \|x\|^q$$

for all $x \in \mathcal{X}$.

Proof. Taking $\psi(x, y, z) := \theta\big(\|x\|^q + \|y\|^q + \|z\|^q\big)$ and applying Theorem 2.2, one can obtain the result.

Corollary 2.4. *Let $\mathcal{X}$ be a quasi-normed space and $\mathcal{Y}$ a p-Banach space. Let θ be a real number. Suppose that a mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfies*

$$\|Df(x, y, z)\| \leq \theta$$

for all $x, y \in \mathcal{X}$. Then there exists a unique quadratic mapping $Q : \mathcal{X} \to \mathcal{Y}$ satisfying (1.2) *such that*

$$\|f(x) - Q(x)\| \leq \frac{\theta}{(3^{2p} - 1)^{\frac{1}{p}}}$$

for all $x \in \mathcal{X}$.

Proof. Taking $\psi(x, y, z) := \theta$ and applying Theorem 2.2, one can obtain the result.

3. Stability of (1.2) in 2-Banach spaces

In this section, we prove the generalized Hyers-Ulam stability of the quadratic functional equation (1.2) in 2-Banach spaces.

Theorem 3.1. *Let $\psi : \mathcal{X}^3 \to [0, \infty)$ be a function such that*

$$\widetilde{\psi}(x, y, z) := \sum_{i=1}^{\infty} \frac{\psi(3^{i-1}x, 3^{i-1}y, 3^{i-1}z)}{3^{2i}} < \infty \tag{3.1}$$

for all $x, y, z \in \mathcal{X}$. If $f : \mathcal{X} \to \mathcal{Y}$ is a mapping satisfying $f(0) = 0$ such that

$$\|Df(x, y, z), w\| \leq \psi(x, y, z) \tag{3.2}$$

for all $x, y, z \in \mathcal{X}$ and all $w \in \mathcal{Y}$, then there exists a unique quadratic mapping $Q : \mathcal{X} \to \mathcal{Y}$ satisfying the equation (1.2) such that

$$\|f(x) - Q(x), w\| \leq \widetilde{\psi}(x, x, x) \tag{3.3}$$

for all $x \in \mathcal{X}$ and all $w \in \mathcal{Y}$.

Proof. Letting $x = y = z$ in (3.2) and dividing by 3^2, we have

$$\left\|f(x) - \frac{1}{3^2} f(3x), w\right\| \leq \frac{1}{3^2}\, \psi(x, x, x) \tag{3.4}$$

for all $x \in \mathcal{X}$ and all $w \in \mathcal{Y}$. Replacing x by $3x$ in (3.4) and then dividing by 3^2, we get

$$\left\|\frac{1}{3^2} f(3x) - \frac{1}{3^4} f(3^2 x), w\right\| \leq \frac{1}{3^4} \psi(3x, 3x, 3x) \tag{3.5}$$

for all $x \in \mathcal{X}$ and all $w \in \mathcal{Y}$. Adding (3.4) and (3.5), we have

$$\left\|f(x) - \frac{1}{3^4} f(3^2 x), w\right\| \leq \frac{1}{3^2}\, \psi(x, x, x) + \frac{1}{3^4} \psi(3x, 3x, 3x)$$

for all $x \in \mathcal{X}$ and all $w \in \mathcal{Y}$. Continuing in this way, one can obtain that

$$\left\|f(x) - \frac{1}{3^{2k}} f(3^k x), w\right\| \leq \sum_{i=1}^{k} \frac{1}{3^{2i}} \psi(3^{i-1}x, 3^{i-1}x, 3^{i-1}x) \tag{3.6}$$

for all $k \in \mathbb{N}$, $x \in \mathcal{X}$ and all $w \in \mathcal{Y}$. Now, for $j \in \mathbb{N}$, dividing the preceding inequality by 3^{2j} and then substituting x by $3^j x$, we see that

$$\begin{aligned}
&\left\|\frac{1}{3^{2j}} f(3^j x) - \frac{1}{3^{2(j+k)}} f(3^{j+k} x), w\right\| \\
&\qquad \leq \sum_{i=1}^{k} \frac{1}{3^{2(i+j)}} \psi(3^{i+j-1}x, 3^{i+j-1}x, 3^{i+j-1}x)
\end{aligned}$$

for all $k \in \mathbb{N}$, $x \in \mathcal{X}$ and all $w \in \mathcal{Y}$. Taking $j, k \to \infty$ in the previous inequality, by (3.1), we conclude that $\left\{\frac{1}{3^{2k}} f(3^k x)\right\}$ is a Cauchy sequence in $\mathcal{Y}$ for all $x \in \mathcal{X}$. Because of the completeness of $\mathcal{Y}$, we can define a mapping $Q : \mathcal{X} \to \mathcal{Y}$ by

$$Q(x) := \lim_{k \to \infty} \frac{1}{3^{2k}} f(3^k x)$$

for all $x \in \mathcal{X}$. By (3.1), we obtain that

$$\begin{aligned}
\|DQ(x, y, z), w\| &= \lim_{k \to \infty} \frac{1}{3^{2k}} \left\|Df(3^k x, 3^k y, 3^k z), w\right\| \\
&\leq \lim_{k \to \infty} \frac{1}{3^{2k}} \psi(3^k x, 3^k y, 3^k z) = 0
\end{aligned}$$

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for all $x, y, z \in \mathcal{X}$ and all $w \in \mathcal{Y}$. Hence the mapping $Q : \mathcal{X} \to \mathcal{Y}$ satisfies (1.2). Taking $k \to \infty$ in (3.6), we get the inequality (3.3). To prove the uniqueness of the quadratic mapping Q, let us assume that there exists a quadratic mapping $Q' : \mathcal{X} \to \mathcal{Y}$ satisfying (1.2) and (3.3). We have

$$\begin{aligned} \|Q(x) - Q'(x), w\| &\leq \frac{1}{3^{2k}} \big\|Q(3^k x) - Q'(3^k x), w\big\| \\ &= \frac{2}{3^{2k}} \widetilde{\psi}(3^k x, 3^k x, 3^k x) \to 0 \quad \text{as} \quad k \to \infty \end{aligned}$$

for all $x \in \mathcal{X}$ and all $w \in \mathcal{Y}$. Therefore Q is unique.

Corollary 3.2. *Let $\mathcal{X}$ be a normed space and $\mathcal{Y}$ a 2-Banach space. Let θ, q be real numbers such that $\theta \geq 0$, $0 < q < 2$ and Suppose that a mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfies*

$$\|Df(x, y, z), w\| \leq \theta\big(\|x\|^q + \|y\|^q + \|z\|^q\big)$$

for all $x, y, z \in \mathcal{X}$ and all $w \in \mathcal{Y}$. Then there exists a unique quadratic mapping $Q : \mathcal{X} \to \mathcal{Y}$ satisfying (1.2) such that

$$\|f(x) - Q(x), w\| \leq \frac{\theta\,\|x\|^q}{3^{q-1}(3^{2-q} - 1)}$$

for all $x \in \mathcal{X}$ and all $w \in \mathcal{Y}$.

Proof. Taking $\psi(x, y, z) := \theta\big(\|x\|^q + \|y\|^q + \|z\|^q\big)$ and applying Theorem 3.1, one can obtain the result.

Corollary 3.3. *Let $\mathcal{X}$ be a normed space and $\mathcal{Y}$ a 2-Banach space. Let θ be a real number. Suppose that a mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfies*

$$\|Df(x, y, z), w\| \leq \theta$$

for all $x, y, z \in \mathcal{X}$ and all $w \in \mathcal{Y}$. Then there exists a unique quadratic mapping $Q : \mathcal{X} \to \mathcal{Y}$ satisfying (1.2) such that

$$\|f(x) - Q(x), w\| \leq \frac{\theta}{8}$$

for all $x \in \mathcal{X}$ and all $w \in \mathcal{Y}$.

Proof. Taking $\psi(x, y, z) := \theta$ and applying Theorem 3.1, one can obtain the result.

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Global dynamics of virus infection model with humoral immune response and distributed delays

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Abstract

In this paper, we investigate the global dynamics of virus infection model with humoral immune response and distributed intracellular delays. The model is 4-dimensional nonlinear delay differential equations that describe the interaction of the virus with target cells, taking into account the humoral immune system response. The model has two types of distributed time delays which describe the time needed for infection of target cell and virus replication. Lyapunov functionals are constructed to establish the global asymptotic stability of the steady states of the model. We have proven that if the basic reproduction number R_0 is less than or equal unity then the uninfected steady state is globally asymptotically stable (GAS), and if the antibody immune response reproduction number R_1 is less than or equal unity and $R_0 > 1$, then the infected steady state without immune response exists and it is GAS; if $R_1 > 1$ then the infected steady state with immune response exists and it is GAS.

Keywords: Humoral infection; Virus dynamics; Global stability; Distributed delay.
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1 Introduction

In the last decade, several mathematical models have been developed to describe the interaction of virus and target cells, such as HIV, HBV etc [1]. Mathematical modeling and model analysis of the viral infection process, estimating key parameter values, and guiding development efficient anti-viral drug therapies. Some of these models take into account the main role of immune system of human body. The immune system is described as having two “arms”: the cellular arm, which depends on T cells to mediate attacks on virally infected or cancerous cells; and the humoral arm, which depends on B cells to make antibodies to clear antigens circulating in blood and lymph. The humoral immunity is more effective than the cell-mediated immune in some diseases like in malaria infection [2].

Some of the existing mathematical models of viral infection are given by nonlinear ODEs by assuming that the infection could occur and the viruses are produced from infected target cells instantaneously, once the uninfected target cells are contacted by the virus particles (see e.g. [3]-[5], [12], [14], [16]). Other accurate models incorporate the delay between the time, the viral entry into the target cell, and the time the production of new virus particles, modeled with discrete time delay or distributed time delay using functional differential equations (see e.g. [6]-[10], [20]-[22]).

Many authors present and develop mathematical models for the humoral immunity [11]-[15], [17]-[19]. The basic model with humoral immunity response was introduced by Akiko [12],

$$\dot{x}(t) = \lambda - dx(t) - \beta x(t)v(t), \tag{1}$$
$$\dot{y}(t) = \beta x(t)v(t) - \delta y(t), \tag{2}$$
$$\dot{v}(t) = N\delta y(t) - cv(t) - qv(t)z(t), \tag{3}$$
$$\dot{z}(t) = gv(t)z(t) - \mu z(t), \tag{4}$$

where x, y, v and z are the population of the uninfected cells, infected cells, virus and B cells. λ and d are the birth rate and death rate of uninfected cells, respectively. β is the infection rate. N is the number of free virus produced during the average infected cell life span. δ is the death rate of infected cells and c is the clearance rate of the virus. g and μ are the birth rate and death rate of B cells. q is the B cells neutralize rate. This model is based on the assumption that, once the virus contacts a target cell, the cell begins producing new virus

particles. The global stability of this model was studied in [17]. In [17], [19], the viral infection models with humoral immunity were incorporated with discrete time delays.

In this paper, we incorporate two types of distributed delays into the model to account the time delay between the time that target cells are contacted by the virus particle and the time the emission of infectious (matures) virus particles. The global stability of the model is established using Lyapunov functionals, which are similar in nature to those used in [22] and [24]. We prove that the global dynamics of the model is determined by the basic reproduction number R_0 and antibody immune response reproduction number R_1. If $R_0 \leq 1$, then the uninfected steady state is globally asymptotically stable (GAS), if $R_1 \leq 1 < R_0$, then the infected steady state without humoral immune response exists and it is GAS, if $R_1 > 1$ then the infected steady state with humoral immune response exists and it is GAS.

2 Virus infection model with humoral immune response and two distributed delays

In this section we propose a mathematical model of virus infection which describes the interaction of the virus with target cells, taking into account the effect of humoral immune system.

$$\dot{x}(t) = \lambda - dx(t) - \beta x(t)v(t), \tag{5}$$

$$\dot{y}(t) = \beta \int_0^h f(\tau)e^{-m\tau}x(t-\tau)v(t-\tau)d\tau - \delta y(t), \tag{6}$$

$$\dot{v}(t) = N\delta \int_0^\omega g(\tau)e^{-n\tau}y(t-\tau)d\tau - cv(t) - qv(t)z(t), \tag{7}$$

$$\dot{z}(t) = gv(t)z(t) - \mu z(t). \tag{8}$$

All the variables and parameters of the model have the same meanings as given in (1)-(4). To account for the time lag between viral contacting a target cell and the production of new virus particles, two distributed intracellular delays are introduced. It assumed that the target cells are contacted by the virus particles at time $t-\tau$ becomes infected cells at time t, where τ is a random variable with a probability distribution $f(\tau)$ over the interval $[0, h]$ and h is limit superior of this delay. The factor $e^{-m\tau}$ account for the probability of surviving the time period of delay, where m is the death rate of infected cells but not yet virus producer cells. On the other hand, it is assumed that, a cell infected at time $t-\tau$ starts to yield new infectious virus at time t where τ is distributed according to a probability distribution $g(\tau)$ over the interval $[0, \omega]$ and ω is limit superior of this delay. The factor $e^{-n\tau}$ account for the probability of surviving the time period of delay, where n is constant. All the parameters are supposed to be positive.

The probability distribution functions $f(\tau)$ and $g(\tau)$ are assumed to satisfy $f(\tau) > 0$ and $g(\tau) > 0$, and

$$\int_0^h f(\tau)d\tau = 1, \quad \int_0^\omega g(\tau)d\tau = 1, \quad \int_0^h f(r)e^{sr}dr < \infty, \quad \int_0^\omega g(r)e^{sr}dr < \infty,$$

where s is a positive number. Then

$$0 < \int_0^h f(\tau)e^{-m\tau}d\tau \leq 1, \quad 0 < \int_0^\omega g(\tau)e^{-n\tau}d\tau \leq 1 \text{for } m \geq 0, \ n \geq 0.$$

The initial conditions for system (5)-(8) take the form

$$\begin{aligned}
&x(\theta) = \varphi_1(\theta), \ y(\theta) = \phi_2(\theta), \\
&v(\theta) = \varphi_3(\theta), z(\theta) = \varphi_4(\theta), \\
&\varphi_j(\theta) \geq 0, \ \theta \in [-\rho, 0), \quad j = 1, ..., 4, \\
&\varphi_j(0) > 0, \ j = 1, ..., 4,
\end{aligned} \tag{9}$$

where $\rho = \max\{h, \omega\}$, $(\varphi_1(\theta), \varphi_2(\theta), ..., \varphi_4(\theta)) \in C([-\rho, 0], \mathbb{R}_+^4)$, where $C([-\rho, 0], \mathbb{R}_+^4)$ is the Banach space of continuous functions mapping the interval $[-\rho, 0]$ into $\mathbb{R}_+^4$. By the fundamental theory of functional differential equations [23], system (5)-(8) has a unique solution satisfying the initial conditions (9).

2.1 Non-negativity and boundedness of solutions

In the following, we establish the non-negativity and boundedness of solutions of (5)-(8) with initial conditions (9).

Proposition 1. Let $(x(t), y(t), v(t), z(t))$ be any solution of (5)-(8) satisfying the initial conditions (9), then $x(t), y(t), v(t)$ and $z(t)$ are all non-negative for $t \geq 0$ and ultimately bounded.

Proof. First, we prove that $x(t) > 0$, for all $t \geq 0$. Assume that $x(t)$ lose its non-negativity on some local existence interval $[0, \ell]$ for some constant ℓ and let $t_1 \in [0, \ell]$ be such that $x(t_1) = 0$. From Eq. (5) we have $\dot{x}(t_1) = \lambda > 0$. Hence $x(t) > 0$ for some $t \in (t_1, t_1 + \varepsilon)$, where $\varepsilon > 0$ is sufficiently small. This leads to contradiction and hence $x(t) > 0$, for all $t \geq 0$. Now from Eqs. (6), (7) and (8) we have

$$y(t) = y(0)e^{-\delta t} + \beta \int_0^t e^{-\delta(t-\eta)} \int_0^h f(\tau)e^{-m\tau}x(\eta-\tau)v(\eta-\tau)d\tau d\eta,$$

$$v(t) = v(0)e^{-\int_0^t (c+qz(\xi))d\xi} + N\delta \int_0^t e^{-\int_\eta^t (c+qz(\xi))d\xi} \int_0^\omega g(\tau)e^{-n\tau}y(\eta-\tau)d\tau d\eta,$$

$$z(t) = z(0)e^{-\int_0^t (\mu-gv(\xi))d\xi},$$

confirming that $y(t) \geq 0$, $v(t) \geq 0$ and $z(t) \geq 0$ for all $t \in [0, \rho]$. By a recursive argument, we obtain $y(t) \geq 0, v(t) \geq 0$ and $z(t) \geq 0$ for all $t \geq 0$.

Next we show the boundedness of the solutions. From Eq. (5) we have $\dot{x}(t) \leq \lambda - dx(t)$. This implies $\limsup_{t\to\infty} x(t) \leq \frac{\lambda}{d}$.

Let $X(t) = \int_0^h f(\tau)e^{-m\tau}x(t-\tau)d\tau + y(t)$, then

$$\begin{aligned}
\dot{X}(t) &= \int_0^h f(\tau)e^{-m\tau}\left(\lambda - dx(t-\tau) - \beta x(t-\tau)v(t-\tau)\right)d\tau \\
&+ \int_0^h f(\tau)e^{-m\tau}\beta x(t-\tau)v(t-\tau)d\tau - \delta y(t), \\
&= \lambda \int_0^h f(\tau)e^{-m\tau}d\tau - d\int_0^h f(\tau)e^{-m\tau}x(t-\tau)d\tau - \delta y(t) \\
&\leq \lambda \int_0^h f(\tau)e^{-m\tau}d\tau - \sigma_1\left[\int_0^h f(\tau)e^{-m\tau}x(t-\tau)d\tau + y(t)\right] \\
&= \lambda \int_0^h f(\tau)e^{-m\tau}d\tau - \sigma_1 X(t) \leq \lambda - \sigma_1 X(t),
\end{aligned}$$

where $\sigma_1 = \min\{d, \delta\}$. Hence $\limsup_{t\to\infty} X(t) \leq L_1$, where $L_1 = \dfrac{\lambda}{\sigma_1}$. Since $\int_0^h f(\tau)e^{-m\tau}x(t-\tau)d\tau > 0$ then $\limsup_{t\to\infty} y(t) \leq L_1$. On the other hand,

Let $Z(t) = v(t) + \frac{q}{g}z(t)$, then

$$\begin{aligned}
\dot{Z}(t) &= N\delta \int_0^\omega g(\tau)e^{-n\tau}y(t-\tau)d\tau - cv(t) - \frac{q\mu}{g}z(t) \\
&\leq N\delta L_1 \int_0^\omega g(\tau)e^{-n\tau}d\tau - \sigma_2(v(t) + \frac{q}{g}z(t)) \\
&= N\delta L_1 \int_0^\omega g(\tau)e^{-n\tau}d\tau - \sigma_2 Z(t) \leq N\delta L_1 - \sigma_2 Z(t),
\end{aligned}$$

where $\sigma_2 = \min\{c, \mu\}$. Hence $\limsup_{t\to\infty} Z(t) \leq L_2$, where $L_2 = \dfrac{N\delta L_1}{\sigma_2}$. Since $v(t) \geq 0$ and $y(t) \geq 0$ then $\limsup_{t\to\infty} v(t) \leq L_2$ and $\limsup_{t\to\infty} z(t) \leq L_2$.

Therefore, $x(t), y(t), v(t)$ and $z(t)$ are ultimately bounded.

2.2 Steady states

We define the basic reproduction number for system (5)-(8) as

$$R_0 = \frac{NFG\beta x_0}{c},$$

and the antibody immune response reproduction number

$$R_1 = \frac{R_0}{1 + \dfrac{\beta\mu}{dg}}.$$

Clearly $R_1 < R_0$. It is clear that, system (5)-(8) has an uninfected steady state $E_0 = (x_0, 0, 0, 0)$, where $x_0 = \dfrac{\lambda}{d}$. In addition to E_0, the system can has an infected steady state without immune response $E_1(x_1, y_1, v_1, 0)$ and infected steady state with immune response $E_2(x_2, y_2, v_2, z_2)$ where

$$\begin{aligned}
&x_1 = \frac{x_0}{R_0}, \qquad y_1 = \frac{c}{N\delta G}v_1, \qquad v_1 = \frac{d}{\beta}(R_0 - 1),\\
&x_2 = \frac{\lambda}{d + \dfrac{\beta\mu}{g}}, \quad y_2 = \frac{\beta F\lambda\mu}{\delta(dg + \beta\mu)}, \quad v_2 = \frac{\mu}{g}, \quad z_2 = \frac{c}{q}(R_1 - 1),\\
&F = \int_0^h f(\tau)e^{-m\tau}d\tau, \qquad G = \int_0^\omega g(\tau)e^{-n\tau}d\tau.
\end{aligned}$$

From above we have the following:

(i) If $R_0 > 1$, then there exists a positive steady state $E_1(x_1, y_1, v_1, 0)$.

(ii) If $R_1 > 1$, then there exists a positive steady state $E_2(x_2, y_2, v_2, z_2)$.

2.3 Global stability

In this section, we prove the global stability of the steady states of system (5)-(8) employing the method of Lyapunov functional which is used in [24] for SIR epidemic model with distributed delay. Next we shall use the following notation: $r = r(t)$, for any $r \in \{x, y, v, z\}$. We also define a function $H : (0, \infty) \to [0, \infty)$ as $H(r) = r - 1 - \ln r$. It is clear that $H(r) \geq 0$ for any $r > 0$ and H has the global minimum $H(1) = 0$.

Theorem 1. If $R_0 \leq 1$, then E_0 is GAS.

Proof. Define a Lyapunov functional W_0 as follows:

$$\begin{aligned}
W_0 = NFG &\left[x_0 H\left(\frac{x}{x_0}\right) + \frac{1}{F}y + \frac{\beta}{F}\int_0^h f(\tau)e^{-m\tau}\int_0^\tau x(t-\theta)v(t-\theta)d\theta d\tau\right.\\
&\left. + \frac{\delta}{FG}\int_0^\omega g(\tau)e^{-n\tau}\int_0^\tau y(t-\theta)d\theta d\tau\right] + v + \frac{q}{g}z. \qquad (10)
\end{aligned}$$

The time derivative of W_0 along the trajectories of (5)-(8) satisfies

$$\begin{aligned}
\frac{dW_0}{dt} &= NFG\Bigg[\left(1-\frac{x_0}{x}\right)(\lambda - dx - \beta xv) + \frac{\beta}{F}\int_0^h f(\tau)e^{-m\tau}x(t-\tau)v(t-\tau)d\tau - \frac{\delta}{F}y \\
&+\frac{\beta}{F}\int_0^h f(\tau)e^{-m\tau}\left(xv - x(t-\tau)v(t-\tau)\right)d\tau + \frac{\delta}{FG}\int_0^\omega g(\tau)e^{-n\tau}(y - y(t-\tau))d\tau\Bigg] \\
&+ N\delta\int_0^\omega g(\tau)e^{-n\tau}y(t-\tau)d\tau - cv - qvz + qvz - \frac{q\mu}{g}z, \\
&= NFG\left[-d\frac{(x-x_0)^2}{x} + \beta x_0 v\right] - cv - \frac{q\mu}{g}z \\
&= -NFGd\frac{(x-x_0)^2}{x} + c(R_0-1)v - \frac{q\mu}{g}z.
\end{aligned} \tag{11}$$

If $R_0 \le 1$ then $\frac{dW_0}{dt} \le 0$ for all $x, v, z > 0$. By Theorem 5.3.1 in [23], the solutions of system (5)-(8) limit to M, the largest invariant subset of $\left\{\frac{dW_0}{dt} = 0\right\}$. Clearly, it follows from (11) that $\frac{dW_0}{dt} = 0$ if and only if $x = x_0$, $v = 0, z = 0$. Noting that M is invariant, for each element of M we have $v = 0$, and $z = 0$, then $\dot{v} = 0$. From Eq. (7) we drive that

$$0 = \dot{v} = N\delta\int_0^\omega g(\tau)e^{-n\tau}y(t-\tau)d\tau.$$

This yields $y = 0$. Hence $\frac{dW_0}{dt} = 0$ if and only if $x = x_0$, $y = 0$, $v = 0$ and $z = 0$. From LaSalle's Invariance Principle, E_0 is GAS.

Theorem 2. If $R_1 \le 1 < R_0$, then E_1 is GAS.

Proof. We construct the following Lyapunov functional

$$\begin{aligned}
W_1 &= NFG\Bigg[x_1 H\left(\frac{x}{x_1}\right) + \frac{1}{F}y_1 H\left(\frac{y}{y_1}\right) + \frac{1}{F}\beta x_1 v_1\int_0^h f(\tau)e^{-m\tau}\int_0^\tau H\left(\frac{x(t-\theta)v(t-\theta)}{x_1 v_1}\right)d\theta d\tau \\
&+\frac{\delta y_1}{FG}\int_0^\omega g(\tau)e^{-n\tau}\int_0^\tau H\left(\frac{y(t-\theta)}{y_1}\right)d\theta d\tau\Bigg] + v_1 H\left(\frac{v}{v_1}\right) + \frac{q}{g}z.
\end{aligned} \tag{12}$$

The time derivative of W_1 along the trajectories of (5)-(8) is given by

$$\begin{aligned}
\frac{dW_1}{dt} &= NFG\Bigg[\left(1-\frac{x_1}{x}\right)(\lambda - dx - \beta xv) + \frac{1}{F}\left(1-\frac{y_1}{y}\right)\left(\beta\int_0^h f(\tau)e^{-m\tau}x(t-\tau)v(t-\tau)d\tau - \delta y\right) \\
&+ \frac{\beta}{F}\int_0^h f(\tau)e^{-m\tau}\left(xv - x(t-\tau)v(t-\tau) + x_1 v_1 \ln\frac{x(t-\tau)v(t-\tau)}{xv}\right)d\tau \\
&+\frac{\delta}{FG}\int_0^\omega g(\tau)e^{-n\tau}\left(y - y(t-\tau) + y_1\ln\frac{y(t-\tau)}{y}\right)d\tau\Bigg] \\
&+ \left(1-\frac{v_1}{v}\right)\left(N\delta\int_0^\omega g(\tau)e^{-n\tau}y(t-\tau)d\tau - cv - qvz\right) + qvz - \frac{q\mu}{g}z.
\end{aligned} \tag{13}$$

Using the steady state conditions for E_1:

$$\lambda = dx_1 + \beta x_1 v_1, \quad F\beta x_1 v_1 = \delta y_1, \quad cv_1 = N\delta G y_1,$$

we have

$$\begin{aligned}\frac{dW_1}{dt} &= NFG\left[-d\frac{(x-x_1)^2}{x} + \beta x_1 v_1 - \beta x_1 v_1 \frac{x_1}{x} + \beta x_1 v - \frac{\beta x_1 v_1}{F}\int_0^h f(\tau)e^{-m\tau}\frac{y_1 x(t-\tau)v(t-\tau)}{y x_1 v_1}d\tau + \frac{\delta}{F}y_1\right.\\ &\left.+\frac{\beta x_1 v_1}{F}\int_0^h f(\tau)e^{-m\tau}\ln\frac{x(t-\tau)v(t-\tau)}{xv}d\tau + \frac{\delta y_1}{FG}\int_0^\omega g(\tau)e^{-n\tau}\ln\frac{y(t-\tau)}{y}d\tau\right]\\ &- N\delta y_1\int_0^\omega g(\tau)e^{-n\tau}\frac{v_1 y(t-\tau)}{v y_1}d\tau - cv + cv_1 + qv_1 z - \frac{q\mu}{g}z. \end{aligned} \tag{14}$$

Using the following equalities:

$$\ln\left(\frac{x(t-\tau)v(t-\tau)}{xv}\right) = \ln\left(\frac{y_1 x(t-\tau)v(t-\tau)}{y x_1 v_1}\right) + \ln\left(\frac{x_1}{x}\right) + \ln\left(\frac{v_1 y}{v y_1}\right),$$
$$\ln\left(\frac{y(t-\tau)}{y}\right) = \ln\left(\frac{v y_1}{v_1 y}\right) + \ln\left(\frac{v_1 y(t-\tau)}{v y_1}\right),$$

we obtain

$$\begin{aligned}\frac{dW_1}{dt} &= NFG\left[-d\frac{(x-x_1)^2}{x} - \frac{\delta y_1}{F}\left(\frac{x_1}{x} - 1 - \ln\frac{x_1}{x}\right)\right.\\ &- \frac{\delta y_1}{F^2}\int_0^h f(\tau)e^{-m\tau}\left(\frac{y_1 x(t-\tau)v(t-\tau)}{y x_1 v_1} - 1 - \ln\frac{y_1 x(t-\tau)v(t-\tau)}{y x_1 v_1}\right)d\tau\\ &\left.-\frac{\delta y_1}{FG}\int_0^\omega g(\tau)e^{-n\tau}\left(\frac{v_1 y(t-\tau)}{v y_1} - 1 - \ln\frac{v_1 y(t-\tau)}{v y_1}\right)d\tau\right] + qz\left(v_1 - \frac{\mu}{g}\right),\\ &= NFG\left[-d\frac{(x-x_1)^2}{x} - \frac{\delta y_1}{F}H\left(\frac{x_1}{x}\right) - \frac{\delta y_1}{F^2}\int_0^h f(\tau)e^{-m\tau}H\left(\frac{y_1 x(t-\tau)v(t-\tau)}{y x_1 v_1}\right)d\tau\right.\\ &\left.-\frac{\delta y_1}{FG}\int_0^\omega g(\tau)e^{-n\tau}H\left(\frac{v_1 y(t-\tau)}{v y_1}\right)d\tau\right] + qz\left(v_1 - \frac{\mu}{g}\right).\end{aligned}$$

Also, we have

$$\begin{aligned}qz\left(v_1 - \frac{\mu}{g}\right) &= qz\left[\frac{d}{\beta}(R_0 - 1) - \frac{\mu}{g}\right] = qz\frac{d}{\beta}\left[R_0 - 1 - \frac{\beta\mu}{dg}\right]\\ &= qz\frac{d}{\beta}\left(1 + \frac{\beta\mu}{dg}\right)\left[\frac{R_0}{1 + \frac{\beta\mu}{dg}} - 1\right] = qz\frac{d}{\beta}\left(1 + \frac{\beta\mu}{dg}\right)[R_1 - 1],\end{aligned}$$

then if $R_0 > 1$ then x_1, y_1 and $v_1 > 0$ and hence, if $R_1 \le 1$ then $\frac{dW_1}{dt} \le 0$ for all $x, y, v > 0$. By Theorem 5.3.1 in [23], the solutions of system (5)-(8) limit to M, the largest invariant subset of $\left\{\frac{dW_1}{dt} = 0\right\}$. It can be seen that $\frac{dW_1}{dt} = 0$ if and only if $x = x_1$, $z = 0$ and $H = 0$ i.e.

$$\frac{y_1 x(t-\tau)v(t-\tau)}{y x_1 v_1} = \frac{v_1 y(t-\tau)}{v y_1} = 1 \text{ for almost all } \tau \in [0, \rho].$$

From Eq. (5), if $x = x_1$ then $\dot{x} = 0$ and $\lambda - dx_1 - \beta x_1 v = 0$, so $v = v_1$ and then $y = y_1$. It follows that $\frac{dW_1}{dt}$ equal to zero at E_1. LaSalle's Invariance Principle implies global stability of E_1.

Theorem 3. If $R_1 > 1$, then E_2 is GAS.

Proof. We construct the following Lyapunov functional

$$W_2 = NFG\left[x_2 H\left(\frac{x}{x_2}\right) + \frac{1}{F}y_2 H\left(\frac{y}{y_2}\right) + \frac{1}{F}\beta x_2 v_2 \int_0^h f(\tau)e^{-m\tau}\int_0^\tau H\left(\frac{x(t-\theta)v(t-\theta)}{x_2 v_2}\right) d\theta d\tau \right.$$
$$\left. + \frac{\delta y_2}{FG}\int_0^\omega g(\tau)e^{-n\tau}\int_0^\tau H\left(\frac{y(t-\theta)}{y_2}\right) d\theta d\tau\right] + v_2 H\left(\frac{v}{v_2}\right) + \frac{q}{g}z_2 H\left(\frac{z}{z_2}\right). \tag{15}$$

The time derivative of W_2 along the trajectories of (5)-(8)

$$\frac{dW_2}{dt} = NFG\left[\left(1-\frac{x_2}{x}\right)(\lambda - dx - \beta xv) + \frac{1}{F}\left(1-\frac{y_2}{y}\right)\left(\beta\int_0^h f(\tau)e^{-m\tau}x(t-\tau)v(t-\tau)d\tau - \delta y\right)\right.$$
$$+\frac{\beta}{F}\int_0^h f(\tau)e^{-m\tau}\left(xv - x(t-\tau)v(t-\tau) + x_2 v_2 \ln\frac{x(t-\tau)v(t-\tau)}{xv}\right)d\tau$$
$$\left.+\frac{\delta}{FG}\int_0^\omega g(\tau)e^{-n\tau}\left(y - y(t-\tau) + y_2\ln\frac{y(t-\tau)}{y}\right)d\tau\right]$$
$$+\left(1-\frac{v_2}{v}\right)\left(N\delta\int_0^\omega g(\tau)e^{-n\tau}y(t-\tau)d\tau - cv - qvz\right) + \left(1-\frac{z_2}{z}\right)\left(qvz - \frac{q\mu}{g}z\right). \tag{16}$$

Using the steady state conditions for E_2:

$$\lambda = dx_2 + \beta x_2 v_2, \quad cv_2 = \ NFG(\frac{\delta}{F}y_2) - qv_2 z_2, \ \ \mu = gv_2,$$

we obtain

$$\frac{dW_2}{dt} = NFG\left[-d\frac{(x-x_2)^2}{x} + \beta x_2 v_2 - \beta x_2 v_2\frac{x_2}{x} + \beta x_2 v\right.$$
$$-\frac{\beta x_2 v_2}{F}\int_0^h f(\tau)e^{-m\tau}\frac{y_2 x(t-\tau)v(t-\tau)}{y x_2 v_2}d\tau + \frac{\delta}{F}y_2$$
$$\left.+\frac{\beta x_2 v_2}{F}\int_0^h f(\tau)e^{-m\tau}\ln\frac{x(t-\tau)v(t-\tau)}{xv}d\tau + \frac{\delta y_2}{FG}\int_0^\omega g(\tau)e^{-n\tau}\ln\frac{y(t-\tau)}{y}d\tau\right]$$
$$+ N\delta y_2\int_0^\omega g(\tau)e^{-n\tau}\frac{v_2 y(t-\tau)}{v y_2}d\tau - cv + cv_2 + qv_2 z - qvz_2 - \frac{q\mu}{g}z + \frac{q\mu}{g}z_2. \tag{17}$$

Using the following equalities:

$$cv = cv_2\frac{v}{v_2} = NFG(\frac{\delta}{F}y_2\frac{v}{v_2}) - qvz_2,$$
$$\ln\left(\frac{x(t-\tau)v(t-\tau)}{xv}\right) = \ln\left(\frac{y_2 x(t-\tau)v(t-\tau)}{y x_2 v_2}\right) + \ln\left(\frac{x_2}{x}\right) + \ln\left(\frac{v_2 y}{v y_2}\right),$$
$$\ln\left(\frac{y(t-\tau)}{y}\right) = \ln\left(\frac{v y_2}{v_2 y}\right) + \ln\left(\frac{v_2 y(t-\tau)}{v y_2}\right),$$

we obtain

$$\begin{aligned}\frac{dW_2}{dt} &= NFG\left[-d\frac{(x-x_2)^2}{x} - \frac{\delta y_2}{F}\left(\frac{x_2}{x} - 1 - \ln\frac{x_2}{x}\right)\right.\\ &- \frac{\delta y_2}{F^2}\int_0^h f(\tau)e^{-m\tau}\left(\frac{y_2 x(t-\tau)v(t-\tau)}{yx_2v_2} - 1 - \ln\frac{y_2x(t-\tau)v(t-\tau)}{yx_2v_2}\right)d\tau\\ &\left.-\frac{\delta y_2}{FG}\int_0^\omega g(\tau)e^{-n\tau}\left(\frac{v_2y(t-\tau)}{vy_2} - 1 - \ln\frac{v_2y(t-\tau)}{vy_2}\right)d\tau\right].\\ &= NFG\left[-d\frac{(x-x_2)^2}{x} - \frac{\delta y_2}{F}H\left(\frac{x_2}{x}\right) - \frac{\delta y_2}{F^2}\int_0^h f(\tau)e^{-m\tau}H\left(\frac{y_2x(t-\tau)v(t-\tau)}{yx_2v_2}\right)d\tau\right.\\ &\left.-\frac{\delta y_2}{FG}\int_0^\omega g(\tau)e^{-n\tau}H\left(\frac{v_2y(t-\tau)}{vy_2}\right)d\tau\right].\end{aligned}$$

Thus, if $R_1 > 1$ then x_2, y_2, v_2 and $z_2 > 0$, and $\frac{dW_2}{dt} \leq 0$. By Theorem 5.3.1 in [23], the solutions of system (5)-(8) limit to M, the largest invariant subset of $\left\{\frac{dW_2}{dt} = 0\right\}$. It can be seen that $\frac{dW_2}{dt} = 0$ if and only if $x = x_2$, and $H = 0$ i.e.

$$\frac{y_2x(t-\tau)v(t-\tau)}{yx_2v_2} = \frac{v_2y(t-\tau)}{vy_2} = 1 \text{ for almost all } \tau \in [0,\rho].$$

From Eq. (5), if $x = x_2$ then $\dot{x} = 0$ and $\lambda - dx_2 - \beta x_2 v = 0$, so $v = v_2$ and then $y = y_2$, and hence $\frac{dW_2}{dt}$ equal to zero at E_2. LaSalle's Invariance Principle implies global stability of E_2.

3 Conclusion

In this paper, we have proposed an virus infection model which describe the interaction of the virus with target cell taking into account the humoral immune response. Two types of distributed time delays describing time needed for infection of target cell and virus replication have been incorporated into the model. Using the method of Lyapunov functional, we establish that the global dynamics are determined be two threshold parameters R_0 and R_1. The basic reproduction number viral infection R_0 determines whether a chronic infection can be established, and the basic reproduction number R_1 for B cells response determines whether a persistent B cells response can be established. We have proven that if $R_0 \leq 1$, the uninfected steady state E_0 is GAS, and the viruses are cleared. If $R_1 \leq 1 < R_0$, the infected steady without humoral immune response E_1 is GAS, and the infection becomes chronic but with no persistent B cells response. If $R_1 > 1$, the infected steady state immune response E_2 is GAS, and the infection is chronic with persistent B cells response.

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A fast and robust algorithm for image restoration with periodic boundary conditions

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Abstract

A new Tikhonov regularization method of Fuhry and Reichel [*A new Tikhonov regularization method, Numerical Algorithms*, 59:433-445, 2011] exhibits the excellent properties for ill-posed problems, but it can only deal with small or moderate size problems because of the expensive computation of singular value decomposition (SVD). In this paper, we extend the above new Tikhonov regularization method to solve large-scale problems, e.g., image restoration problem with periodic boundary conditions, and realize this extending by applying Fast Fourier Transformation (FFT) algorithm to the spectral decomposition of the block circulant with circulant blocks (BCCB) matrices. Experimental results confirm the superiority of our new method.

Key words: Periodic boundary conditions; FFT algorithm; Tikhonov regularization method; Image restoration

1 Introduction

The Fredholm integral equation of the first kind which arises from many image or signal restoration problems is formulated as follows

$$\int_a^b \kappa(s,t) f(t) dt = g(s), \tag{1}$$

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where $\kappa(s,t)$ is integral kernel and $g(s)$ is obtained by the known $\kappa(s,t)$ and $f(t)$. We can get the following linear system by discretization of integral equation (1),

$$Ax_{true} = b_{true}, \tag{2}$$

where $A \in \mathcal{R}^{m\times n}$ is blurring matrix, for simple notation, we consider $m \geq n$ and $x_{true} \in \mathcal{R}^n$ represents original signal with noise-free, blurred signal $b_{true} \in \mathcal{R}^m$ is formulated by blurring matrix A acting on original signal x_{true}.

Random noise $e \in \mathcal{R}^m$ is added to the right side of (2), so the final linear system is as follows

$$Ax = b = b_{true} + e, \tag{3}$$

where $x \in \mathcal{R}^n$ is an approximate solution of x_{true}, but it is just inaccessible to the true solution x_{true} generally. Our goal is to utilize an applicable method to make the relative error between x and x_{true} minimum. Typically, this is a large-scale ill-posed problem.

Tikhonov regularization methods are promising ways for ill-posed problems (see, e.g., [1, 2]), the general form is as follows

$$\min_{x\in R^n}\{\| Ax - b \|_2^2 + \| L_\lambda x \|_2^2\}, \tag{4}$$

where scalar $\lambda > 0$ is called regularization parameter and L_λ is regularization matrix. The regularization matrix is generally λI, where I represents identity matrix. A closely related Tikhonov regularization approach [3] by Fuhry and Reichel showed a novel construction of the regularization matrix, that is $L_\lambda = D_\lambda V^T$, where V^T is an unitary matrix and D_λ is a diagonal matrix containing the regularization parameter and some singular values. The numerical and visual experiments demonstrated that the new Tikhonov regularization method [3] is an excellent method for small or moderate size problems. However, the above Tikhonov regularization method is based on SVD which is expensive consuming for large-scale problems.

Since the fast algorithm such as FFT algorithm are good at doing spectral decomposition of structure matrices (see, e.g., [2, 4]), we can deal with the above large-scale problems using this property. We intentionally gain the BCCB matrices as the blurring operators A by setting circularly symmetry point spread functions (PSF) and assuming periodic boundary conditions (see other boundary conditions in [5, 6, 7], where the reference [6] showed a fast algorithm for deblurring models with Neumann boundary conditions, and [7] proposed a note on antireflective boundary conditions and fast deblurring models). Then we can exploit FFT algorithm (e.g., [8, 9]) to get the eigenvalues of BCCB matrices fast. FFT is an efficient algorithm which is widely used in many fields such as image filtering, image saving, image enhancement and image restoration and so on. Zhu

et al. [10] introduced that FFT algorithm can be used for solving compressed sensing to accelerate the computing process. Li et al. [11] showed that FFT is an effective method in signal sparse decomposition. Since Matching Pursuit (MP) adaptively decomposes signals in the redundant of dictionary to achieve some sparse representations, and it is very time consuming, FFT-based MP implementation runs significantly faster than greedy MP implementation. Furthermore, Hu et al. [12] showed that FFT can also be used in image compression. The authors adopted Radix-4 FFT to realize the limit distortion for image coding, and to discuss the feasibility and advantage of Fourier transform for image compression. Using Radix-4 FFT can reduce data storage, computing complexity and time-consuming.

The contributions of the paper are as follows: firstly, motivated by [3], we extend the new Tikhonov regularization method to solve the large-scale ill-posed image restoration problem. Secondly, we exploit FFT algorithm to fast spectral decomposition of the BCCB matrices. Finally, we test several kinds of blurs and noises to show the robustness of our algorithms. Experimental results indicate the advantages of the proposed method.

The organization of this paper is given as follows. Section 2 is mainly a recall of the new Tikhonov regularization method proposed by Fuhry and Reichel. Section 3 exhibits our method based on the FFT algorithm. Computational results will be shown in section 4. Finally, section 5 shows a conclusion about our method.

2 New Tikhonov regularization method

For completeness, we include in this section the known new Tikhonov regularization method [3] applied to the ill-posed problem (4).

Tikhonov regularization method is a popular and classical method for ill-posed problems, the general form is to solve the following least squares problem

$$\min_{x\in R^n}\{\| Ax-b \|_2^2 + \| L_\lambda x \|_2^2\}, \tag{5}$$

where L_λ is the regularization matrix and scalar $\lambda > 0$ is called the regularization parameter. In general, the regularization matrix L_λ is chosen to be λI where I is the identity matrix, and the resulting method is called standard Tikhonov regularization method. Furthermore, the finite differential operators are also used when the desired solution x has some particular properties (see [13, 14, 15, 16]). The least squares problem (5) is equivalent to the following normal equation

$$(A^TA + L_\lambda^T L_\lambda)x = A^Tb. \tag{6}$$

A closely related approach with a novel regularization matrix has been pro-

posed by Fuhry and Reichel [3]. We exploit singular value decomposition

$$A = U\Sigma V^T,$$

where $U \in \mathcal{R}^{m\times m}$, $V \in \mathcal{R}^{n\times n}$ are two unitary matrices to construct the new regularization matrix and $\Sigma = \text{diag}[\sigma_1, \sigma_2, \cdots, \sigma_n]$ where σ_i represents the i-th singular value of A. The new regularization matrix is presented as follows

$$L_\lambda = D_\lambda V^T, \tag{7}$$

where

$$D_\lambda^2 = \begin{pmatrix} \max(\lambda^2-\sigma_1^2, 0) & & & \\ & \max(\lambda^2-\sigma_2^2, 0) & & \\ & & \ddots & \\ & & & \max(\lambda^2-\sigma_n^2, 0) \end{pmatrix},$$

and matrix V^T is the unitary matrix from the SVD of matrix A, and σ_i is the i-th singular value of A.

In the light of the SVD of A and equations (6) and (7), we obtain the following equivalent equation

$$x = V^T(\Sigma^T\Sigma + D_\lambda^2)^{-1}\Sigma^T U^T b. \tag{8}$$

The solving of equation (8) needs the regularization parameter λ which is determined by discrepancy principle in [3].

It is easy to know that the regularization parameter λ satisfies $\sigma_{k+1} < \lambda < \sigma_k$ which the σ_k represents the k-th singular value and $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \cdots \geq \sigma_n \geq 0$. So we have

$$\Sigma^T\Sigma + D_\lambda^2 = \text{diag}[\sigma_1^2, \sigma_2^2, \cdots, \sigma_k^2, \lambda^2, \cdots, \lambda^2] \in \mathcal{R}^{n\times n}. \tag{9}$$

In order to avoid the propagation of the random noise e in (3) into the computed approximate solution x_{true}, the smallest eigenvalue of $A^TA + L_\lambda^T L_\lambda$ has to be large sufficiently. Also, since our model is minimization problem, we hope L_λ to be a small norm in order to help us decide a more accurate approximation of x. The following two properties demonstrate that the new Tikhonov method is a good one.

a. The smallest eigenvalue of the matrix $A^TA + L_\lambda^T L_\lambda$ should be λ^2 where $\lambda^2 \geq \sigma_i^2, i = k+1, k+2, \cdots, n$. Since $A^TA = V\Sigma^T U^T U\Sigma V^T = V\Sigma^T\Sigma V^T$,$L_\lambda^T L_\lambda = VD_\lambda^T D_\lambda V^T$, then $A^TA + L_\lambda^T L_\lambda = V(\Sigma^T\Sigma + D_\lambda^T D_\lambda)V^T$.

b. The regularization matrix L_λ has smaller norm than λI in Frobenius norm $||.||_F$. Since λ, σ_1 are strictly positive and $||L_\lambda||_F^2 = ||D_\lambda||_F^2 = \sum_{\sigma_j^2 \leq \lambda^2}(\lambda^2 - \sigma_j^2) < n\lambda^2 = ||\lambda I||_F^2$, then more accurate approximation of x can be reached.

4

The smallest eigenvalue of the matrix $A^T A + L_\lambda^T L_\lambda$ is equal to the smallest element of the diagonal matrix (9), i.e., λ^2 where $\sigma_{k+1}^2 < \lambda^2 < \sigma_k^2$. The corresponding Theorem 2.1 and Corollary 2.2 in [3] demonstrate the new Tikhonov regularization method indeed can achieve better balance for the above two aspects.

3 The new method combined FFT algorithm with new Tikhonov regularization method

Motivated by the idea of new Tikhonov regularization method proposed in [3], and due to the fast FFT algorithm, we use FFT algorithm to accelerate the spectral decomposition of BCCB matrices in the process of new Tikhonov regularization method. Particularly, where the BCCB matrix which is gained by imposing circularly symmetric PSF and periodic boundary conditions (see [2, 8, 9]). The detailed FFT algorithm is showed in this section.

3.1 FFT algorithm applied to BCCB matrices

It is well known that BCCB matrices which are normal matrices have the particular spectral decomposition

$$A = \mathcal{F}^* \Lambda \mathcal{F}, \tag{10}$$

where $\mathcal{F} \in \mathcal{C}^{n\times n}$ is 2D unitary discrete Fourier transform (DFT) matrix, $*$ represents conjugate transpose and the diagonal matrix $\Lambda = \mathrm{diag}[\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n]$ contains all eigenvalues of $A \in \mathcal{R}^{n\times n}$. This matrix $\mathcal{F}$ has a very convenient property which can perform fast matrix-vector multiplications without constructing $\mathcal{F}$ explicitly. In MATLAB, the function `fft2` and `ifft2` are used for matrix-vector multiplications of $\mathcal{F}$ and $\mathcal{F}^*$, respectively.

Since the implicit matrix $\mathcal{F}$ is a unitary matrix, we have the following equation according to the properties of Fourier transforms,

$$A = \mathcal{F}^* \Lambda \mathcal{F} \Rightarrow \mathcal{F} A = \Lambda \mathcal{F} \Rightarrow \mathcal{F} a_1 = \Lambda f_1 = \boldsymbol{\lambda}/\sqrt{N}, \tag{11}$$

where $\boldsymbol{\lambda} \in \mathcal{R}^{n\times 1}$ is a vector which contains all eigenvalues of A. It is well known that the first column of $\mathcal{F}$, f_1, is a vector of all ones, and the first column of A, a_1, can be gained by PSF and MATLAB function `circshift` (see [2]). We assume the matrix A^{-1} exists, so the final computing form is as follows

$$b = Ax = \mathcal{F}^* \Lambda \mathcal{F} x \Rightarrow x = A^{-1} b = \mathcal{F}^* \Lambda^{-1} \mathcal{F} b, \tag{12}$$

where b is the observed image and A is the BCCB matrix which can exploit the FFT algorithm.

3.2 The new Tikhonov regularization method using FFT algorithm (NTRF)

Tikhonov regularization method is a classical and promising method for image deblurring, but it shows disadvantages if we impose the random noise on the images. Fuhry and Reichel recently proposed a novel construction of the regularization matrix $L_\lambda = D_\lambda V^T$ introduced in section 2 (see [3]), called new Tikhonov regularization method.

Combining section 2 with section 3.1, we extend the new Tikhonov regularization method for small or moderate size problems to new Tikhonov regularization method for large-scale problems. Similar to section 2, for solving the least squares problem (5), we get the normal equation (6) easily. Differently, the regularization matrix is as follows

$$\tilde{L}_\mu = \tilde{D}_\mu \mathcal{F}, \tag{13}$$

where $\mathcal{F} \in \mathcal{C}^{n\times n}$ is the 2D unitary discrete Fourier transform (DFT) matrix and $*$ represents conjugate transpose. And regularization matrix $\tilde{D}_\mu \in \mathcal{R}^{n\times n}$ is as follows

$$\tilde{D}_\mu^2 = \begin{pmatrix} \max(\mu^2-\lambda_1^2,0) & & & \\ & \max(\mu^2-\lambda_2^2,0) & & \\ & & \ddots & \\ & & & \max(\mu^2-\lambda_n^2,0) \end{pmatrix},$$

where $\mu \in \mathcal{R}$ is also the regularization parameter just like the λ in section 2 and λ_i is the i-th eigenvalue of matrix A.

From least square problem (5), the following equation can be gained and A is a real matrix,

$$(A^*A + \tilde{L}_\mu^* \tilde{L}_\mu)x = A^*b.$$

We exploit spectral decomposition $A = \mathcal{F}^*\Lambda\mathcal{F}$ and $\tilde{L}_\mu = \tilde{D}_\mu\mathcal{F}$ to gain the following eqution

$$\mathcal{F}^*(\Lambda^*\Lambda + \tilde{D}_\mu^*\tilde{D}_\mu)\mathcal{F}x = \mathcal{F}^*\Lambda^*\mathcal{F}b.$$

Similar to equation (12), it is easy to get the following equation

$$x = \mathcal{F}^*(\Lambda^*\Lambda + \tilde{D}_\mu^*\tilde{D}_\mu)^{-1}\Lambda^*\mathcal{F}b. \tag{14}$$

The above equation (14) is our final computing scheme. The following experiments in section 4 demonstrate the equation (14) is indeed a promising way for image restoration. The following Theorem shows that if the smallest eigenvalue of $A^*A + \tilde{L}_\mu^*\tilde{L}_\mu^*\tilde{L}_\mu$ is sufficiently large, this can avoid propagation of the noise. Moreover, since our goal is to get smaller norm, the choosing of the regularization matrix $\tilde{L}_\mu^*\tilde{L}_\mu$ is proper which can help to determine the approximation solution.

Theorem 3.1 *Let $M = A^*A + \tilde{L}_\mu^*\tilde{L}_\mu, M \in R^{n\times n}$, where $A \in R^{n\times n}$ satisfies equation (10) and $\tilde{L}_\mu \in C^{n\times n}$ satisfies equation (13). Let $\mu > 0$ be the regularization parameter, then*

i) *The smallest eigenvalue of the matrix M is μ^2 where $\mu^2 \geq \lambda_i^2, i \in \bar{S}$ where index set $S = \{j|\lambda_j^2 > \mu^2, j = 1, 2, \cdots, n\}$.*

ii) *The regularization matrix $\tilde{L}_\mu$ has smaller Frobenius norm than μI, where μ here is the regularization parameter of Tikhonov model.*

Proof. i) According to the definitions of M, A, $\tilde{L}_\mu$, we have

$$M = A^*A + \tilde{L}_\mu^*\tilde{L}_\mu = \mathcal{F}^*\Lambda^*\Lambda\mathcal{F} + \mathcal{F}^*\tilde{D}_\mu^*\tilde{D}_\mu\mathcal{F} = \mathcal{F}^*(\Lambda^*\Lambda + \tilde{D}_\mu^*\tilde{D}_\mu)\mathcal{F} = \mathcal{F}^*D\mathcal{F},$$

where D is a diagonal matrix (i.e., $\Lambda^*\Lambda + \tilde{D}_\mu^*\tilde{D}_\mu$) that includes diagonal elements λ_i^2 and μ^2, $i \in S, S = \{j|\lambda_j^2 > \mu^2, j = 1, 2, \cdots, n\}$.

Due to the symmetric matrix M and unitary matrix $\mathcal{F}$, matrix M has the smallest eigenvalue μ^2.

ii) We have

$$||\tilde{L}_\mu||_F^2 = ||\tilde{D}_\mu\mathcal{F}||_F^2 = ||\tilde{D}_\mu||_F^2 = \Sigma_{\lambda_j^2<\mu^2}(\mu^2 - \lambda_j^2), j \in \bar{S},$$

and $0 < \mu^2 - \lambda_j^2 < \mu^2$,

$$||\tilde{L}_\mu||_F^2 = \Sigma_{\lambda_j^2<\mu^2}(\mu^2 - \lambda_j^2) < n\mu^2 = ||\mu I||_F^2.$$

□

The new algorithm is shown as follows:

Algorithm 1 (New Tikhonov regularization method using FFT algorithm (NTRF))

1. Compute Λ by spectral decomposition $A = \mathcal{F}^*\Lambda\mathcal{F}$ where A is a BCCB matrix.
2. Compute parameter μ where $\mu = 5\mu_{gcv}$, μ_{gcv} is obtained by GCV method.
3. Construct
$$\tilde{D}_\mu^2 = \text{diag}[\max(u^2 - \lambda_1^2, 0), \max(u^2 - \lambda_2^2, 0), \cdots, \max(u^2 - \lambda_n^2, 0)].$$
4. Directly compute
$$x = \mathcal{F}^*(\Lambda^*\Lambda + \tilde{D}_\mu^*\tilde{D}_\mu)^{-1}\Lambda^*\mathcal{F}b,$$
where $\mathcal{F}$ is not explicit, but matrix-vector multiplication $\mathcal{F}b$ and $\mathcal{F}^*b$ can be obtained by fft2(b) and ifft2(b) fastly in practical MATLAB implementation.

4 Experimental results

In this section, we present four different images *synthetic*, *cameraman*, *lena*, *einstein* in Figure 1 which are all of size 256 × 256 pixels to show the ef-

fectiveness and feasibility of our proposed method. The Tikhonov regularization method based on FFT (TRF), called `tik_ fft` in the MATLAB package HON[1] from [2], is compared with our method NTRF by imposing periodic boundary conditions. Particularly, the traditional Tikhonov regularization method based on FFT method (TRF) is different from NTRF method. Mainly due to the different construction $\tilde{D}_\mu^2$ where the diagonal matrix of TRF method is diag$[\mu_{gcv}^2, \mu_{gcv}^2, \cdots, \mu_{gcv}^2]$, and the diagonal matrix of NTRF method is diag$[\max(u^2-\lambda_1^2,0), \max(u^2-\lambda_2^2,0), \cdots, \max(u^2-\lambda_n^2,0)]$.

In the following examples, we mainly compare visual quality of restored image and the peak signal-to-noise ratio ($PSNR$) value which is defined as follows:

$$PSNR(u,v) = 10 \cdot log_{10} \frac{255^2}{\frac{1}{mn}\sum_{i,j}(u_{i,j}-v_{i,j})^2}$$

where $v_{i,j}$ and $u_{i,j}$ denote the pixel values of the restored and the original images, respectively. Mainly we have that larger $PSNR$ means better restored image.

The noise-free blurred image b_{true} is computed as $b_{true} = Ax_{true}$ (see equation (2)). The elements of the noise vector e are normally distributed with zero mean, and if we set $b = b_{true} + \alpha \cdot \|b_{true}\|_2 \cdot e$ where b_{true} is blurred signal. In this case, we say that the level of noise is α. For example, if $b = b_{true} + 0.01 \cdot ||b_{true}|| \cdot e$, the level of noise is 1%

The corresponding regularization parameters α of all examples are generated by Generalized Cross Validation (GCV) for TRF, for simplicity, five times α for NTRF due to empirical estimation. The following numerical examples are all implemented with MATLAB (R2010a) and the computer of test has 1G RAM and Intel(R) Pentium(R) D CPU @2.80GHz @2.79GHz.

Here, we consider two kinds of blur in our experiments, i.e., Gaussian blur and Moffat blur. The blurred images also are corrupted by additive noise-Gaussian noise. We not only compare the visual quality, but also compare the $PSNR$ values of TRF method and NTRF method. From the following tables and restored images, we can easily get the fair comparisons of our NTRF method and TRF method. We also get that our method is more effective and stable.

4.1 Example 1

We consider images which are corrupted by blur and noise, where the blur is 10×10 pixels Gaussian-shaped PSFs with standard deviation(σ^2) with 1, 1.5 and 2, meanwhile, two kinds of Gaussian noise level are 0.5% and 1% . Table 1 shows the results obtained by TRF method and NTRF method. From the table, we can see that NTRF method gets larger $PSNR$ values than TRF method. That

[1] www.siam.org/books/fa03.

Fig. 1: Original images.

demonstrates the better numerical results of our NTRF method. Furthermore, from the *PSNR* comparisons of the two methods, we obtain that the numerical difference of *PSNR* in our NTRF method becomes larger with the standard deviation σ^2 decreasing and Gaussian noise increasing. For example, the *PSNR* value difference of $\sigma^2 = 1$ and 1% noise between TRF and NTRF is larger than $\sigma^2 = 2$ and 0.5% noise. It demonstrates that our NTRF method behaves better under the condition of lower blur and higher noise. Figure 2 shows the images degenerated by $\sigma^2 = 1.5$ blur and 0.5% noise and the restored results by TRF method and NTRF method. Figure 3 displays the images degenerated by $\sigma^2 = 2$ blur and 1% noise and the restored images by TRF method and NTRF method. Evidently, the visual results with TRF method leave more noise (see the black region of second column in Figure 2) than our method. And our NTRF method shows the favorable denoising ability.

Table 1: Corresponding PSNR values using TRF method and NTRF method under the different Gaussian blurs and Gaussian white noises.

Examples	Variance	$\sigma^2 = 1$		$\sigma^2 = 1.5$		$\sigma^2 = 2$	
	Noise	0.5%	1%	0.5%	1%	0.5%	1%
synthetic	TRF	74.9845	72.2842	74.4708	72.7537	73.0333	71.6071
	NTRF	78.7617	77.2222	75.7437	75.0221	74.0365	73.5287
lena	TRF	79.1305	76.7826	77.3220	76.0470	75.1652	74.2478
	NTRF	81.2388	79.8669	77.4890	76.8047	75.4911	74.9820
einstein	TRF	79.8850	77.3747	78.0915	76.7371	76.0327	74.9479
	NTRF	82.3276	80.8633	78.2457	77.4736	76.1387	75.6452

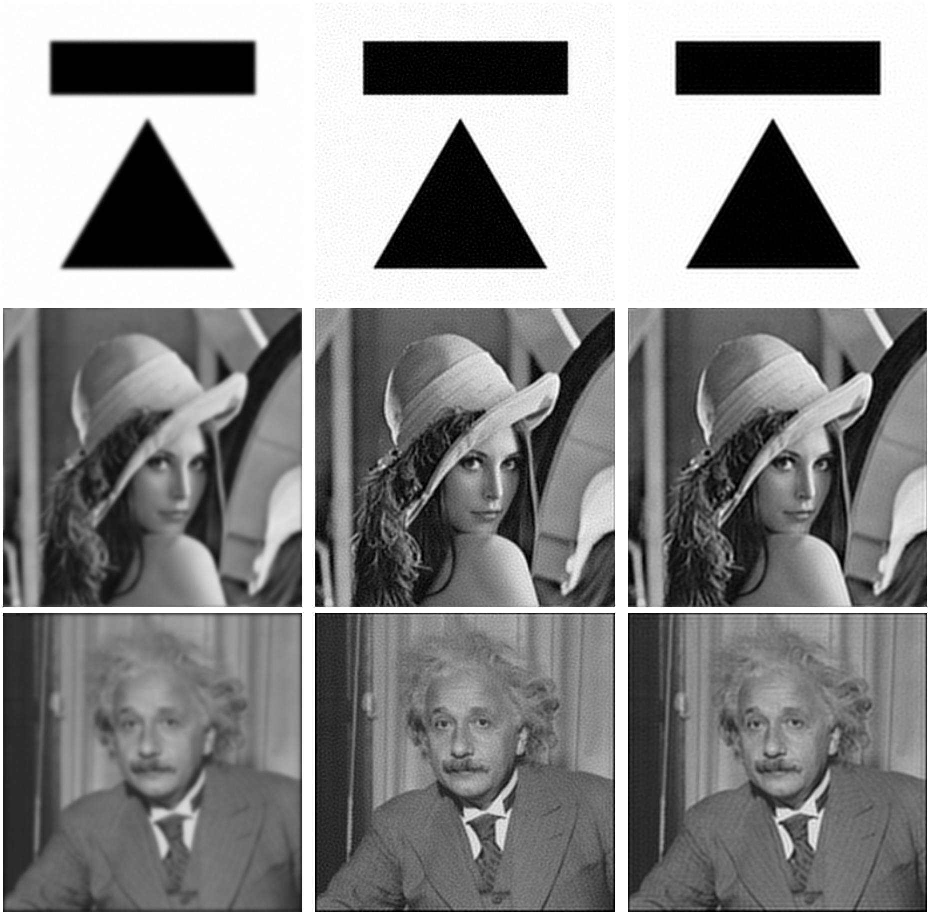

Fig. 2: First column: blurred and noisy images with Gaussian blur($\sigma^2 = 1.5$) and Gaussian noise(0.5%); Second column: restored images using TRF method ; Third column: restored images using NTRF method.

4.2 Example 2

We add Moffat blur into the images in this subsection. The detail of Moffat blur can be got from [17, 18]. And we use (x, y, z) to denote blur, where x represents the size of Moffat blur, y denotes standard deviation (σ^2) of the blur, z is a parameter. Here the noisy-blurry images have $\sigma^2 = 1$ and $\sigma^2 = 1.5$ blur and 0.5% and 1% Gaussian noise. Table 2 shows the excellence of NTRF method duo to the larger *PSNR* values using NTRF method for all test images. Similar as example 1, the numerical difference of *PSNR* in our NTRF method becomes larger with the standard deviation decreasing and Gaussian noise increasing. So we can also conclude that our NTRF method behaves better under the condition of lower blur and higher noise. Figure 4 shows the blurred and noisy images with

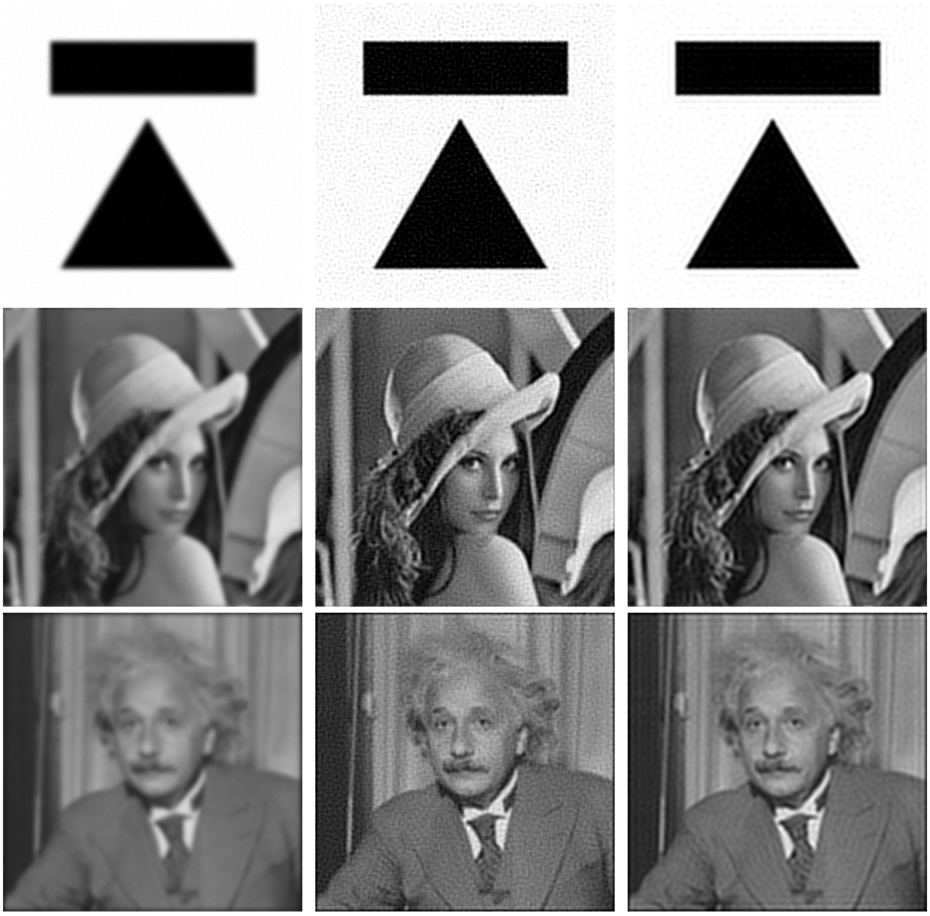

Fig. 3: First column: Blurred and noisy images with Gaussian blur ($\sigma^2 = 2$) and Gaussian noise (1%); Second column: Restored images using TRF method ; Third column: Restored images using NTRF method.

Moffat blur ($\sigma^2 = 1$) and Gaussian noise (0.5%). Figure 5 shows the blurred and noisy images with Moffat blur ($\sigma^2 = 1.5$) and Gaussian noise (0.5%). From the restored images in Figure 4 and 5, the more residual noise using TRF and less residual noise using our NTRF method demonstrate the better visual results of our method.

5 Conclusions

In this paper, we apply the new Tikhonov regularization method with FFT algorithm to generate a novel method , i.e. NTRF method, for dealing with large-scale ill-posed image restoration problems, since FFT algorithm is good at computing

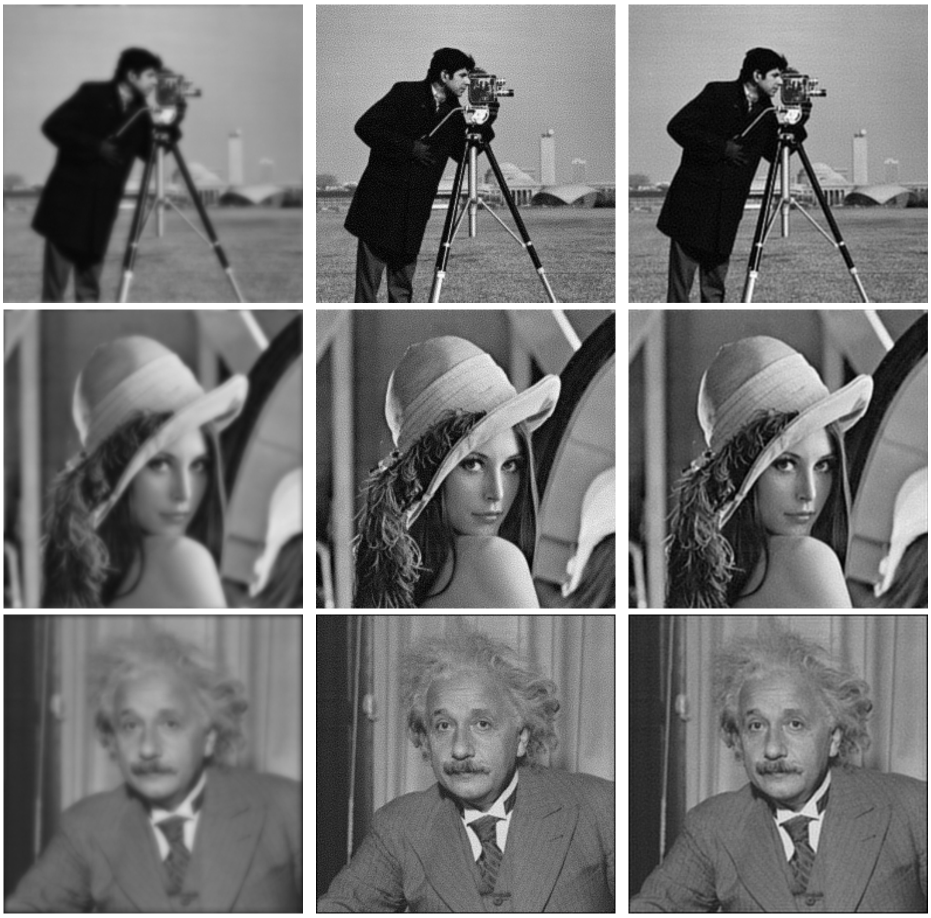

Fig. 4: First column: blurred and noisy images with Moffat blur ($\sigma^2 = 1$) and Gaussian noise (0.5%); Second column: restored images using TRF method ; Third column: restored images using NTRF method.

the spectral decomposition. Our new method retains the stability and effectiveness of the method in [3], and reduces time-consuming by using FFT algotithm. The structure of blurring matrix is a key step and should be BCCB structure that generated by circularly symmetric PSF and periodic boundary conditions. In the numerical tests, we employed different variances of different types blur and Gaussian noise to compare the effectiveness of TRF method and NTRF method, respectively. Meanwhile, the comparison results show that our NTRF method works better than TRF method under different blurs and noises. Furthermore, it is easy to discover that the difference of *PSNR* values using our NTRF method becomes bigger if we set more Gaussian noise and the smaller standard deviation of blur. It demonstrates that our NTRF method behaves better under the condition of lower blur and higher noise, and shows the favourable denoising abil-

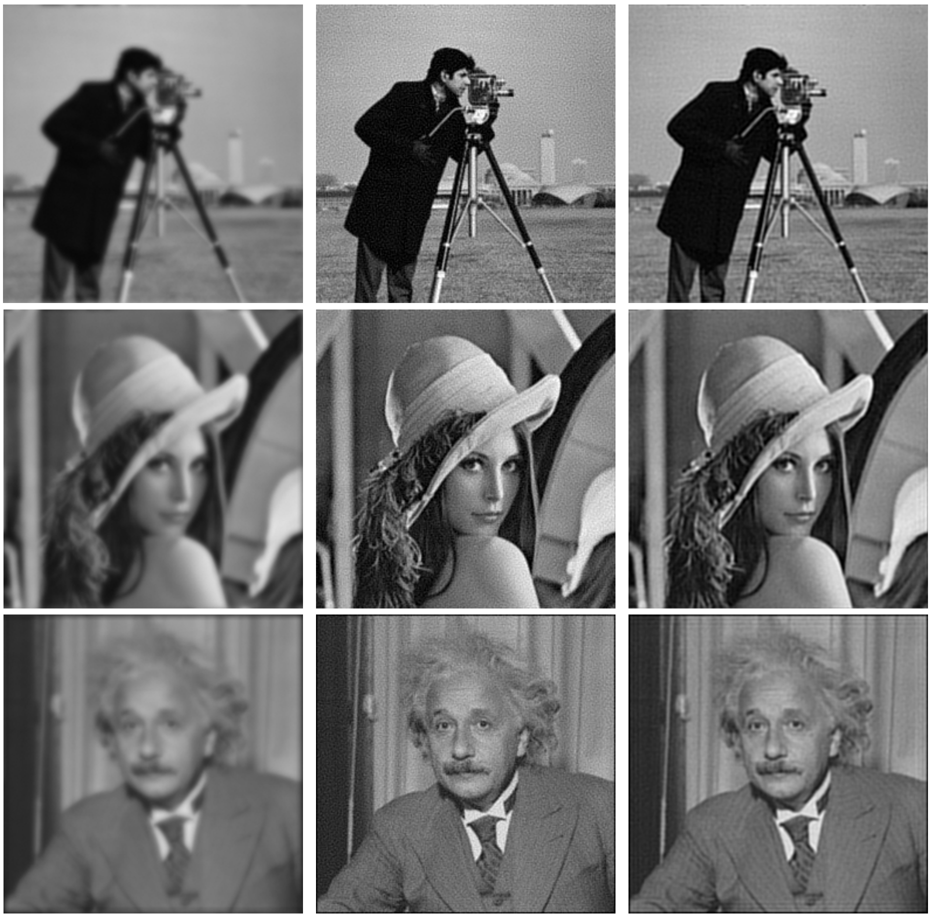

Fig. 5: First column: blurred and noisy images with Moffat blur ($\sigma^2 = 1,5$) and Gaussian noise (0.5%); Second column: restored images using TRF method ; Third column: restored images using NTRF method.

ity. Moreover, the restored images processed by TRF method contain more noise from the visual results while our NTRF method is not. Due to this, the proposed NTRF method performs better than the TRF method in the denoising process.

The reason why our new method for large-scale problems can be implemented is that we can exploit the fast algorithm of structure matrix to gain the spectral decomposition, e.g., FFT algorithm. Similar as the idea of our method, the another structure matrix which can also gain the spectral decomposition by fast algorithm discrete cosine transformation (DCT) will be gained if we impose reflexive boundary conditions and circularly symmetric PSF on the images. This work will be considered in the following paper.

Table 2: Corresponding PSNR values using TRF method and NTRF method under the different Moffat blurs and Gaussian white noises.

Examples	Variance	$\sigma^2 = 1$		$\sigma^2 = 1.5$	
	Noise	0.5%	1%	0.5%	1%
cameraman	TRF	74.6245	72.1259	74.8573	73.0050
	NTRF	78.5629	76.2789	75.5309	73.8735
lena	TRF	75.9755	73.3959	76.5289	74.5310
	NTRF	80.5133	78.3322	77.6106	75.9278
einstein	TRF	76.5182	73.9135	77.2480	75.1816
	NTRF	81.3081	79.2094	78.7393	76.7393

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Approximation by Spherical de la Vallée-Poussin Mean Operators*

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Abstract

The classical de la Vallée-Poussin mean operators have played important roles in approximation theory and harmonic analysis. This paper considers two kinds of spherical de la Vallée-Poussin mean operators. The first class is constructed by the convolution with a generalizing de la Vallée-Poussin kernel on the unit sphere, and the second one proposed in this paper is the convolution's discrete version on some regular grid points of the unit sphere. The errors, these operators approximating target functions in Sobolev space, are estimated. Furthermore, the upper bound and the convergence of the discrete type operators are discussed.

Keywords de la Vallée-Poussin mean; sphere; filter; approximation

MSC 65D10, 41A30, 41A63

1 Introduction

Considering that there exists complete orthogonal system on the unit sphere, so for a square integrable function under the Lebesgue measure, it has a Fourier-Laplace series expansion. Therefore, it is natural to study the approximation by partial summations of the spherical Fourier-Laplace series. Also, some known summation methods of the series, for example de la Vallée-Poussin mean, Cesáro summation, Riesz summation, and linear summations, especially attract researchers' attention [14]. As a particular case of linear summation, de la Vallée-Poussin mean has some remarkable features. For example, on the unit circle $\mathbb{T}$, the de la Vallée-Poussin mean is exact for polynomials of certain degree, and on the other hand it is uniformly bounded(see [12]).

Naturally, an interesting issue is to define and explore the de la Vallée-Poussin mean on the unit sphere. Berens and Li [1] introduced a class of de la Vallée-Poussin mean operators by using spherical convolution scheme, and gave the upper and lower bounds of approximation error. Filbir and Themistoclakis [3] introduced a kind of de la Vallée-Poussin mean by using the multiplication of two Fourier kernels to study the spherical scattered data approximation. In [13], Sloan and Womerersley employed the technique of hyperinterpolation developed in [9, 11] and constructed a class of discrete filtered de la Vallée-Poussin mean operators with positive weights. In [12], Sloan introduced spheres-generalizing de la Vallée-Poussin mean operators by using a filtered version of the Fourier-Laplace series partial summation, and proved that the mean operators are uniformly bounded.

In this paper we first construct a discrete form of spherical de la Vallée-Poussin mean operators by using filtered de la Vallée-Poussin kernel and cubature rule, which are similar to, to some extent, quasi-interpolation operators on the unit sphere. Then, we estimate the errors of the mean operators approximating the function in Sobolev space. Also, we discuss the upper bound and the convergence of the mean operators.

The outline of this paper is as follows. In the next section, we state some backgrounds and related notations. In Section 3, we mainly prove two lemmas. In last section, we state our results and give their proofs.

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2 Backgrounds and Notations

Let $\mathbb{S}^2 := \{x := (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$ be the two-dimensional unit sphere embedded in $\mathbb{R}^3$, and the space $L^2 := L^2(\mathbb{S}^2)$ be the Hilbert space of square integrable functions on $\mathbb{S}^2$ with the inner product $(f, g) := \int_{\mathbb{S}^2} f(x)g(x)dx$, and the norm $\|f\|_2 := \sqrt{(f, g)}$, where dx denotes the Lebesgue surface measure on $\mathbb{S}^2$. The space of continuous functions on $\mathbb{S}^2$ is denoted by $C(\mathbb{S}^2)$ with the supremum norm $\|f\| := \sup_{x\in\mathbb{S}^2} |f(x)|$.

For integer $l \geq 0$, the restriction to $\mathbb{S}^2$ of a homogeneous harmonic polynomial of degree l is called a spherical harmonic with degree l. The class of all spherical harmonics with degree l is denoted by $\mathbb{H}_l$, and the class of all spherical harmonics with degree $l \leq n$ is denoted by Π_n. Of course, we have $\Pi_n = \oplus_{l=0}^{n}\mathbb{H}_l$. The dimension of $\mathbb{H}_l$ is given by $d_l := 2l+1$ for $l \geq 1$, and $d_l := 2l+1$ for $l = 0$. For any $l \in \mathbb{N}_0$, the set $\{Y_{l,k} : k = 1, 2, \ldots, d_l\}$ denotes a real L^2 orthogonal basis of $\mathbb{H}_l$. The addition formula establishes a connection between the spherical harmonics with degree k and the Legendre polynomial P_k, which is given by

$$\sum_{k=1}^{2l+1} Y_{l,k}(x)Y_{l,k}(y) = \frac{2l+1}{4\pi} P_l(x \cdot y), \tag{2.1}$$

where P_l is the Legendre polynomial with degree k, and $P_l(1) = 1$. From (1), we can define the kernel K_l as

$$K_l(x, y) := K_l(x \cdot y) := \sum_{k=1}^{2l+1} Y_{l,k}(x)Y_{l,k}(y) = \frac{2l+1}{4\pi} P_l(x \cdot y). \tag{2.2}$$

Since $L^2(\mathbb{S}^2) = \text{closure}\{\oplus_l \mathbb{H}_l\}$, any function $f \in L^2$ can be expanded into a Fourier-Laplace series with respect to orthogonal system $\{Y_{l,k} : k = 1, 2, \ldots, 2l+1, l = 0, 1, \ldots\}$ as follows: $f = \sum_{l=0}^{\infty}\sum_{k=1}^{2l+1} \hat{f}_{l,k}Y_{l,k}$, where$\hat{f}_{l,k}$ are Fourier coefficients given by $\hat{f}_{l,k} := (f, Y_{l,k}) := \int_{\mathbb{S}^2} f(x)Y_{l,k}(x)dx$.

For more backgrounds related to spherical harmonic analysis, we refer the reader to [4], [8], and [14].

We here also need to deal with Sobolev space $H^r(\mathbb{S}^2)$:

$$H^r(\mathbb{S}^2) := \left\{ f \in L^2(\mathbb{S}^2) : \|f\|_r^2 := \sum_{l=0}^{\infty} \left(l + \frac{1}{2}\right)^{2r} \sum_{k=0}^{2l+1} \hat{f}_{l,k}^2 < \infty \right\}, \quad r \geq 0.$$

Let $h : \mathbb{R}^+ \to \mathbb{R}^+$ be a bounded function satisfying

$$h(x) = \begin{cases} 1, & x \in [0, 1], \\ 0, & x \in [2, +\infty), \end{cases}$$

which can be called "filter function" by following a lead from signal analysis. So we use the "filter function" and the kernels $K_i(t) (i = 0, 1, \ldots, 2L)$, and construct a new kernel $H_L(t)$:

$$H_L(t) := \sum_{l=0}^{2L} h\left(\frac{l}{L}\right) K_l(t), \quad t \in [-1, 1]. \tag{2.3}$$

Now we construct the de la Vallée-Poussin mean operators for given $f \in L^2(\mathbb{S}^2)$ (see [12]):

$$V_L f(x) := (f, H_L(x, \cdot)) = \int_{\mathbb{S}^2} f(y)H_L(x \cdot y)dy, \tag{2.4}$$

which can be written as $V_L f(x) = \sum_{l=0}^{2L} h\left(\frac{l}{L}\right) \sum_{k=1}^{2l+1} \hat{f}_{l,k}Y_{l,k}(x)$.

For $x \in \mathbb{S}^2$, it can be represented in polar coordinates by θ, ϕ as $x := (\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta)$, where θ is the colatitude angle measured down from the z-axis and varying between 0 and π, and ϕ is the latitude angle varying from 0 to 2π measured from the x-axis.

We now are concerned with the approximation based on the following equiangular grid points $\Gamma_N := \{(\theta_m,\phi_l),\ 0\le m\le 2N-1,\ 0\le l\le 2N-1\}$, where $\theta_m := \frac{m\pi}{2N}$, and $\varphi_l = \frac{l\pi}{N}$. For $f\in C(\mathbb{S}^2)$, Driscoll and Healy [2] proposed an approximation scheme for the discrete form operators:

$$Q_{N-1}f(\theta,\phi) := \sum_{n=0}^{N-1}\sum_{j=0}^{2l+1}\tilde{f}_{n,j}Y_{n,j}(\theta,\phi),$$

where

$$\tilde{f}_{n,j} := \frac{\sqrt{2\pi}}{2N}\sum_{m=0}^{2N-1}a_m^N\sum_{l=0}^{2N-1}f(\theta_m,\phi_l)Y_{n,j}(\theta_m,\phi_l),$$

and

$$a_m^N := \frac{2^{\frac{3}{2}}}{N}\sin\left(\frac{m\pi}{N}\right)\sum_{l=0}^{N-1}\frac{1}{2l+1}\sin\left((2l+1)\frac{m\pi}{N}\right),\quad 0\le m\le 2N.$$

The coefficients a_m^N are chosen so that the trapezoidal rule

$$\int_{\mathbb{S}^2}f(\omega)d\omega\approx\frac{1}{2N}\sum_{m=0}^{2N-1}a_m^N\sum_{l=0}^{2N-1}f(\theta_m,\phi_l)$$

is exact for all polynomials of degree at most $\le 2N-1$.

In this paper we also consider the following discrete version of operators (4) based on the grid points Γ_N, which can be written as

$$\begin{aligned} V_N^L f(x) &:= \frac{\sqrt{2\pi}}{2N}\sum_{s=0}^{2N-1}a_s^N\sum_{j=0}^{2N-1}f(\theta_s,\phi_t)H_L(\omega_{s,t}\cdot x) &(2.5)\\ &= \frac{\sqrt{2\pi}}{2N}\sum_{s=0}^{2N-1}a_s^N\sum_{j=0}^{2N-1}f(\theta_s,\phi_t)\sum_{l=0}^{2L}h\left(\frac{l}{L}\right)K(\omega_{s,t}\cdot x). &(2.6)\end{aligned}$$

In the following, C denotes an absolute positive constant and its value may be different at different occurrences, even within the same formula.

3 Preliminary Results

We introduce the notation $(f,g)_N := \frac{1}{2N}\sum\limits_{m=0}^{2N-1}a_m^N\sum\limits_{l=0}^{2N-1}f(\theta_m,\phi_l)g(\theta_m,\phi_l)$, as the discrete inner product of f and g. Then we have
(1) $\tilde{f}_{n,j}=\sum_{p=0}^{\infty}\sum_{s=0}^{2p+1}\hat{f}_{p,s}(Y_{n,j},Y_{p,s})_N$;
(2) $(Y_{n,j},Y_{p,s})_N=(Y_{n,j},Y_{p,s})=\delta_{n,p}\delta_{j,s}$,
where $\delta_{x,y}$ is the Kronecker symbol defined by $\delta_{x,y}=0$ for $x\ne y$, and $\delta_{x,y}=1$ for $x=y$.

We denote by $I_{n,j}(p)$ the set of all indices $-p\le s\le p$ such that $s=j(\mathrm{mod}\ 2N)$. Then, for $0\le n\le N$, and $0\le p\le N-1$, it follows that

$$I_{n,j}(p)=\begin{cases}\phi, & \text{if } |j|>p,\\ \{j\}, & \text{if } |j|\le p.\end{cases}$$

The following Lemma 1 and Lemma 2 can be found in [5].

Lemma 1 (see [5]). If f has Fourier-Laplace series expansions, then for $n\le N$ there holds $\tilde{f}_{n,j}=\hat{f}_{n,j}+\sum_{p=N}^{\infty}\sum_{s\in I_{n,j}(p)}\hat{f}_{p,s}(Y_{n,j},Y_{p,s})_N$.

Lemma 2 (see [5]). Let $0\le n\le N$ and $|j|\le p$. Then for $p\ge N$, there holds that $|I_{n,j}(p)|\le\frac{2p}{N}$, where $|I_{n,j}(p)|$ denotes the cardinal number of set $I_{n,j}(p)$.

From the two lemmas, we can prove the following Lemma 3.

Lemma 3 If $f \in H^q$, then $\sum_{n=0}^{N}\sum_{j=-n}^{n}|\tilde{f}(n,j)-\hat{f}(n,j)|^2 \le C_q N^{2(1-q)}\|f\|_q^2$, where C_q denotes a positive constant depending only on q.

Proof From Lemma 1, we get

$$|\tilde{f}(n,j)-\hat{f}(n,j)|^2 \le \left|\sum_{p\ge N}\sum_{s\in I_{n,j}(p)}\hat{f}(p,s)(Y_{n,j},Y_{p,s})_N\right|^2$$
$$\le \left(\sum_{p\ge N}\sum_{s\in I_{n,j}(p)}p^{-2q}\right)\left(\sum_{p\ge N}\sum_{s\in I_{n,j}(p)}p^{2q}|\hat{f}(p,s)|^2|(Y_{n,j},Y_{p,s})_N|^2\right).$$

By Lemma 2 we have

$$\sum_{p\ge N}\sum_{s\in I_{n,j}(p)}p^{-2q}=\sum_{p\ge N}|I_{n,j}(p)|p^{-2q}\le \frac{C}{N}\sum_{p=-N}^{\infty}p^{-2q+1}\le C_q N^{1-2q}.$$

On the other hand,

$$\sum_{n=0}^{N}\sum_{j=-n}^{n}\sum_{p\ge N}\sum_{s\in I_{n,j}(p)}p^{2q}|\hat{f}(p,s)|^2|(Y_{n,j},Y_{p,s})_N|^2 \le \sum_{n=0}^{N}\sum_{j=-n}^{n}\sum_{p\ge N}\sum_{s\in I_{n,j}(p)}p^{2q}|\hat{f}(p,s)|^2$$
$$\le CN\sum_{p\ge N}(p+\frac{1}{2})^{2q}\sum_{s\in I_{n,j}(p)}|\hat{f}(p,s)|^2 \le CN\sum_{p\ge N}(p+\frac{1}{2})^{2q}\sum_{s=-p}^{p}|\hat{f}(p,s)|^2 \le CN\|f\|_q^2.$$

Hence $\sum_{n=0}^{N}\sum\limits_{j=-n}^{n}|\tilde{f}(n,j)-\hat{f}(n,j)|^2 \le C_q N^{2(1-q)}\|f\|_q^2$. The proof of Lemma 1 is complete.

Lemma 4. Let $H_L(t)$ be the kernel defined by (3). If there exists a positive constant C such that

$$\sum_{l=0}^{2L-1}\left|\triangle^3 h\left(\frac{l}{L}\right)\right| \le \frac{C}{L^2}, \tag{3.7}$$

then $|H_L(t)|\le CL^2$.

Proof For the kernel sequence $K_0(t),K_1(t),\ldots$, the partial summations $Z_l^{(0)},Z_l^{(1)}$, and $Z_l^{(2)}$ can be defined as (see [9])

$$\begin{aligned} Z_l^{(0)}(t) &= K_0(t)+\cdots+K_l(t),\\ Z_l^{(1)}(t) &= Z_0^{(0)}(t)+\cdots+Z_l^{(0)}(t),\\ Z_l^{(2)}(t) &= Z_0^{(1)}(t)+\cdots+Z_l^{(1)}(t), \end{aligned}$$

respectively. Applying the summation formula by parts to the formally infinite series

$$H_L(t)=\sum_{l=0}^{\infty}h\left(\frac{l}{L}\right)K_l(t)$$

yields

$$\begin{aligned} H_L(t) &= -\sum_{l=0}^{\infty}\left(\triangle h\left(\frac{l}{L}\right)\right)Z_l^{(0)}(t)=\sum_{l=0}^{\infty}\left(\triangle^2 h\left(\frac{l}{L}\right)\right)Z_l^{(1)}(t)\\ &= -\sum_{l=0}^{\infty}\left(\triangle^3 h\left(\frac{l}{L}\right)\right)Z_l^{(2)}(t)=-\sum_{l=0}^{2L-1}\left(\triangle^3 h\left(\frac{l}{L}\right)\right)Z_l^{(2)}(t), \end{aligned} \tag{3.8}$$

where $\triangle h(\frac{l}{L}), \triangle^2 h(\frac{l}{L}), \triangle^3 h(\frac{l}{L})$ denotes the first, the second, and the 3st forward difference of $h(\frac{l}{L})$, respectively, and the fact that all forward differences of $h(\frac{l}{L})$ vanish for $l \geq 2L$ is used in the last step.

By Cesáro summation formula, we have

$$Z_l^{(2)}(t) = Z_l^{(0)}(t) + 2Z_{l-1}^{(0)}(t) + \frac{2\cdot 3}{2!}Z_{l-2}^{(0)}(t) + \cdots + \frac{2\cdot 3\cdots(l+1)}{l!}Z_0^{(0)}(t). \tag{3.9}$$

Furthermore,

$$Z_i^{(0)}(t) = K_0(t) + K_1(t) + \cdots + K_i(t) = \frac{1}{4\pi}(1 + 3P_1(t) + \cdots + (2i+1)P_i(t)).$$

So it follows from the inequality $|P_i(t)| \leq 1$ (see [4]) that

$$|Z_i^{(0)}(t)| \leq \frac{1}{4\pi}(1 + 3 + \cdots + (2i+1)) \leq C(i+1)^2. \tag{3.10}$$

Combining (9) with (10), we get

$$\begin{aligned} |Z_l^{(2)}(t)| &\leq C\left((l+1)^2 + 2l^2 + \cdots + (l+1)\right) \\ &\leq C\left((2l+3)^3 + (2l+2)^3 + \cdots + (l+3)^3\right) \leq Cl^4. \end{aligned} \tag{3.11}$$

So from (8), (11), and the condition (7) we obtain

$$|H_L(t)| \leq CL^2. \tag{3.12}$$

This finishes the proof of Lemma 3.

4 Main Results and Proofs

We first estimate the approximation errors of operators (4) and (5) when the approximated functions are in Sobolev space $H^r(\mathbb{S}^2)$.

Theorem 1 If the filter function h is bounded by M, then for $f \in H^q$ $(q > 0)$, there holds

$$\|f - V_l f\|_q \leq (M+1)\left(L + \frac{3}{2}\right)^{-q}\|f\|_q.$$

Proof Since $f \in H^q$, f has the expansion: $f(x) = \sum_{l=0}^{\infty}\sum_{k=1}^{2l+1}\hat{f}_{l,k}Y_{l,k}(x)$, and $V_L f(x) = \sum_{l=0}^{2L} h\left(\frac{l}{L}\right)\sum_{k=1}^{2l+1}\hat{f}_{l,k}Y_{l,k}(x)$. Hence,

$$\begin{aligned} \|f - V_L f\|^2 &= \sum_{l=0}^{2L}\left(h\left(\frac{l}{L}\right) - 1\right)^2 \sum_{k=1}^{2l+1}\hat{f}_{l,k}^2 + \sum_{l=2L+1}^{\infty}\sum_{k=1}^{2l+1}\hat{f}_{l,k}^2 \\ &= \sum_{l=L+1}^{2L}\left(h\left(\frac{l}{L}\right) - 1\right)^2 \sum_{k=1}^{2l+1}\hat{f}_{l,k}^2 + \sum_{l=2L+1}^{\infty}\sum_{k=1}^{2l+1}\hat{f}_{l,k}^2 \\ &\leq \left(L + \frac{3}{2}\right)^{-2q}\sum_{l=L+1}^{2L}\left(l + \frac{1}{2}\right)^{2q}\left(h\left(\frac{l}{L}\right) - 1\right)^2\sum_{k=1}^{2l+1}\hat{f}_{l,k}^2 \\ &+ \left(2L + \frac{3}{2}\right)^{-2q}\sum_{l=2L+1}^{\infty}\left(l + \frac{1}{2}\right)^{2q}\sum_{k=1}^{2l+1}\hat{f}_{l,k}^2 \\ &\leq (M+1)^2\left(L + \frac{3}{2}\right)^{-2q}\sum_{l=L+1}^{\infty}\left(l + \frac{1}{2}\right)^{2q}\sum_{k=1}^{2l+1}\hat{f}_{l,k}^2 \\ &\leq (M+1)^2\left(L + \frac{3}{2}\right)^{-2q}\|f\|_q^2. \end{aligned}$$

Therefore,

$$\|f - V_L f\| \le (M+1)\left(L + \frac{3}{2}\right)^{-q} \|f\|_q.$$

The proof of Theorem 1 is complete.

Theorem 2 If $f \in H^q$ $(q > 1)$, and the filter function h is bounded by M, then

$$\|f - V_N^L f\| \le C_q M L^{-(q-1)} \|f\|_q.$$

Proof For the operators (5), we apply Funck-Hecke formula (see [4, 8, 14]), and obtain

$$\begin{aligned}
\widehat{V_N^L f}(n,j) &= \frac{\sqrt{2\pi}}{2N} \sum_{s=0}^{2N-1} a_s^N \sum_{j=0}^{2N-1} f(\theta_s, \phi_t) H_L(\omega_{s,t}\cdot x) Y_{n,j}(x)dx \\
&= \frac{\sqrt{2\pi}}{2N} \sum_{s=0}^{2N-1} a_s^N \sum_{j=0}^{2N-1} f(\theta_s, \phi_t) \widehat{H}_L(n) Y_{n,j}(\omega_{s,t}) \\
&= \frac{\sqrt{2\pi}}{2N} \widehat{H}_L(n) \sum_{s=0}^{2N-1} a_s^N \sum_{j=0}^{2N-1} f(\theta_s, \phi_t) Y_{n,j}(\omega_{s,t}) = \widehat{H}_L(n) \tilde{f}(n,j).
\end{aligned}$$

Hence,

$$\begin{aligned}
\|f - V_N^L f\|^2 &= \sum_{n=0}^{\infty} \sum_{j=-n}^{n} \left[\hat{f}(n,j) - \widehat{V_N^L f}(n,j)\right]^2 \\
&= \sum_{n=0}^{\infty} \sum_{j=-n}^{n} \left[\hat{f}(n,j) - \hat{f}(n,j)\hat{H}_L(n) + \hat{f}(n,j)\hat{H}_L(n) - \hat{H}_L(n)\tilde{f}(n,j)\right]^2,
\end{aligned}$$

which follows from the basic inequality $(a+b)^2 \le 2(a^2+b^2)$ that

$$\begin{aligned}
\|f - V_N^L f\|^2 &\le 2\left\{\sum_{n=0}^{\infty} \sum_{j=-n}^{n} |\hat{f}(n,j)|^2 [1 - \hat{H}_L(n)]^2 \sum_{n=0}^{\infty} \sum_{j=-n}^{n} |\hat{H}_L(n)|^2 |\hat{f}(n,j) - \tilde{f}(n,j)|^2\right\} \\
&= 2\sum_{n=0}^{\infty} \sum_{j=-n}^{n} |\hat{f}(n,j)|^2 [1 - \hat{H}_L(n)]^2 + 2\sum_{n=0}^{2L} \sum_{j=-n}^{n} |\hat{H}_L(n)|^2 |\hat{f}(n,j) - \tilde{f}(n,j)|^2 \\
&\quad +2 \sum_{n=2L+1}^{\infty} \sum_{j=-n}^{n} |\hat{H}_L(n)|^2 |\hat{f}(n,j) - \tilde{f}(n,j)|^2.
\end{aligned}$$

From (2) and (3) we know $H_L(t) = \frac{1}{4\pi}\sum_{l=0}^{2L} h\left(\frac{l}{L}\right)(2l+1)P_l(t)$. So we use the facts (see [4]) $\hat{H}_L(n) = 2\pi\int_{-1}^{1} H_L(t)P_n(t)dt$, and $\int_{-1}^{1} P_n(t)P_m(t)dt = \frac{2}{2n+1}\delta_{nm}$, and obtain

$$\hat{H}_L(n) = \begin{cases} h(\frac{l}{L}), & n = 0, 1, \ldots, 2L, \\ 0, & n = 2L+1, 2L+2, \ldots. \end{cases} \tag{4.13}$$

Hence, by Lemma 3 and (13) we see

$$\begin{aligned}
\|f - V_N^L f\|^2 &\le 2\left(\sum_{n=L+1}^{\infty} \sum_{j=-n}^{n} |\hat{f}(n,j)|^2 [1 - \hat{H}_L(n)]^2 + \sum_{n=0}^{2L} \sum_{j=-n}^{n} |\hat{H}_L(n)|^2 |\hat{f}(n,j) - \tilde{f}(n,j)|^2\right) \\
&\le CM^2 L^{-2q} \|f\|_q^2 + C_q M^2 L^{2(1-q)} \|f\|_q^2 \le C_q M^2 L^{-2(q-1)} \|f\|_q^2,
\end{aligned}$$

which follows

$$\|f - V_N^L f\| \le C_q M L^{-(q-1)} \|f\|_q.$$

The proof of Theorem 2 is complete.

Sloan [12] studied the convergence of operators (4) in the metric of uniform norm, and obtained a profound result (here we only give the case of $\mathbb{S}^2$):

Proposition 1 Let $h \in C^1(\mathbb{R}^+)$ be the unique piecewise polynomial constructed as follows

$$h(x) = \begin{cases} 1, & x \in [0,1], \\ 1-2(x-1)^2, & x \in (1, \frac{3}{2}], \\ 2(2-x)^2, & x \in (\frac{3}{2}, 2], \\ 0, & x \in (2, +\infty). \end{cases}$$

And let V_L be the linear operators defined by (4). Then $\sup_{L\geq 0} \|V_L\| < \infty$, where $C^1(\mathbb{R}^+)$ is a class of functions which are defined on $[0, +\infty)$ and have continuous derivatives.

Sloan [12] reduced the restriction of $h \in C^\infty(\mathbb{R}^+)$ (see [10], [14]) to $h \in C^1(\mathbb{R}^+)$. In fact, the key inequality (7) also holds when h only satisfies the above conditions in Proposition 1. Motivated by this, we introduce a function class $\bar{M}W^3$, which is composed of the function satisfying

$$\omega_3(f,u) \leq \bar{M}u^3, \quad 0 < u \leq 1, \tag{4.14}$$

where $\bar{M}$ is a positive constant, and $\omega_3(f,u)$ is the modulus of smoothness of order 3 of function f (see [6]). Clearly, from the definition of $\omega_3(f,u)$ we find for $h \in \bar{M}W^3$,

$$\sum_{l=0}^{2L-1} \left| \triangle^3 h\left(\frac{l}{L}\right)\right| \leq \bar{M} \sum_{l=0}^{2L-1} \frac{1}{L^3} \leq \frac{\bar{M}C}{L^2}.$$

So we have

Theorem 3 Let $h \in C(\mathbb{R}^+)$ be a filter function, and it satisfies the smoothness condition (14). Then for linear operators (4), we have $\sup_{L\geq 0} \|V_L\| < \infty$.

Next we estimate the upper bound of uniform norm for the operators V_N^L.

Theorem 4 If the filter function h satisfies (7), then for the operators (5), there holds $\|V_N^L\| \leq CL^2$.

Proof From 3.5.5 and 3.5.7 in [7], we have for $x \in [0, 2\pi]$,

$$\left|\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots + \frac{\sin(2N-1)x}{2N-1}\right| \leq \sqrt{2}|x| \leq 2\sqrt{2}\pi,$$

$$\left|\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \frac{\sin 4x}{4} + \cdots + \frac{\sin(2N-1)x}{2N-1}\right| \leq \frac{\pi}{2} + 1.$$

Thus,

$$\left|\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots + \frac{\sin(2N-1)x}{2N-1}\right| \leq \frac{1}{2}(2\sqrt{2}\pi + \frac{\pi}{2} + 1).$$

Hence,

$$|a_m^N| = \left|\frac{2^{\frac{3}{2}}}{N} \sin\left(\frac{m\pi}{N}\right) \sum_{l=0}^{N-1} \frac{1}{2l+1} \sin\left((2l+1)\frac{m\pi}{N}\right)\right| \leq \frac{C}{N}. \tag{4.15}$$

Using Lemma 4 and (15), we obtain

$$\begin{aligned} |V_N^L f(x)| &= \left|\frac{\sqrt{2\pi}}{2N} \sum_{s=0}^{2N-1} a_s^N \sum_{j=0}^{2N-1} f(\theta_s, \phi_t) H_L(\omega_{s,t} \cdot x)\right| \leq \frac{C}{N}\|f\| \sum_{s=0}^{2N-1} \sum_{j=0}^{2N-1} \left|a_s^N H_L(\omega_{s,t} \cdot x)\right| \\ &\leq \frac{C}{N^2}\|f\| L^2 \sum_{s=0}^{2N-1} \sum_{j=0}^{2N-1} 1 \leq CL^2\|f\|, \end{aligned}$$

that is

$$\|V_N^L\| \leq CL^2.$$

The proof of Theorem 4 is completed.

From Theorem 4 we see that for fixed kernel H_L the norms of operators sequence $\{\|V_N^L\|\}$ are uniformly bounded. Since $H_L(\omega \cdot x)$ is a spherical polynomial with degree $2L$, and

$$\int_{\mathbb{S}^2} f(y)H(x\cdot y)dy \approx \frac{\sqrt{2\pi}}{2N}\sum_{s=0}^{2N-1} a_s^N \sum_{j=0}^{2N-1} f(\theta_s,\phi_t)H_L(\omega_{s,t}\cdot x),$$

is exact when $f(y)H(x \cdot y)$ is a polynomial with degree $2N-1$, we have

$$\begin{aligned}|f(x)-V_N^L f(x)| &\leq |f(x)-p_{2(N-L)-1}(x)|+|p_{2(N-L)-1}(x)-V_N^L f(x)|\\ &\leq E_{2(N-L)-1}(f)+\|V_N^L\|E_{2(N-L)-1}(f)\leq CL^2E_{2(N-L)-1}(f),\end{aligned}$$

where $p_{2(N-L)-1}(x)$ is the best approximation polynomial with degree $2(N-L)-1$ of f, and $E_{2(N-L)-1}(f)$ denotes the best polynomial approximation of f.

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A finite iterative algorithm for solving the least-norm generalized (P, Q) reflexive solution of the matrix equations $A_iXB_i = C_i$ ☆

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Abstract

In this paper, based on the idea of conjugate gradient method, an algorithm for solving the least-norm generalized (P, Q) reflexive solution of the matrix equations $A_iXB_i = C_i(i = 1, \cdots, N)$. According to the algorithm, the solvability of the problem can be determined automatically, i.e., the solutions of Problem I can be obtained within finite iterative steps in the absence of roundoff errors. The unique least-norm generalized (P, Q) reflexive iterative solution can be derived when an appropriate initial iterative matrices is chosen. In addition, in the solution set of above problem, the unique optimal approximation solution X^* to a given matrix Y in Frobenius norm can be obtained. Finally, one numerical experiment is given to verify the theoretical results of this paper, which shows the robustness and efficiency of the algorithm.

Keywords: Matrix equations; least-norm solution; generalized reflexive matrix; finite iterative algorithm.
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1. Introduction

Throughout the paper, R^n will denote the complex $n-$ vector space and the set of $n \times m$ matrices by $R^{n\times m}$. For a matrix $A \in R^{m\times n}$, $\|A\|$ represents its Frobenius norm, $R(A)$ represents its column space, $tr(A)$ represents its trace and $vec(\cdot)$ represents the vec operator, i.e., $vec(A) = (a_1^T, a_2^T, \cdots, a_n^T)^T$ for the matrix $A = (a_1, a_2, \cdots, a_n) \in R^{m\times n}, a_i \in R^m, i = 1, 2, \cdots, n. A \otimes B$ stands for the Kronecker product of matrices A and B.

In [28], Peng and Hu presented the conditions for the solvability of matrix equation $AX = B$ over reflexive or anti-reflexive matrices and the numerical methods for solving the matrix equation $AX = B$ over generalized (P, Q) reflexive or anti-reflexive matrices have been proposed in [49]. In [15], the authors presented a new iteration method for $AX = B$. Then, the conditions for the solvability of matrix equation $AXB = C$ over special matrices such as symmetric matrices (anti-symmetric matrices), reflexive matrices (anti-reflexive matrices) and so on, have been presented in [4, 16, 19, 29, 30, 31, 32, 33, 37, 48, 50]. By using the generalized singular value decomposition, and gradient based method, the conditions for the solvability and numerical methods for Sylvester and generalized Sylvester equation have been proposed in [1, 5, 9, 17, 26, 34, 36, 39, 40, 41, 42, 43, 44]. The coupled matrix equations have been considered in [6, 7, 8, 10, 11, 12, 13, 18, 20, 21, 22, 38, 45] and the other kinds of matrix equations have been considered in [14, 23, 24, 25, 46].

In this paper, we will consider the following matrix equations

$$A_i X B_i = C_i (i = 1, 2, \cdots, N) \tag{1}$$

where $A_i \in R^{p\times n}, B_i \in R^{m\times q}, C_i \in R^{p\times q}$ and $X \in R^{n\times m}$ is the generalized (P, Q) reflexive matrix.

In [3], the definition and some properties of generalized reflexive (anti-reflexive) matrix have been presented.

Definition 1. [3] *A matrix $P \in R^{n\times n}$ is called a generalized reflection matrix if $P^T = P$ and $P^2 = I$. A matrix $A \in R^{n\times m}$ is said to be a generalized reflexive (anti-reflexive) matrix with respect to the generalized reflection matrix $P \in R^{n\times n}, Q \in R^{m\times m}$, if $A = PAQ (A = -PAQ)$. We denote the set of all generalized (P, Q) reflexive (or anti-reflexive) matrices by $R_r^{n\times m}(P, Q)$(or $R_a^{n\times m}(P, Q)$).*

According to the definition above, the proof of the following lemma is trivial.

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Lemma 2. *For an arbitrary matrix $A \in R^{n\times m}$, we have*

$$A + PAQ \in R_r^{n\times m}(P, Q), A - PAQ \in R_a^{n\times m}(P, Q),$$

where P, Q are generalized reflection matrices.

Lemma 3. *If $A \in R_r^{n\times m}(P, Q), B \in R_a^{n\times m}(P, Q)$, then we have $tr(A^T B) =$ 0.*

Now we consider the following problems.

Problem I. For given matrices $A_i \in R^{p\times n}, B_i \in R^{m\times q}, C_i \in R^{p\times q}$, find matrices $X \in R_r^{n\times m}(P, Q)$ such that

$$\left\| \begin{pmatrix} A_1XB_1 - C_1 \\ A_2XB_2 - C_2 \\ \vdots \\ A_NXB_N - C_N \end{pmatrix} \right\| = \min$$

Obviously, Problem I is equivalent to solving $\sum\limits_{i=1}^{N} \|A_iXB_i - C_i\|^2 = \min\{X \in R_r^{n\times m}(P, Q)\}$. We denote $F(X)$ by $\sum\limits_{i=1}^{N} \|A_iXB_i - C_i\|^2$.

Problem II. Let S_R denote the set of solutions of Problem I, for a given $Y \in R_r^{n\times m}(P, Q)$, find $\overline{X} \in S_R$ such that

$$\|\overline{X} - Y\| = \min_{X\in S_R} \|X - Y\|.$$

Problem II occurs frequently in experimental design, see for instance [27]. Here the matrix Y may be obtained from experiments, but it may not be the solution of Problem I. The best estimate $\overline{X}$ is the matrix that not only is the solution of Problem I, but also is the best approximation of the matrix Y.

As to Problem I, an iterative method is presented to solve it. By this method, the solvability of Problem I can be determined automatically, i.e., for any generalized (P, Q) reflexive initial iterative matrices X_1, the solutions of Problem I can be obtained within finite iterative steps in the absence of roundoff errors. The unique least-norm generalized (P, Q) reflexive iterative solution of Eq. (1) can be derived when an appropriate initial iterative

matrices is chosen. Furthermore, the optimal approximate solution of Eq. (1) for a given matrices Y can be derived by finding the least-norm generalized (P, Q) reflexive solutions of a new corresponding matrix equations. Finally, a random numerical example is given to support the theoretical results of this paper.

2. An iterative algorithm for solving Problem I and Problem II

In this section, we firstly introduce some definitions, lemmas and theorems which are required for solving Problem I. The proof of the following lemma is similar with [2] and its proof is omitted.

Lemma 4. *Let $F(X)$ be a continuous, differential and convex function on $R_r^{n\times m}$ (P, Q), then there exists $X^* \in R_r^{n\times m}(P,Q)$ such that*

$$F(X^*) = \min_{X\in R_r^{n\times m}(P,Q)} F(X),$$

if and only if $\nabla F(X^*) = 0$.

According to the Taylor series expansion, we have

$$F(X+\epsilon H) = F(X) + \epsilon(\nabla F(X), H)_F + o(\epsilon), \quad \forall X, H \in R_r^{n\times m}(P,Q), \epsilon \in R, \tag{2}$$

where $(,)_F$ is the F-inner product denoted by $(A,B)_F = tr(B^TA)$ and $\nabla F(X)$ is the gradient of the function $F(X)$.

By concrete deduction, we can get

$$F(X+\epsilon H) = F(X) + 2\epsilon \sum_{i=1}^{N} (A_iXB_i - C_i, A_iHB_i)_F + \epsilon^2 \sum_{i=1}^{N} \|A_iHB_i\|_F^2. \tag{3}$$

Then by Lemma 2, we have

$$\begin{aligned}
&2\epsilon(A_iXB_i - C_i, A_iHB_i)_F \\
&= 2\epsilon(A_i^TA_iXB_iB_i^T - A_i^TC_iB_i^T, H)_F \\
&= 2\epsilon(\tfrac{1}{2}(A_i^TA_iXB_iB_i^T - A_i^TC_iB_i^T + P(A_i^TA_iXB_iB_i^T - A_i^TC_iB_i^T)Q), H)_F \\
&+\tfrac{1}{2}(A_i^TA_iXB_iB_i^T - A_i^TC_iB_i^T - P(A_i^TA_iXB_iB_i^T - A_i^TC_iB_i^T)Q), H)_F) \\
&= \epsilon(A_i^TA_iXB_iB_i^T - A_i^TC_iB_i^T + P(A_i^TA_iXB_iB_i^T - A_i^TC_iB_i^T)Q, H)_F.
\end{aligned}$$

4

So we can get

$$F(X+\epsilon H)=F(X)+\epsilon\sum_{i=1}^{N}(A_i^TA_iXB_iB_i^T-A_i^TC_iB_i^T$$

$$+P(A_i^TA_iXB_iB_i^T-A_i^TC_iB_i^T)Q,H)_F+\epsilon^2\sum_{i=1}^{N}\|A_iHB_i\|_F^2.$$

By comparing the above equality with (2) we have

$$\nabla F(X)=\sum_{i=1}^{N}(A_i^TA_iXB_iB_i^T-A_i^TC_iB_i^T+P(A_i^TA_iXB_iB_i^T-A_i^TC_iB_i^T)Q)_F. \tag{4}$$

Therefore, we can get the following lemma

Lemma 5. *$X^*\in R_r^{n\times m}(P,Q)$ is a solution of Problem I, if and only if* $\nabla F(X^*)=\sum\limits_{i=1}^{N}(A_i^TA_iXB_iB_i^T-A_i^TC_iB_i^T+P(A_i^TA_iXB_iB_i^T-A_i^TC_iB_i^T)Q)=0.$

For convenience, we denote

$$\begin{aligned}
&D(H)=\sum_{i=1}^{N}(A_i^TA_iHB_iB_i^T+PA_i^TA_iHB_iB_i^TQ),\\
&E=\sum_{i=1}^{N}(A_i^TC_iB_i^T+PA_i^TC_iB_i^TQ),\\
&F(X)=-\nabla F(X)=E-D(X),F_k=F(X_k).
\end{aligned}$$

Algorithm 1: (An iterative algorithm for solving Problem I)

Step 1. Input matrices $A_i\in R^{p\times n},B_i\in R^{m\times q},C_i\in R^{p\times q}$ $(i=1,\cdots,N)$ and the generalized reflexive matrices $P\in R^{n\times n},Q\in R^{m\times m}$; Choose any matrix $X_1\in R_r^{n\times m}(P,Q)$;

Step 2. Compute $F_1=E-D(X_1),G_1=D(F_1),k:=1$

Step 3. If $F_k=0$, stop; else

Step 4. Compute

$$\begin{aligned}
&X_{k+1}=X_k+\tfrac{\|F_k\|_F^2}{\|G_k\|_F^2}G_k;\\
&F_{k+1}=F_k-\tfrac{\|F_k\|_F^2}{\|G_k\|_F^2}D(G_k);\\
&G_{k+1}=F_{k+1}-\tfrac{\|F_{k+1}\|_F^2}{\|F_k\|_F^2}G_k;
\end{aligned}$$

5

Step 5. If $F_k \neq 0, k := k+1$, go to Step 4.

Obviously, we know that $X_k \in R_r(P,Q), G_k \in R_r(P,Q), F_k \in R_r(P,Q)$. According to Algorithm 1,the following lemmas can be verified easily by using similar methods of [35].

Lemma 6. *For the sequences F_k, G_k which are produced by Algorithm 1, if there exists an integer number k such that $F_i \neq 0$ for all $i = 1, 2, \cdots, k$, then we have that*

$$(F_i, F_j)_F = 0, (G_i, D(G_j))_F = 0 \quad (i, j = 1, 2, \cdots, k, i \neq j) \tag{5}$$

Lemma 7. *If $X^* \in R_r^{n\times m}(P,Q)$ is a solution of Problem I, the following equality will hold for any initial matrix $X_1 \in R_r^{n\times p}(P,Q)$*

$$(X^* - X_i, D(G_i))_F = a\|F_i\|_F^2,$$

where a is a positive number.

Theorem 8. *For an arbitrary initial matrix $X_1 \in R_r^{n\times m}(P,Q)$, a generalized (P,Q) reflexive least-norm solution of Problem I can be obtained by Algorithm 1 within finite iterative steps in the absence of roundoff errors.*

Proof: If there exists a positive integer $k \leq mpq$ such that $F_k = 0$, then Algorithm 1 will stop and X_k exactly is the solution of Problem I. Otherwise, $F_k \neq 0 (k = 1, 2, \cdots, mpq)$, i.e., $G_k \neq 0$ by Lemma 7. Therefore, X_{mpq+1} and F_{mpq+1} can be computed by Algorithm 1. Now, by Lemma 7 we have $(F_k, F_{mpq+1})_F = 0, k = 1, 2, \cdots, mpq$ and $(F_k, F_j)_F = 0, k, j = 1, 2, \cdots, mpq, k \neq j$. So, $F_1, \cdots, F_{mpq+1}$ is an orthogonal basis of the matrix space $R_r^{mp\times q}(P,Q)$, which implies $F_{mpq+1} = 0$, that is, X_{mpq+1} is a solution of Problem I.

Lemma 9. [30] *Assume that the consistent system of linear equations $My = b$ has a solution $y_0 \in R(M^T)$, then y_0 is the least-norm solution of the system of linear equations.*

Theorem 10. *If Problem I is consistent and $X_1 = \sum_{i=1}^{N}(A_i^T H_i B_i^T + P A_i^T H_i B_i^T Q)$ is the initial iterative matrix, where $H_i (i = 1, 2, \cdots, N)$ are arbitrary, or especially let $X_1 = 0$, then the unique least-norm generalized (P,Q) reflexive solution of Problem I can be obtained by Algorithm 1.*

Proof: The proof is similar with the method of [47] and so is omitted.

Now we consider Problem II.

For arbitrary matrix $Y \in R^{n\times m}$ and $X \in S_R$, by Lemma 2 we can get

$$\begin{aligned}\min_{X\in S_R} \|X-Y\|^2 &= \min_{X\in S_R} \|X - \tfrac{1}{2}(Y+PYQ) - \tfrac{1}{2}(Y-PYQ)\|^2 \\ &= \min_{X\in S_R} \|X - \tfrac{1}{2}(Y+PYQ)\|^2 + \|\tfrac{1}{2}(Y-PYQ)\|^2.\end{aligned}$$

As the set of solutions of Problem I denoted by S_R is not empty. Hence, to find $X^* \in R_r^{n\times m}(P,Q)$ such that $\min\limits_{X\in R_r^{n\times m}(P,Q)} \|X-Y\|^2$ is equivalent to find $X^* \in R_r^{n\times m}(P,Q)$ such that

$$\min_{X\in R_r^{n\times m}(P,Q)} \sum_{i=1}^{N} \|A_i(X-Y)B_i - C_i - A_iXB_i\|^2. \tag{6}$$

Let $\overline{X} = X - Y, \overline{C^{(i)}} = C^{(i)} - A_iYB_i$, then Problem II is equivalent to finding the least-norm reflexive solution X^* of the following minimum residual problem

$$\min_{X\in R_r^{n\times m}(P,Q)} \sum_{i=1}^{N} \|A_i\overline{X}B_i - \overline{C}\|_F^2. \tag{7}$$

According to Theorem 10, if we take the initial iterative matrix $X_1 = \sum\limits_{i=1}^{N}(A_i^TH_iB_i^T + PA_i^TH_iB_i^TQ)$ for (7), where $H_i \in R^{m\times q}(i = 1, 2, \cdots, N)$ are arbitrary, or especially let $X_1 = 0$, then the least-norm solution $\overline{X}^*$ of (7) can be obtained by Algorithm 1. Thus, the solution of Problem II is $\widetilde{X} = \overline{X}^* + Y$.

3. Numerical example

The stopping criterion used is that the Frobenius norm of F_k less than ϵ, where $\epsilon = 10^{-8}$.

Example 1. We consider the linear systems of matrix equations

$$\begin{cases} A_1XB_1 = C_1 \\ A_2XB_2 = C_2 \\ A_3XB_3 = C_3 \end{cases} \tag{8}$$

with

$$A_1 = \begin{pmatrix} 1 & 3 & -5 & 7 & -5 \\ 3 & 0 & 4 & 1 & -1 \\ 0 & -2 & 3 & -4 & 1 \\ 11 & 6 & 2 & 7 & -1 \\ -5 & 5 & -2 & -1 & -1 \\ 3 & 4 & -6 & -2 & -1 \end{pmatrix}, A_2 = \begin{pmatrix} 4 & 3 & -3 & 3 & 1 \\ 3 & -2 & 1 & -4 & 5 \\ 4 & 2 & -1 & 0 & 1 \\ 2 & -1 & 3 & -1 & 4 \\ 4 & 3 & -1 & 3 & 1 \\ 3 & -5 & 0 & -1 & 3 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 5 & 3 & 0 & 4 & -2 \\ 7 & 4 & 6 & -3 & 4 \\ 5 & -5 & 3 & 1 & 6 \\ -3 & 6 & 3 & 4 & 2 \\ 6 & 5 & -5 & 4 & 2 \\ 2 & 0 & 3 & 5 & -2 \end{pmatrix}, B_1 = \begin{pmatrix} -1 & 5 & 0 & -2 & 4 & -5 \\ 3 & -1 & 0 & 3 & 2 & 2 \\ 0 & 3 & 1 & 2 & 1 & 3 \\ -2 & 4 & 3 & 1 & 3 & 4 \\ -1 & 3 & 4 & 3 & 2 & 2 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} -1 & 5 & 0 & -2 & 4 & -5 \\ 3 & -1 & 0 & 3 & 2 & 2 \\ 0 & 3 & 2 & 5 & 4 & 1 \\ 1 & 4 & -3 & 2 & 3 & 3 \\ 4 & 3 & 3 & 4 & 1 & 5 \end{pmatrix}, B_3 = \begin{pmatrix} -1 & 5 & 0 & -2 & 4 & -5 \\ 3 & -1 & 0 & 3 & 2 & 2 \\ 0 & 3 & 3 & 2 & 6 & 2 \\ 2 & 3 & -4 & 1 & 2 & 3 \\ 3 & 4 & 2 & 3 & 4 & 6 \end{pmatrix},$$

$$C_1 = \begin{pmatrix} -77 & -196 & -5 & 90 & 45 & 60 \\ 32 & 94 & -4 & 36 & 70 & 5 \\ 40 & 54 & 18 & -126 & -54 & -78 \\ 26 & 172 & -96 & 230 & 68 & 18 \\ -118 & 114 & -40 & 4 & 15 & 20 \\ -46 & -139 & -39 & -60 & -49 & 5 \end{pmatrix},$$

$$C_2 = \begin{pmatrix} 10 & 47 & -52 & 106 & 57 & 42 \\ 113 & -13 & -52 & -225 & 46 & 76 \\ 25 & -5 & -53 & -35 & 58 & -51 \\ 12 & -54 & 39 & -15 & 9 & 40 \\ 10 & 47 & -52 & 106 & 38 & 47 \\ 107 & -119 & 9 & -142 & 49 & 92 \end{pmatrix},$$

$$C_3 = \begin{pmatrix} 15 & -51 & 30 & 61 & -80 & 40 \\ 119 & 42 & -42 & 40 & 51 & 210 \\ 56 & -60 & 26 & 15 & 40 & -40 \\ 46 & -81 & 49 & 16 & -46 & 53 \\ 10 & 25 & 115 & -78 & 35 & 41 \\ 23 & 113 & -34 & -60 & 79 & 18 \end{pmatrix}.$$

8

(a) Find the solution of Problem I and the least-norm generalized (P,Q) reflexive solution of the matrix equation with respect to P,Q

(b) Let S_R denote the set of solutions of Problem I with respect to P,Q. For a given matrix $Y \in R^{n\times m}$, find $X^* \in S_R$ such that $\|X^* - Y\|_F^2 = \min\limits_{X\in S_R} \|X - Y\|_F^2$.

Firstly, we can compute the solution and least-norm generalized (P,Q) reflexive solution of Problem I by using Algorithm 1. Initial iterative matrix is chosen as zero matrix and P and Q are chosen to be

$$P = Q = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

By Algorithm 1, we can get

$$X_{92} = \begin{pmatrix} 0.5650 & 1.0337 & -0.1602 & 0.5760 & 0.6146 \\ -1.6275 & -0.9054 & 1.6275 & 0 & 0 \\ -0.1602 & -1.0337 & 0.5650 & 0.5760 & 0.6146 \\ 0.2782 & 0 & 0.2782 & -0.5220 & 0.2000 \\ -0.7231 & 0 & -0.7231 & -0.6815 & 0.4693 \end{pmatrix},$$

$\|F_{92}\|_F = \| - \nabla F(X_{92})\|_F = 0.9002 \times 10^{-10}$,

$$\min_{X\in R_r^{n\times m}(P,P)} \left\| \begin{pmatrix} A_1XB_1 - C_1 \\ A_2XB_2 - C_2 \\ A_3XB_3 - C_3 \end{pmatrix} \right\| \approx \left\| \begin{pmatrix} A_1X_{92}B_1 - C_1 \\ A_2X_{92}B_2 - C_2 \\ A_3X_{92}B_3 - C_3 \end{pmatrix} \right\| = 696.4168.$$

Let

$$Y = \begin{pmatrix} 3 & 6 & 2 & 4 & 3 \\ -7 & 4 & 7 & 0 & 0 \\ 2 & -6 & 3 & 4 & 3 \\ 5 & 0 & 5 & 8 & 6 \\ 2 & 0 & 2 & 7 & 5 \end{pmatrix},$$

then we can get the unique least-norm generalized (P,Q) reflexive solution

as follows

$$\overline{X}^*_{85} = \begin{pmatrix} 2.3691 & 4.7706 & 2.2389 & 3.4314 & 2.3798 \\ -5.2381 & 5.2942 & 5.2381 & 0 & 0 \\ 2.2389 & -4.7706 & 2.3691 & 3.4314 & 2.3798 \\ 4.7156 & 0 & 4.7156 & 8.4878 & 5.8015 \\ 2.6203 & 0 & 2.6203 & 7.5433 & 4.6373 \end{pmatrix},$$

$\|F_{85}\|_F = \| - \nabla F(X_{85})\|_F = 0.8222 \times 10^{-10}$.

Therefore, the solution of Problem II is

$$\widetilde{X}^*_{85} = \overline{X}^*_{85} + Y = \begin{pmatrix} 5.3691 & 10.7706 & 4.2389 & 7.4314 & 5.3798 \\ -12.2381 & 9.2942 & 12.2381 & 0 & 0 \\ 4.2389 & -10.7706 & 5.3691 & 7.4314 & 5.3798 \\ 9.7156 & 0 & 9.7156 & 16.4878 & 11.8015 \\ 4.6203 & 0 & 4.6203 & 14.5433 & 9.6373 \end{pmatrix}.$$

In this case, we have

$$\min_{X \in S_R} \|X - Y\|_F = \|\widetilde{X}_{85} - Y\|_F = 20.6606.$$

4. Conclusion

In this paper, a finite iterative algorithm for solving the generalized (P, Q) reflexive least-norm solutions of a class of matrix equations has been present. A very appealing feature of this algorithm is that a solution can be obtained within finite iteration steps in the absence of roundoff errors. Also, the unique least-norm generalized (P, Q) reflexive solution of the matrix pair nearness problem has been considered. However, the convergence rate and error analysis of Algorithm 1 are very complicated and have not been considered in this paper. We leave these cases as a topic for further research. Of course, for the problem with large and not sparse matrices, Algorithm 1 may not be finite termination because of errors.

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DIFFERENCE BETWEEN THE INTEGRAL MEANS ARISING FROM MONTGOMERY'S IDENTITY AND APPLICATIONS

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Abstract. Some new estimates of the difference between the integral means for convex, s-convex in the second sense, and quasi-convex functions are established. New estimates of errors in approximating probability density function involving general moments are also given.

1. Introduction

In [1], Cerone and Dragomir established an identity of Montgomery to obtain the following result.

Theorem 1. *Let $f : [a, b] \to R$ be an absolutely continuous mapping and as is also $g : [x, y] \to R$ with $[x, y] \subseteq [a, b]$. Then the following inequalities hold,*

$$\left|\int_x^y g(s)f(s)ds - \frac{1}{b-a}\int_a^b f(s)ds\int_x^y g(s)ds\right| \tag{1.1}$$
$$\leq \begin{cases} \frac{\|f'\|_\infty}{(b-a)}\left\{\frac{\int_x^y g(s)ds}{2}\left[(x-a)^2+(b-y)^2\right]+\int_x^y |\phi(s;x,y;g)|ds\right\}, \\ \hfill \text{if } f' \in L_\infty[a,b]; \\ \frac{\|f'\|_p}{(b-a)}\left\{\frac{\left|\int_x^y g(s)ds\right|^q}{q+1}\left[(x-a)^{q+1}+(b-y)^{q+1}\right]\right. \\ \left.+\int_x^y |\phi(s;x,y;g)|^q ds\right\}^{\frac{1}{q}}, \qquad \text{if } f' \in L_p[a,b], p>1, \frac{1}{p}+\frac{1}{q}=1; \\ \frac{\|f'\|_1}{(b-a)}\max\left\{\Theta\int_x^y |g(s)|ds, \sup_{s\in[x,y]}|\phi(s;x,y;g)|\right\}, \text{if } f' \in L_1[a,b]; \end{cases}$$

where, for $s \in [a, b]$,

$$\phi(s;x,y;g) = (s-a)\int_s^y g(u)du - (b-s)\int_x^s g(u)du,$$
$$\Theta = \frac{b-a}{2} - \frac{y-x}{2} + \left|\frac{a+b}{2} - \frac{x+y}{2}\right|,$$

and $\|\cdot\|_p, p \geq 1$ are the usual Lebesgue norms on $[a, b]$. More precisely, $\|g\|_\infty := ess\sup_{t\in[a,b]} |g(t)|$ and $\|g\|_p = \left(\int_a^b |g(s)|^p ds\right)^{\frac{1}{p}}, 1 \leq p < \infty$.

The above results are obtained for a generalized Chebychev functional, see [2, Ch. IX] invoving the integral mean of functions over different intervals. The special

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case of Theorem 1, it produces a generalization of the following Mahajani type inequalities, see [3, p.474]. If f has a bounded derivative on [a,b] and $\int_a^b f(x)dx = 0$ then, for $x \in [a,b]$, the following inequlity holds :

$$\Big|\int_a^x f(t)dt\Big| \leq \frac{(b-a)^2}{8}\|f'\|_\infty.$$

In [4], Fink has given some generalization of Mahajani type inequality as well.

In what follow, we recall the definition of s-convex function in the second sense, usually denoted by K_s^2, that was introduced by Hudzik and Maligranda [5]. This class is defined in the following way : $f : [0,\infty) \to R$ is said to be s-convex function in the second sense if

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y)$$

for all $x, y \in [0,\infty), \alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and for some fixed $s \in (0,1]$. For example, the function $f : [0,1] \to [0,1]$ defined by $f(t) = t^s, s \in (0,1]$, is a s-convex function in the second sense. It can be easily seen that for $s = 1$, s-convexity reduces to ordinary convexity of functions defined on $[0,\infty)$.

In [6], Dragomir and Fitzpatrick proved a variant of Hadamard's inequality which holds for s-convex functions in the second sense. Suppose that $f : [0,\infty) \to [0,\infty)$ is a s-convex function in the second sense and $a, b, \in [0,\infty)$ with $a < b$. If $f \in L^1[a,b]$, then the following inequality holds:

$$2^{s-1}f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{s+1}. \tag{1.2}$$

The constant is the best possible in the second inequality (1.2).

Recently, Alomari et al. [7] have established some Ostrowski type inequalities for the class of functions whose derivatives in absolute value are s-convex functions in the second sense.

In the following, we recall the definition of quasi-convex functions. In [8], this class is defined in the following way : $f : [a,b] \to R$ is a quasi-convex functions if

$$f(\alpha x + (1-\alpha)y) \leq \max\{f(x), f(y)\}$$

for all $x, y \in [a,b], \alpha \in [0,1]$. Clearly, any convex function is a quasi-convex function, and there exist a quasi-convex function which is not convex, see [9].

The purpose of this article is to establish some new results related to the inequality (1.1) for the functions whose absolute value of the first derivative are convex. The corresponding versions in the case that the power of the absolute value of the first derivative is s-convex in the second sense, s-concave in the second sense and quasi-convex, respectively, are also obtained. Applying the obtained results, some new Mahajani type inequalities over any subinterval and some new inequalities for the probability density functions involving moments will be also given.

For convenience, for $a \le x < y \le b$, we use the following notations throughtout this paper:

$$\begin{aligned}
&A(x,y;g) = \int_x^y g(s)ds;\\
&\phi(s;x,y;g) = (s-a)A(s,y;g) - (b-s)A(x,s;g), s \in [x,y];\\
&I(x,y;g) = \frac{(x-a)^2}{3(b-a)} + \frac{1}{|A(x,y;g)|(b-a)(y-x)} \int_x^y |\phi(s;x,y;g)|(y-s)ds;\\
&J(x,y;g) = \frac{(b-y)^2}{3(b-a)} + \frac{1}{|A(x,y;g)|(b-a)(y-x)} \int_x^y |\phi(s;x,y;g)|(s-x)ds;\\
&K(x,y,p;g) = \frac{y-x}{|A(x,y;g)|}\left(\frac{p+1}{y-x}\int_x^y |\phi(s;x,y;g)|^p ds\right)^{\frac{1}{p}}, p > 1.
\end{aligned}$$

where $f : [a,b] \to R$ is an absolutely continuous mapping and as is also $g : [x,y] \to R$.

2. Difference between the integral means

Theorem 2. *Let $f : [a,b] \to R$ be an absolutely continuous mapping and as is also $g : [x,y] \to R$ with $[x,y] \subseteq [a,b]$. Then the following inequalities hold,*

$$(2.1)\quad \left|\frac{1}{A(x,y;g)}\int_x^y g(s)f(s)ds - \frac{1}{b-a}\int_a^b f(s)ds\right|$$
$$\le \begin{cases}
\frac{(x-a)^2}{6(b-a)}|f'(a)| + I(x,y;g)|f'(x)| + J(x,y;g)|f'(y)| + \frac{(b-y)^2}{6(b-a)}|f'(b)|, \\
\qquad\qquad \text{if } |f'| \text{ is convex on } [a,b]; \\[2mm]
\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\frac{\|f'\|_\infty}{b-a}\left[(x-a)^2 + K(x,y,p;g) + (b-y)^2\right], \\
\qquad\qquad \text{if } |f'|^q \in K_s^2 \text{ on } [a,b] \subseteq [0,\infty), q > 1, \frac{1}{p}+\frac{1}{q} = 1\ ; \\[2mm]
\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\frac{2^{\frac{s-1}{q}}}{b-a}\Big[(x-a)^2\left|f'(\frac{a+x}{2})\right| + K(x,y,p;g)\left|f'(\frac{x+y}{2})\right| \\
+(b-y)^2\left|f'(\frac{y+b}{2})\right|\Big], \\
\qquad\qquad \text{if } |f'|^q \text{ is } s\text{-concave on } [a,b] \subseteq [0,\infty), q > 1, \frac{1}{p}+\frac{1}{q} = 1; \\[2mm]
\frac{(x-a)^2}{2(b-a)}\big(\max\{|f'(a)|^q, |f'(x)|^q\}\big)^{\frac{1}{q}} + \frac{(b-y)^2}{2(b-a)}\big(\max\{|f'(y)|^q, |f'(b)|^q\}\big)^{\frac{1}{q}} \\
+\frac{1}{|A(x,y;g)|(b-a)}\int_x^y |\phi(s;x,y;g)|ds\big(\max\{|f'(x)|^q, |f'(y)|^q\}\big)^{\frac{1}{q}}, \\
\qquad\qquad \text{if } |f'|^q \text{ is quasi-convex on } [a,b], q \ge 1.
\end{cases}$$

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Proof. The following first identity has been obtained in [1]. By suitable substitution of variables, we get the following identities.

$$\begin{aligned}
&\frac{1}{A(x,y;g)}\int_x^y g(s)f(s)ds-\frac{1}{b-a}\int_a^b f(s)ds \\
&=\int_a^x \frac{(x-a)}{b-a}f'(s)ds-\frac{1}{(b-a)A(x,y;g)}\int_x^y \phi(s;x,y;g)f'(s)ds+\int_0^1 \frac{(b-y)}{b-a}f'(s)ds \\
&=\frac{(x-a)^2}{b-a}\int_0^1 tf'((1-t)a+tx)dt \\
&-\frac{y-x}{(b-a)A(x,y;g)}\int_0^1 \phi((1-t)x+ty;x,y;g)f'((1-t)x+ty)dt \\
&+\frac{(b-y)^2}{b-a}\int_0^1 (1-t)f'((1-t)y+tb)dt.
\end{aligned} \tag{2.2}$$

Firstly, from (2.2), we obtain

$$\begin{aligned}
&\left|\frac{1}{A(x,y;g)}\int_x^y g(s)f(s)ds-\frac{1}{b-a}\int_a^b f(s)ds\right| \\
&\le \frac{(x-a)^2}{b-a}\int_0^1 t|f'((1-t)a+tx)|dt \\
&+\frac{y-x}{(b-a)|A(x,y;g)|}\int_0^1 \big|\phi((1-t)x+ty;x,y;g)\big|\cdot\big|f'((1-t)x+ty)\big|dt \\
&+\frac{(b-y)^2}{b-a}\int_0^1 (1-t)\big|f'((1-t)y+tb)\big|dt.
\end{aligned} \tag{2.3}$$

Using the convexity of $|f'|$, we get

$$\begin{aligned}
&\frac{(x-a)^2}{b-a}\int_0^1 t|f'((1-t)a+tx)|dt \\
&\le \frac{(x-a)^2}{b-a}\int_0^1 \Big[t(1-t)|f'(a)|+t^2|f'(x)|\Big]dt \\
&=\frac{(x-a)^2}{6(b-a)}|f'(a)|+\frac{(x-a)^2}{3(b-a)}|f'(x)|,
\end{aligned} \tag{2.4}$$

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$$
\begin{aligned}
(2.5)\qquad & \frac{y-x}{(b-a)|A(x,y;g)|}\int_0^1 \big|\phi((1-t)x+ty;x,y;g)\big|\cdot\big|f'((1-t)x+ty)\big|dt \\
&\le \frac{y-x}{(b-a)|A(x,y;g)|}\Bigg[\bigg(\int_0^1 \big|\phi((1-t)x+ty;x,y;g)\big|(1-t)dt\bigg)\big|f'(x)\big| \\
&+\bigg(\int_0^1 \big|\phi((1-t)x+ty;x,y,g)\big|tdt\bigg)\big|f'(y)\big|\Bigg] \\
&\le \frac{1}{|A(x,y;g)|(b-a)(y-x)}\bigg(\int_x^y |\phi(s;x,y;g)|(y-s)ds\bigg)|f'(x)| \\
&+\frac{1}{|A(x,y;g)|(b-a)(y-x)}\bigg(\int_x^y |\phi(s;x,y;g)|(s-x)ds\bigg)|f'(y)|
\end{aligned}
$$

and

$$
\begin{aligned}
(2.6)\qquad & \frac{(b-y)^2}{b-a}\int_0^1 (1-t)f'((1-t)y+tb)dt \\
&\le \frac{(b-y)^2}{b-a}\int_0^1 \Big[(1-t)^2|f'(y)|+(1-t)t|f'(b)|\Big]dt \\
&= \frac{(b-y)^2}{3(b-a)}|f'(y)|+\frac{(b-y)^2}{6(b-a)}|f'(b)|.
\end{aligned}
$$

By combining inequalities (2.3), (2.4), (2.5) and (2.6), we obtain the inequality (2.1) for $|f'|$ is convex.

Secondly, continuing (2.3) and using the Hölder inequality, we obtain that

(2.7)

$$
\begin{aligned}
&\left|\frac{1}{A(x,y)}\int_x^y g(s)f(s)ds-\frac{1}{b-a}\int_a^b f(s)ds\right| \\
&\le \frac{(x-a)^2}{b-a}\bigg(\int_0^1 t^p dt\bigg)^{\frac{1}{p}}\bigg(\int_0^1 |f'((1-t)a+tx)|^q dt\bigg)^{\frac{1}{q}} \\
&+\frac{(y-x)}{(b-a)|A(x,y)|}\bigg(\int_0^1 \big|\phi((1-t)x+ty;x,y;g)\big|^p dt\bigg)^{\frac{1}{p}}\bigg(\int_0^1 |f'((1-t)x+ty)|^q dt\bigg)^{\frac{1}{q}} \\
&+\frac{(b-y)^2}{b-a}\bigg(\int_0^1 (1-t)^p dt\bigg)^{\frac{1}{p}}\bigg(\int_0^1 |f'((1-t)y+tb)|^q dt\bigg)^{\frac{1}{q}}.
\end{aligned}
$$

Now, using the s-convexity of $|f'|^q$ and $f'\in L_\infty[a,b]$, we get

$$
\begin{aligned}
(2.8)\qquad & \int_0^1 |f'((1-t)a+tx)|^q dt \\
&\le \int_0^1 \Big[(1-t)^s|f'(a)|^q+t^s|f'(x)|^q\Big]dt \\
&= \frac{1}{s+1}\Big(|f'(a)|^q+|f'(x)|^q\Big)\le \frac{2\|f'\|_\infty^q}{s+1},
\end{aligned}
$$

$$\begin{aligned} &\int_0^1 |f'((1-t)x+ty)|^q dt \\ &\le \int_0^1 \Big[(1-t)^s|f'(x)|^q + t^s|f'(y)|^q\Big]dt \\ &= \frac{1}{s+1}\Big(|f'(x)|^q + |f'(y)|^q\Big) \le \frac{2\|f'\|_\infty^q}{s+1} \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} &\int_0^1 |f'((1-t)y+tb)|^q dt \\ &\le \int_0^1 \Big[(1-t)^s|f'(y)|^q + t^s|f'(b)|^q\Big]dt \\ &= \frac{1}{s+1}\Big(|f'(y)|^q + |f'(b)|^q\Big) \le \frac{2\|f'\|_\infty^q}{s+1}. \end{aligned} \tag{2.10}$$

By simple computation, we obtain

$$\int_0^1 t^p dt = \frac{1}{p+1}, \tag{2.11}$$

$$\int_0^1 (1-t)^p dt = \frac{1}{p+1} \tag{2.12}$$

and

$$\int_0^1 \big|\phi((1-t)x+ty)\big|^p dt = \frac{1}{y-x}\int_x^y |\phi(s)|^p ds, \tag{2.13}$$

and combining inequalities (2.7)-(2.13), we get the inequality (2.1) for $|f'|^q$ is s-convex.

Thirdly, using the s-concavity of $|f'|^q$ and the first inequality of (1.2), we obtain

$$\int_0^1 |f'((1-t)a+tx)|^q dt \le 2^{s-1}\Big|f'\big(\frac{a+x}{2}\big)\Big|^q, \tag{2.14}$$

$$\int_0^1 |f'((1-t)x+ty)|^q dt \le 2^{s-1}\Big|f'\big(\frac{x+y}{2}\big)\Big|^q, \tag{2.15}$$

and

$$\int_0^1 |f'((1-t)y+tb)|^q dt \le 2^{s-1}\Big|f'\big(\frac{y+b}{2}\big)\Big|^q. \tag{2.16}$$

By combining inequalities (2.7), (2.11)-(2.16), we have the inequality (2.1) for the case $|f'|^q$ is s-concave.

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Finally, by (2.3), and using Hölder inequality and the quasi-convexity of $|f'|^q$, we have

$$\begin{aligned}
&\left|\frac{1}{A(x,y)}\int_x^y g(s)f(s)ds - \frac{1}{b-a}\int_a^b f(s)ds\right| \\
&\le \frac{(x-a)^2}{b-a}\left(\int_0^1 tdt\right)^{\frac{q-1}{q}}\left(\int_0^1 t|f'((1-t)a+tx)|^q dt\right)^{\frac{1}{q}} \\
&+ \frac{y-x}{(b-a)|A(x,y;g)|}\left(\int_0^1 \big|\phi((1-t)x+ty;x,y;g)\big|dt\right)^{\frac{q-1}{q}} \\
&\times\left(\int_0^1 \big|\phi((1-t)x+ty;x,y;g)\big||f'((1-t)x+ty)|^q dt\right)^{\frac{1}{q}} \\
&+ \frac{(b-y)^2}{b-a}\left(\int_0^1 (1-t)dt\right)^{\frac{q-1}{q}}\left(\int_0^1 |f'((1-t)y+tb)|^q dt\right)^{\frac{1}{q}} \\
&\le \frac{(x-a)^2}{b-a}\left(\int_0^1 tdt\right)\max\{|f'(a)|^q,|f'(x)|^q\}^{\frac{1}{q}} \\
&+ \frac{(b-y)^2}{b-a}\left(\int_0^1 (1-t)dt\right)\max\{|f'(y)|^q,|f'(b)|^q\}^{\frac{1}{q}} \\
&+ \frac{y-x}{|A(x,y;g)|(b-a)}\left(\int_0^1 \big|\phi((1-t)x+ty;x,y;g)\big|dt\right)\max\{|f'(x)|^q,|f'(y)|^q\}^{\frac{1}{q}}.
\end{aligned}$$

By combining above inequalities and (2.11)-(2.16) for $p=1$, we have the inequality (2.1) for the case $|f'|^q$ is quasi-convex. This completes the proofs of Theorem 2. ∎

Remark 1. *Using the definitions of* $\|f'\|_\infty, I(x,y;g)$ *and* $J(x,y;g)$ *, we obtain*

$$\begin{aligned}
&\frac{(x-a)^2}{6(b-a)}|f'(a)| + I(x,y;g)|f'(x)| + J(x,y;g)|f'(y)| + \frac{(b-y)^2}{6(b-a)}|f'(b)|, \\
&\le \|f'\|_\infty\left[\frac{(x-a)^2}{6(b-a)} + \frac{(x-a)^2}{3(b-a)} + \frac{1}{|A(x,y;g)|(b-a)(y-x)}\int_x^y |\phi(s;x,y;g)|(y-s)ds\right. \\
&\left.+ \frac{(b-y)^2}{3(b-a)} + \frac{1}{|A(x,y;g)|(b-a)(y-x)}\int_x^y |\phi(s;x,y;g)|(s-x)ds + \frac{(b-y)^2}{6(b-a)}\right] \\
&= \frac{\|f'\|_\infty}{(b-a)}\left\{\frac{\int_x^y g(s)ds}{2}\left[(x-a)^2+(b-y)^2\right] + \int_x^y |\phi(s;x,y;g)|ds\right\}.
\end{aligned}$$

Therefore, for the strick convex mapping, we get the bound in the first inequality of (2.1) *is smaller than the one in the first inequality* (1.1). *By the similar computation, we also have the bound in the last inequality of* (2.1) *is smaller than the one in the first inequality* (1.1).

The following corollary gives an new estimate for the difference between weighted and unweighted integral means.

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Corollary 1. *Assume that hypotheses in Theorem 2 hold. Then we have*

$$
\begin{aligned}
&\left|\frac{1}{\int_a^b g(u)du}\int_a^b g(u)f(u)du - \frac{1}{b-a}\int_a^b f(u)du\right| \qquad (2.17)\\
&\leq \begin{cases}
I(a,b;g)|f'(a)| + J(a,b;g)|f'(b)|, \\
\hfill \text{if } |f'| \text{ is convex on } [a,b]; \\
\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\frac{\|f'\|_\infty}{b-a}K(a,b,p;g), \\
\hfill \text{if } |f'|^q \in K_s^2 \text{ on } [a,b]\subseteq[0,\infty),\ q>1, \frac{1}{p}+\frac{1}{q}=1; \\
\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\frac{2^{\frac{s-1}{q}}}{b-a}\left|K(a,b,p;g)\right|\left|f'(\frac{a+b}{2})\right|, \\
\hfill \text{if } |f'|^q \text{ is } s\text{-concave on } [a,b]\subseteq[0,\infty), q>1, \frac{1}{p}+\frac{1}{q}=1; \\
\frac{1}{|A(a,b;g)|(b-a)}\int_a^b\left|\phi(s;x,y;g)\right|ds\big(\max\{|f'(x)|^q,|f'(y)|^q\}\big)^{\frac{1}{q}}, \\
\hfill \text{if } |f'|^q \text{ is quasi-convex on } [a,b], q\geq 1.
\end{cases}
\end{aligned}
$$

Proof. Set $x=a$ and $y=b$ in Theorem 2 produces the result (2.17). ∎

The following corollary gives bounds for the difference between the mean of a function compared to its mean over a subinterval.

Corollary 2. *Let $f:[a,b]\to R$ be an absolutely continuous mapping and $a\leq x<y\leq b$. Then we have*

$$
\begin{aligned}
&\left|\frac{1}{y-x}\int_x^y f(u)du - \frac{1}{b-a}\int_a^b f(u)du\right| \qquad (2.18)\\
&\leq \begin{cases}
\frac{(x-a)^2}{6(b-a)}|f'(a)| + I(x,y;1)|f'(x)| + J(x,y;1)|f'(y)| + \frac{(b-y)^2}{6(b-a)}|f'(b)|, \\
\hfill \text{if } |f'| \text{ is convex on } [a,b]; \\
\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\frac{\|f'\|_\infty}{b-a}\Big[(x-a)^2+K(x,y,p;1)+(b-y)^2\Big], \\
\hfill \text{if } |f'|^q \in K_s^2 \text{ on } [a,b]\subseteq[0,\infty),\ q>1, \frac{1}{p}+\frac{1}{q}=1; \\
\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\frac{2^{\frac{s-1}{q}}}{b-a}\Big[(x-a)^2\left|f'(\frac{a+x}{2})\right| + K(x,y,p;1)\left|f'(\frac{x+y}{2})\right| \\
+(b-y)^2\left|f'(\frac{y+b}{2})\right|\Big], \\
\hfill \text{if } |f'|^q \text{ is } s\text{-concave on } [a,b]\subseteq[0,\infty), q>1, \frac{1}{p}+\frac{1}{q}=1; \\
\frac{(x-a)^2}{2(b-a)}\big(\max\{|f'(a)|^q,|f'(x)|^q\}\big)^{\frac{1}{q}} \\
+\frac{(b-y)^2}{2(b-a)}\big(\max\{|f'(y)|^q,|f'(b)|^q\}\big)^{\frac{1}{q}} \\
+\frac{1}{(y-x)(b-a)}\int_x^y|\phi(s;x,y;1)|ds\big(\max\{|f'(x)|^q,|f'(y)|^q\}\big)^{\frac{1}{q}}, \\
\hfill \text{if } |f'|^q \text{ is quasi-convex on } [a,b], q\geq 1.
\end{cases}
\end{aligned}
$$

Proof. Taking $g\equiv 1$ in Theorem 2, we have the (2.18) immediately. ∎

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Remark 2. *The type of inequality in* (2.18) *has been applied to probability density functions, special means, Jeffreys divergence in Information Theorem and the sampling of continuous streams in Statistics, see* [10, 11, 12, 13].

Corollary 3. *Assume that the hypotheses in Corollary 2 hold and* $\int_a^b f(u)du = 0$. *Then we have*

$$
\left|\int_x^y f(u)du\right| \tag{2.19}
$$

$$
\leq \begin{cases}
(y-x)\Big[\frac{(x-a)^2}{6(b-a)}|f'(a)| + I(x,y;1)|f'(x)| + J(x,y;1)|f'(y)| \\
+\frac{(b-y)^2}{6(b-a)}|f'(b)|\Big], \qquad \textit{if } |f'| \textit{ is convex on } [a,b]; \\[2ex]
(y-x)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\frac{\|f'\|_\infty}{b-a}\Big[(x-a)^2 + K(x,y,p;1) \\
+(b-y)^2\Big], \textit{if } |f'|^q \in K_s^2 \textit{ on } [a,b] \subseteq [0,\infty),\ q>1, \frac{1}{p}+\frac{1}{q}=1; \\[2ex]
\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\frac{2^{\frac{s-1}{q}}(y-x)}{b-a}\Big[(x-a)^2\big|f'(\frac{a+x}{2})\big| + K(x,y,p;1) \\
\times\big|f'(\frac{x+y}{2})\big| + (b-y)^2\big|f'(\frac{y+b}{2})\big|\Big], \\
\qquad \textit{if } |f'|^q \textit{ is s-concave on } [a,b]\subseteq[0,\infty), q>1, \frac{1}{p}+\frac{1}{q}=1; \\[2ex]
(y-x)\Big[\frac{(x-a)^2}{2(b-a)}\max\{|f'(a)|^q, |f'(x)|^q\}^{\frac{1}{q}} \\
+\frac{(b-y)^2}{2(b-a)}\max\{|f'(y)|^q, |f'(b)|^q\}^{\frac{1}{q}}\Big] \\
+\frac{1}{(b-a)}\int_x^y|\phi(s;x,y;1)|ds\big(\max\{|f'(x)|^q, |f'(y)|^q\}\big)^{\frac{1}{q}}, \\
\qquad \textit{if } |f'|^q \textit{ is quasi-convex on } [a,b], q\geq 1.
\end{cases}
$$

Proof. From Corollary 2, putting $\int_a^b f(t)dt = 0$ and multiplying both sides by $y-x$, we have the desire inequality (2.19) immediately. ∎

Remark 3. *The inequality in* (2.19) *is Mahajani type inequality over any subinterval, and if* $\int_a^b f(t)dt = 0$ *in* (2.1) *and* (2.17) *then they may be looked upon as weighted Mahajani type inequalities over arbitrary subintervals.*

3. Applications involving moments

In this section we investigate inequalities involving moments.

Theorem 3. *Let $f : [a, b] \to R$ be an absolutely continuous mapping, $\gamma \in R$ and $[x, y] \subseteq [a, b]$. Then the following inequalities hold,*

$$\left|\int_x^y (t-\gamma)^n f(t)dt - \frac{1}{b-a}\int_a^b f(s)ds \int_x^y (t-\gamma)^n dt\right| \tag{3.1}$$

$$\leq \begin{cases} A(x,y;(t-\gamma)^n)\Big[\frac{(x-a)^2}{6(b-a)}|f'(a)| + I(x,y;(t-\gamma)^n)|f'(x)| \\ +J(x,y;(t-\gamma)^n)|f'(y)| + \frac{(b-y)^2}{6(b-a)}|f'(b)|\Big], \\ \qquad \text{if } |f'| \text{ is convex on } [a,b]; \\ A(x,y;(t-\gamma)^n)\Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\Big(\frac{2}{s+1}\Big)^{\frac{1}{q}}\frac{\|f'\|_\infty}{b-a}\Big[(x-a)^2 \\ +K(x,y,p;(t-\gamma)^n) + (b-y)^2\Big], \\ \qquad \text{if } |f'|^q \in K_s^2 \text{ on } [a,b] \subseteq [0,\infty),\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ A(x,y;(t-\gamma)^n)\Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\frac{2^{\frac{s-1}{q}}}{b-a}\Big[(x-a)^2\big|f'(\frac{a+x}{2})\big| \\ +K(x,y,p;(t-\gamma)^n)\big|f'(\frac{x+y}{2})\big| + (b-y)^2\big|f'(\frac{y+b}{2})\big|\Big], \\ \qquad \text{if } |f'|^q \text{ is } s\text{-concave on } [a,b] \subseteq [0,\infty), q>1, \frac{1}{p}+\frac{1}{q}=1; \\ \frac{(x-a)^2}{2(b-a)}\Big(\max\{|f'(a)|^q, |f'(x)|^q\}\Big)^{\frac{1}{q}} \\ +\frac{(b-y)^2}{2(b-a)}\Big(\max\{|f'(y)|^q, |f'(b)|^q\}\Big)^{\frac{1}{q}} \\ +\frac{1}{(b-a)}\int_x^y |\phi(s;x,y;(t-\gamma)^n)|ds \\ \times\Big(\max\{|f'(x)|^q, |f'(y)|^q\}\Big)^{\frac{1}{q}}, \\ \qquad \text{if } |f'|^q \text{ is quasi-convex on } [a,b], q \geq 1. \end{cases}$$

Proof. Taking $g(t) = (t-\gamma)^n$ in (2.1) and multiplying both sides by $A(x, y; (t-\gamma)^n)$, we have the desire results. ∎

Corollary 4. *Let $f : [a, b] \to R$ be an absolutely continuous mapping, $\gamma \in R$ and $[x, y] \subseteq [a, b]$. Then we have*

$$\left|\int_a^b (t-\gamma)^n f(t)dt - \frac{1}{b-a}\int_a^b f(s)ds \int_a^b (t-\gamma)^n dt\right| \tag{3.2}$$

$$\leq \begin{cases} A(a,b;(t-\gamma)^n)\big[I(a,b;(t-\gamma)^n)|f'(a)| \\ +J(a,b;(t-\gamma)^n)|f'(b)|\big], \qquad \text{if } |f'| \text{ is convex on } [a,b]; \\ A(a,b;(t-\gamma)^n)\Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\Big(\frac{2}{s+1}\Big)^{\frac{1}{q}}\frac{\|f'\|_\infty}{b-a}K(a,b,p;(t-\gamma)^n), \\ \qquad \text{if } |f'|^q \in K_s^2 \text{ on } [a,b] \subseteq [0,\infty),\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ A(a,b;(t-\gamma)^n)\Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\frac{2^{\frac{s-1}{q}}}{b-a}| + K(a,b,p;(t-\gamma)^n)\Big|f'(\frac{a+b}{2})\Big|, \\ \qquad \text{if } |f'|^q \text{ is } s\text{-concave on } [a,b] \subseteq [0,\infty),\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{(b-a)}\int_a^b |\phi(s;a,b;(t-\gamma)^n)|ds\Big(\max\{|f'(a)|^q, |f'(b)|^q\}\Big)^{\frac{1}{q}}, \\ \qquad \text{if } |f'|^q \text{ is quasi-convex on } [a,b], q \geq 1. \end{cases}$$

Proof. Taking $x = a$ and $y = b$ in Theorem 3, inequality (3.1) reduce to inequality (3.2) obviously. ∎

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Remark 4. *The Theorem 3 and Corollary 4 give the new estimates of errors for the general moments over a interral and subinterval, and we note that the first and the last inequality in* (3.2) *are better than the first inequality of (3.10) in* [1].

Corollary 5. *Let* $f : [a, b] \to R$ *be an absolutely continuous p.d.f. associated with a random variable* X*. Then the expectation* $E(X)$ *satisfies the inequalities*

$$\left|E(X) - \frac{a+b}{2}\right| \tag{3.3}$$
$$\leq \begin{cases} \frac{b^2-a^2}{2}\Big[I(a,b;t)|f'(a)| + J(a,b;t)|f'(b)|\Big], \\ \hfill \textit{if } |f'| \textit{ is convex on } [a,b]; \\[2ex] \frac{b^2-a^2}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\frac{\|f'\|_\infty}{b-a}K(a,b,p;t), \\ \hfill \textit{if } |f'|^q \in K_s^2 \textit{ on } [a,b] \subseteq [0,\infty),\ q>1, \frac{1}{p}+\frac{1}{q}=1; \\[2ex] \frac{b^2-a^2}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\frac{2^{\frac{s-1}{q}}}{b-a} + K(a,b,p;t)\big|f'(\frac{a+b}{2})\big|, \\ \hfill \textit{if } |f'|^q \textit{ is s-concave on } [a,b] \subseteq [0,\infty), q>1, \frac{1}{p}+\frac{1}{q}=1; \\[2ex] \frac{1}{b-a}\int_a^b |\phi(s;a,b;t)|ds\Big(\max\{|f'(a)|^q, |f'(b)|^q\}\Big)^{\frac{1}{q}}, \\ \hfill \textit{if } |f'|^q \textit{ is quasi-convex on } [a,b], q \geq 1. \end{cases}$$

Proof. Taking $n = 1$ and $\gamma = 0$ in Corollary 4, gives (3.3). ∎

Remark 5. *The Corollary 5 give a new estimates of errors for expectation, and the bound of the first and the last inequality in* (3.3) *are both smaller than one in the first inequality of (3.22) in* [1].

Corollary 6. *Let* $f : [a, b] \to R$ *be an absolutely continuous p.d.f. associated with a random variable* X*. Then the variance* $\sigma^2(X)$ *satisfies the inequalities*

$$\left|\sigma^2(X) - \frac{(b-E(X))^3 - (a-E(X))^3}{3(b-a)}\right| \tag{3.4}$$
$$\leq \begin{cases} A(a,b;(t-E(X)^2)\big[I(a,b;(t-E(X))^2)|f'(a)| \\ +J(a,b;(t-E(X))^2)|f'(b)|\big], \hfill \textit{if } |f'| \textit{ is convex on } [a,b]; \\[2ex] A(a,b;(t-E(X)^2)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\frac{\|f'\|_\infty}{b-a}K(a,b,p;(t-E(X))^2), \\ \hfill \textit{if } |f'|^q \in K_s^2 \textit{ on } [a,b] \subseteq [0,\infty),\ q>1, \frac{1}{p}+\frac{1}{q}=1; \\[2ex] A(a,b;(t-E(X)^2)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\frac{2^{\frac{s-1}{q}}}{b-a}K(a,b,p;(t-E(X))^2)\Big|f'(\frac{a+b}{2})\Big|, \\ \hfill \textit{if } |f'|^q \textit{ is s-concave on } [a,b] \subseteq [0,\infty), q>1, \frac{1}{p}+\frac{1}{q}=1; \\[2ex] \frac{1}{b-a}\int_a^b |\phi(s;a,b;(t-E(X))^2)|ds\Big(\max\{|f'(a)|^q, |f'(b)|^q\}\Big)^{\frac{1}{q}}, \\ \hfill \textit{if } |f'|^q \textit{ is quasi-convex on } [a,b], q \geq 1. \end{cases}$$

Proof. Taking $n = 2$ and $\gamma = E(X)$ in Corollary 4, gives (3.4). ∎

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Remark 6. *The* (3.4) *give a new estimates of errors for variance, and the first and the last inequality in* (3.4) *are both better than the first inequality of (3.23) in* [1].

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A note on the relaxed Newton-like method for nonsymmetric algebraic Riccati equation *

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Abstract

The non-symmetric algebraic Riccati equation arising in transport theory can be rewritten as a vector equation and the minimal positive solution of the non-symmetric algebraic Riccati equation can be obtained by solving the vector equation. In this paper, a remark on the parameter of the relaxed Newton-like method is given, and the range of the parameter is discussed. In fact, the range of the parameter can be lager, i.e., the modulus of the parameter λ can be more than 1.

Keywords: Non-symmetric algebraic Riccati equation, Relaxed Newton-like method, Parameter

AMS classification: 65F10

1 Introduction

The following non-symmetric algebraic Riccati equation can be obtained by discretizing integral equation in transport theory [1, 2]

$$XCX - XE - AX + B = 0, \tag{1}$$

where $A,\ B,\ C,\ E \in R^{n\times n}$ have the following special form:

$$A = \triangle - eq^T,\ B = ee^T,\ C = qq^T,\ E = D - qe^T. \tag{2}$$

Here and in the following, $e = (1, 1, \ldots, 1)^T$ with $q_i = \frac{c_i}{2w_i}$, $\triangle = diag(\delta_1, \delta_2, \ldots, \delta_n)^T$ with $\delta_i = \frac{1}{c_i w_i(1+\alpha)}$, $D = diag(d_1, d_2, \ldots, d_n)^T$ with $d_i = \frac{1}{c_i w_i(1+\alpha)}$, and $0 < w_n < \cdots < w_2 < w_1$, $\sum_{i=1}^n c_i = 1,\ c_i > 0,\ i = 1, 2, ..., n.$

The form of the Riccati equation (1) arises in Markov models [3], and in nuclear physics [1, 4], and it has many positive solutions in the componentwise sense. The existence of positive solutions of equation (1) has been shown in [1]. Equation (1) has many positive solutions, but only the minimal positive solution is physically meaningful. So it is important to develop some effective and efficient methods to compute the minimal positive solution of equation (1).

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Recently, Lu [5] has shown that the matrix equation (1) is equivalent to a vector equation and has developed a simple and efficient iterative method to compute the minimal positive solution of equation (1). The Newton method has been presented and analyzed by Lu in [6] for solving the vector equation. In [7], Lin et al applied the modified Newton method presented in [8] for solving the vector equation, and proposed the modified Newton method. In [9], based on the relaxation technique, Li et al further studied the Newton method and proposed a relaxed Newton-like method for solving the vector equation. Since the relaxed Newton-like method contains a parameter λ, and the convergence analysis of the relaxed Newton-like method has been given when the modulus of the parameter λ is less than 1. In fact, the modulus of the parameter λ can be more than 1. In this note, we give more detailed analysis on the parameter λ of the relaxed Newton-like method.

2 Some notions and Lemmas

Before we give the main result, firstly, we give some notions and Lemmas For any matrices $A = [a_{i,j}]$ and $B = [b_{i,j}] \in R^{m\times n}$, we write $A \geq B(A > B)$ if $a_{i,j} \geq b_{i,j}(a_{i,j} > b_{i,j})$ holds for all i, j. The Hadamard product of A and B is defined by $A \circ B = [a_{i,j} \cdot b_{i,j}]$. A real square matrix A is called a Z-matrix if all its off-diagonal elements are non-positive. The superscript T denotes the transpose of a vector or a matrix. I denotes the identity matrix with appropriate dimension.

The following Lemma will be used later.

Lemma 1 [10] *For a Z-matrix A, the following statements are equivalent:*
(1) A is a nonsingular M-matrix;
(2) A is nonsingular and $A^{-1} \geq 0$;
(3) $Av > 0$ for some vector $v \geq 0$.

It has been shown in [5, 6] that the solution of (1) must have the following form:

$$X = T \circ (uv^T) = (uv^T) \circ T,$$

where $T = [t_{i,j}] = [1/(\delta_i + d_j)]$ and u, v are two vectors, which satisfy the vector equations:

$$\begin{cases} u = u \circ (Pv) + e, \\ v = v \circ (\tilde{P}u) + e, \end{cases} \tag{3}$$

where $P = [p_{i,j}] = [q_j/(\delta_i + d_j)]$, $\tilde{P} = [\tilde{p}_{i,j}] = [q_j/(\delta_j + d_i)]$. Define $w = [u^T, v^T]^T$. Then the equation (3) can be rewritten equivalently as

$$f(w) = w - w \circ \mathcal{P}w - e = 0, \tag{4}$$

where

$$\mathcal{P} = \begin{bmatrix} 0 & P \\ \tilde{P} & 0 \end{bmatrix}.$$

The minimal positive solution of (1) can be obtained by computing the minimal positive solution of the vector equation (4).

In [9], Li et al proposed the following relaxed Newton-like method for solving the vector equation (4), which is defined as follows:

Algorithm 1 (**The relaxed Newton-like method**) [9] *For $k = 0, 1, 2, ...$ and real parameter $|\lambda| < 1$, the relaxed Newton-like method method is defined as follows:*

$$\begin{cases} \bar{w}_k = w_k + f'(w_k)^{-1} f(w_k), \\ \tilde{w}_k = (1-\lambda) w_k + \lambda \bar{w}_k, \\ w_{k+1} = \tilde{w}_k - f'(w_k)^{-1} f(\tilde{w}_k). \end{cases} \tag{5}$$

In fact, the modulus of parameter λ can range greater than 1. In the following, we will give the analysis. Before we give the convergence analysis of the relaxed Newton-like method with corresponding parameter λ, let us now state some results which indispensable for our subsequent discussions.

Lemma 2 [6] *For any vectors $w_+, w \in R^{2n}$, we have*

$$f(w_+) = f(w) + f'(w)(w_+ - w) + \frac{1}{2} f''(w)(w_+ - w, w_+ - w). \tag{6}$$

In particular, if $w_+ = w_*$, the minimal positive solution of (4), then

$$0 = f(w) + f'(w)(w_* - w) + \frac{1}{2} f''(w)(w_* - w, w_* - w). \tag{7}$$

Furthermore, for any $y > 0$ or $y < 0$,

$$f''(w) y^2 < 0 \tag{8}$$

and $f''(w)y^2$ is independent of w.

Lemma 3 [6] *If $0 \le w < w_*$ and $f(w) < 0$, then $f'(w)$ is a nonsingular M-matrix.*

3 Main result

Now, we give the convergence analysis of the relaxed Newton-like method when the modulus of parameter λ range greater than 1.

Theorem 1 *Given a vector $w_k \in R^{2n}$. w_{k+1} are obtained by the relaxed Newton-like method method (5), where the modulus of the parameter λ can be greater than 1. For appropriate parameter λ, if $0 \le w_k < w_*$ and $f(w_k) < 0$, then $w_k < w_{k+1} < w_*$ and $f(w_{k+1}) < 0$, moreover, $f'(w_{k+1})$ is a nonsingular M-matrix.*

Proof. Since $w_k < w_*$ and $f(w_k) < 0$, by Lemma 3, we can easily obtain that $f'(w_k)$ is a nonsingular M-matrix. By Lemma 1, we have $f'(w_k)^{-1} \ge 0$.

By Eq. (8), we can get that

$$\tilde{w}_k = w_k + \lambda f'(w_k)^{-1} f(w_k),\ w_{k+1} = w_k + f'(w_k)^{-1}[\lambda f(w_k) - f(\tilde{w}_k)].$$

Let $e_k = w_k - w_*,\ h_k = f'(w_k)^{-1} f(w_k),\ r_k = f'(w_k)^{-1}[\lambda f(w_k) - f(\tilde{w}_k)]$, then

$$e_k < h_k < 0,\ \tilde{w}_k = w_k + \lambda h_k,\ w_{k+1} = w_k + r_k \tag{9}$$

3

Let e_{ki}, h_{ki} and r_{ki} be the corresponding i-components $(i = 1, 2, \cdots, 2n)$ of the vectors e_k, h_k and r_k, respectively. It follows from equation (6) that

$$\begin{aligned} f(\tilde{w}_k) &= f(w_k + \lambda h_k) \\ &= f(w_k) + f'(w_k)\lambda h_k + \frac{1}{2}f''(w_k)(\lambda h_k)^2 \\ &= f(w_k) + \lambda f(w_k) + \frac{1}{2}f''(w_k)(\lambda h_k)^2 \\ &= (\lambda + 1)f(w_k) + \frac{1}{2}f''(w_k)(\lambda h_k)^2. \end{aligned} \tag{10}$$

By Eq. (10), we have $\lambda f(w_k) - f(\tilde{w}_k) = -f(w_k) - \frac{1}{2}f''(w_k)(\lambda h_k)^2 > 0$, it means that $r_k > 0$, $i.e., w_{k+1} > w_k$.

By Lemma 2, equation (9) and (10), we can get the following error vector equation

$$\begin{aligned} e_{k+1} &= e_k + r_k \\ &= f'(w_k)^{-1}\left[f(w_k) + \frac{1}{2}f''(w_k)e_k^2\right] + f'(w_k)^{-1}[\lambda f(w_k) - f(\tilde{w}_k)] \\ &= f'(w_k)^{-1}\left[(\lambda + 1)f(w_k) - f(\tilde{w}_k) + \frac{1}{2}f''(w_k)e_k^2\right] \\ &= \frac{1}{2}f'(w_k)^{-1}f''(w_k)[e_k^2 - (\lambda h_k)^2] \end{aligned}$$

If we choose parameter λ satisfied $\lambda^2 < \min\left\{\frac{e_{ki}^2}{h_{ki}^2}\right\}$, then $e_{k+1} < 0$. Hence, $w_{k+1} < w_*$.

By Lemma 2, equation (9) and (10), we obtain that

$$\begin{aligned} f(w_{k+1}) &= f(w_k + r_k) \\ &= f(w_k) + f'(w_k)r_k + \frac{1}{2}f''(w_k)r_k^2 \\ &= (\lambda + 1)f(w_k) - f(\tilde{w}_k) + \frac{1}{2}f''(w_k)r_k^2 \\ &= \frac{1}{2}f''(w_k)\left[r_k^2 - (\lambda h_k)^2\right]. \end{aligned} \tag{11}$$

By simple computation, we know that

$$\begin{aligned} r_k + h_k &= f'(w_k)^{-1}[(\lambda + 1)f(w_k) - f(\tilde{w}_k)] \\ &= f'(w_k)^{-1}\left[-\frac{1}{2}f''(w_k)(\lambda h_k)^2\right] \\ &= f_k > 0, \end{aligned}$$

It means that

$$\frac{r_{ki}}{h_{ki}} = -1 + \frac{f_{ki}}{h_{ki}} < -1.$$

If we choose parameter λ satisfied $\lambda^2 < \min\left\{\frac{r_{ki}^2}{h_{ki}^2}\right\}$, by equation (11), we can get $f(w_{k+1}) < 0$. By Lemma 3, it can be concluded that $f'(w_{k+1})$ is a nonsingular M-matrix. Hence, if we choose parameter λ satisfied $\lambda^2 < \min\left\{\frac{r_{ki}^2}{h_{ki}^2}, \frac{e_{ki}^2}{h_{ki}^2}\right\}$, $(i = 1, 2, \cdots, 2n)$, we know that the modulus of parameter λ can range greater than 1. From above analysis, the proof of the theorem 1 is completed. □

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FIXED POINT THEOREM IN MULTI-BANACH ALGEBRAS

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Abstract. In this paper, we apply a fixed point theorem to solve the operator equation $AxBx = x$ in multi-Banach algebras under a nonlinear contraction.

1. Introduction

It is known that the first important hybrid fixed point theorem due to Krasnoselskii [1] which combines the metric fixed point theorem of Banach with the topological fixed point theorem of Schauder in a Banach space has several applications to nonlinear integral equations that arise in the inversion of the perturbed differential equations. Many attempts have been made to improve and weaken the hypotheses of Krasnoselskii's fixed point theorem. See [2] and the references therein. The case with the Krasnoselskii type fixed point theorem of the present author [3] in Banach algebras is similar. The study of the nonlinear integral equations in Banach algebras was initiated by Dhage [4] via fixed point theorems (see also [3, 4, 5]).

2. Multi-normed spaces

The notion of multi-normed space was introduced by Dales and Polyakov in [6]. This concept is somewhat similar to operator sequence space and has some connections with operator spaces and Banach lattices. Motivations for the study of multi-normed spaces and many examples are given in [6, 7, 8].

Let $(\mathcal{E}, \|.\|)$ be a complex normed space, and let $k \in \mathbb{N}$. We denote by $\mathcal{E}^k$ the linear space $\mathcal{E} \oplus \cdots \oplus \mathcal{E}$ consisting of k-tuples $(x_1, \cdots, x_k)$, where $x_1, \cdots, x_k \in \mathcal{E}$. The linear operations on $\mathcal{E}^k$ are defined coordinate-wise. The zero element of either $\mathcal{E}$ or $\mathcal{E}^k$ is denoted by 0. We denote by $\mathbb{N}_k$ the set $\{1, 2, \cdots, k\}$ and by Σ_k the group of permutations on k symbols.

Definition 2.1. A multi-norm on $\{\mathcal{E}^k : k \in \mathbb{N}\}$ is a sequence

$$(\|.\|_k) = (\|.\|_k : k \in \mathbb{N})$$

such that $\|.\|_k$ is a norm on $\mathcal{E}^k$ for each $k \in \mathbb{N}$:

(A1) $\|(x_{\sigma(1)}, ..., x_{\sigma(k)})\|_k = \|(x_1, ..., x_k)\|_k \qquad (\sigma \in \Sigma_k, x_1, ..., x_k \in \mathcal{E})$;

(A2) $\|(\alpha_1 x_1, ..., \alpha_k x_k)\|_k \le (\max_{i \in \mathbb{N}_k} |\alpha_i|) \|x_1, ..., x_k\|_k$
$(\alpha_1, ..., \alpha_k \in \mathbb{C},\ x_1, ..., x_k \in \mathcal{E})$;

(A3) $\|(x_1, ..., x_{k-1}, 0)\|_k = \|(x_1, ..., x_{k-1})\|_{k-1} \qquad (x_1, ..., x_{k-1} \in \mathcal{E})$;

(A4) $\|(x_1, ..., x_{k-1}, x_{k-1})\|_k = \|(x_1, ..., x_{k-1})\|_{k-1} \qquad (x_1, ..., x_{k-1} \in \mathcal{E})$.

In this case, we say that $((\mathcal{E}^k, \|.\|_k) : k \in \mathbb{N})$ is a multi-normed space.

Lemma 2.2. [8] *Suppose that $((\mathcal{E}^k, \|.\|_k) : k \in \mathbb{N})$ is a multi-normed space, and take $k \in \mathbb{N}$. Then*

(a) $\|(x, ..., x)\|_k = \|x\| \ (x \in \mathcal{E})$;

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(b) $\max_{i\in\mathbb{N}_k}\|x_i\| \le \|(x_1,...,x_k)\|_k \le \sum_{i=1}^k \|x_i\| \le k\max_{i\in\mathbb{N}_k}\|x_i\|$ $(x_1,...,x_k\in\mathcal{E})$.

It follows from (b) that, if $(\mathcal{E},\|.\|)$ is a Banach space, then $(\mathcal{E}^k,\|.\|_k)$ is a Banach space for each $k\in\mathbb{N}$; in this case $((\mathcal{E}^k,\|.\|_k):k\in\mathbb{N})$ is a multi-Banach space.

Now we state two important examples of multi-norms for an arbitrary normed space $\mathcal{E}$; cf. [6].

Example 2.3. *The sequence $(\|.\|_k : k\in\mathbb{N})$ on $\{\mathcal{E}^k : k\in\mathbb{N}\}$ defined by*

$$\|x_1,...,x_k\|_k := \max_{i\in\mathbb{N}_k}\|x_i\| \qquad (x_1,...,x_k\in\mathcal{E})$$

is a multi-norm called the minimum multi-norm. The terminology 'minimum' is justified by property (b).

Example 2.4. *Let $\{(\|.\|_k^\alpha : k\in\mathbb{N}) : \alpha\in A\}$ be the (non-empty) family of all multi-norms on $\{\mathcal{E}^k : k\in\mathbb{N}\}$. For $k\in\mathbb{N}$, set*

$$|||x_1,...,x_k|||_k := \sup_{\alpha\in A}\|(x_1,...x_k)\|_k^\alpha \ (x_1,...,x_k\in\mathcal{E}).$$

Then $(|||.|||_k : k\in\mathbb{N})$ is a multi-norm on $\{\mathcal{E}^k : k\in\mathbb{N}\}$, called the maximum multi-norm.

We need the following observation which can be easily deduced from the triangle inequality for the norm $\|.\|_k$ and the property (b) of multi-norms.

Lemma 2.5. *Suppose that $k\in\mathbb{N}$ and $(x_1,...,x_k)\in\mathcal{E}^k$. For each $j\in\{1,...,k\}$, let $(x_n^j)_{n=1,2,...}$ be a sequence in $\mathcal{E}$ such that $\lim_{n\to\infty}x_n^j = x_j$. Then for each $(y_1,...,y_k)\in\mathcal{E}^k$ we have*

$$\lim_{n\to\infty}(x_n^1-y_1,...,x_n^k-y_k) = (x_1-y_1,...,x_k-y_k).$$

Definition 2.6. Let $((\mathcal{E}^k,\|.\|_k):k\in\mathbb{N})$ be a multi-normed space. A sequence (x_n) in $\mathcal{E}$ is a *multi-null* sequence if, for each $\varepsilon>0$, there exists $n_0\in\mathbb{N}$ such that

$$\sup_{k\in\mathbb{N}}\|(x_n,...,x_{n+k-1})\|_k < \varepsilon \ (n\ge n_0).$$

Let $x\in\mathcal{E}$. We say that the sequence (x_n) is *multi-convergent* to $x\in\mathcal{E}$ and write

$$\lim_{n\to\infty}x_n = x$$

if (x_n-x) is a multi-null sequence.

Definition 2.7. [6, 9] Let $(\mathcal{A},\|.\|)$ be a normed algebra such that $((\mathcal{A}^k,\|.\|_k):k\in\mathbb{N})$ is a multi-normed space. Then $((\mathcal{A}^k,\|.\|_k):k\in\mathbb{N})$ is a multi-normed algebra if

$$\|(a_1b_1,...,a_kb_k)\|_k \le \|(a_1,...,a_k)\|_k\cdot\|(b_1,...,b_k)\|_k$$

for all $k\in\mathbb{N}$ and all $a_1,...,a_k,b_1,...,b_k\in\mathcal{A}$. Further, the multi-normed algebra $((\mathcal{A}^k,\|.\|_k): k\in\mathbb{N})$ is a multi-Banach algebra if $((\mathcal{A}^k,\|.\|_k):k\in\mathbb{N})$ is a multi-Banach space.

Example 2.8. [6, 9] *Let p,q with $1\le p\le q<\infty$, and $\mathcal{A}=\ell^p$. The algebra $\mathcal{A}$ is a Banach sequence algebra with respect to coordinatewise multiplication of sequences. Let $(\|.\|_k : k\in\mathbb{N})$ be the standard (p,q)-multi-norm on $\{\mathcal{A}^k : k\in\mathbb{N}\}$. Then $((\mathcal{A}^k,\|.\|_k) : k\in\mathbb{N})$ is a multi-Banach algebra.*

3. Fixed point theorem

In this section, we assume that $(\mathcal{X}, \|.\|)$ and $(\mathcal{Y}, \|.\|)$ are Banach spaces such that $(\mathcal{X}^k, \|.\|_k)$ and $(\mathcal{Y}^k, \|.\|_k)$ are multi-Banach spaces.

A mapping $T : \mathcal{X} \to \mathcal{X}$ is called multi-$\mathcal{D}$-Lipschitzian if there exists a continuous and nondecreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\begin{aligned} &\|(T(x_1) - T(y_1), \cdots, T(x_k) - T(y_k))\|_k \\ &\leq \phi\left(\|(x_1 - y_1), \cdots, (x_k - y_k)\|_k\right), \end{aligned} \tag{3.1}$$

for x_i, y_i, $1 \leq i \leq k$, where $\phi(0) = 0$.

We call the function ϕ a multi-$\mathcal{D}$-function of T on $\mathcal{X}$. If ϕ is not necessarily nondecreasing and satisfies $\phi(r) < r$, for $r > 0$, the mapping T is called a nonlinear contraction with a contraction function ϕ. If $\phi(t) = \alpha t$ in (3.1), then the mapping $T : \mathcal{X} \to \mathcal{X}$ in (3.1) is called multi-Lipschitzian with Lipschitz constant α.

Let $(\mathcal{X}, \|.\|)$ be a Banach space and let $T : \mathcal{X} \to \mathcal{X}$. Then T is called a compact operator if $\overline{T(\mathcal{X})}$ is a compact subset of $\mathcal{X}$. Again T is called totally bounded if for any bounded subset $\mathcal{S}$ of $\mathcal{X}$, $T(\mathcal{S})$ is a totally bounded set of $\mathcal{X}$. Further, T is called completely continuous if it is continuous and totally bounded. Note that every compact operator is totally bounded, but the converse may not be true.

Theorem 3.1. *Let $\mathcal{S}$ be a closed, convex and bounded subset of a Banach algebra $(\mathcal{X}, \|.\|)$ and let $A : \mathcal{X} \to \mathcal{X}$, $B : \mathcal{S} \to \mathcal{X}$ be two operators such that*

(a) *A is multi-$\mathcal{D}$-Lipschitzian with a multi-$\mathcal{D}$-function ϕ,*

(b) *B is completely continuous, and*

(c) *$x = AxBy \Longrightarrow x \in \mathcal{S}$, for all $y \in \mathcal{S}$.*

Then the operator equation

$$AxBx = x$$

has a solution, whenever $M\phi(r) < r$ for $r > 0$. Here $M = \|B(S)\|$.

Proof. Let $y \in \mathcal{S}$ and define a mapping $A_y : \mathcal{X} \to \mathcal{X}$ by

$$A_y(x) = AxBy, \ x \in \mathcal{X}.$$

For $x_{ij} \in \mathcal{X}$ $(i = 1, 2$ and $1 \leq j \leq k)$,

$$\begin{aligned} & \|(A_y x_{11} - A_y x_{21}, \cdots, A_y x_{1k} - A_y x_{2k})\|_k \\ = \ & \|(Ax_{11}B_y - Ax_{21}By, \cdots, Ax_{11}B_y - Ax_{21}By)\|_k \\ \leq \ & \|(Ax_{11} - Ax_{21}, \cdots, Ax_{1k} - Ax_{2k})\|_k \|B_y\| \\ \leq \ & M\phi(\|(x_{11} - x_{21}, \cdots, x_{1k} - x_{2k})\|_k). \end{aligned}$$

Then A_y is a nonlinear contraction on $\mathcal{X}$ with a contraction function ψ given by $\psi(r) = M\phi(r)$, $r > 0$. Now an application of a fixed point theorem of Boyd and Wong [10] yields that there is a unique point $x^* \in \mathcal{X}$ such that

$$A_y(x^*) = x$$

or, equivalently,

$$x^* = Ax^*By.$$

Since hypothesis (c) holds, we have that $x^* \in \mathcal{S}$. Define a mapping $N : \mathcal{S} \to \mathcal{X}$ by

$$Ny = z,$$

where $z \in X$ is the unique solution of the equation

$$z = AzBy, \ y \in S.$$

We show that N is continuous. Let $\{y_n\}$ be a sequence in $\mathcal{S}$ converging to a point y. Since $\mathcal{S}$ is closed, $y \in \mathcal{S}$. Now,

$$\begin{aligned}\|Ny_n - Ny\| &= \|ANy_nBy_n - ANyBy\| \\ &\leq \|ANy_nBy_n - ANyBy_n\| + \|ANyBy_n - ANyBy\| \\ &\leq \|ANy_n - ANy\|\|By_n\| + \|ANy\|\|y_n - y\| \\ &\leq M\phi(\|Ny_n - Ny\|) + \|ANy\|\|y_n - y\|\end{aligned}$$

This shows that $Ny_n - Ny \to 0$ whenever $n \to \infty$ and consequently N is continuous on $\mathcal{S}$. Next we show that N is a compact operator on $\mathcal{S}$. Now for any $z \in \mathcal{S}$ we have

$$\begin{aligned}\|Az\| &\leq \|Aa\| + \|Az - Aa\| \\ &\leq \|Aa\| + \alpha\|z - a\| \\ &\leq c\end{aligned}$$

where $c = \|Aa\| + diam(\mathcal{S})$ for some fixed $a \in \mathcal{S}$.

Let $\epsilon > 0$ be given. Since B is completely continuous, $B(\mathcal{S})$ is totally bounded. Then there is a set $\mathcal{Y} = \{y_1, ..., y_n\}$ in $\mathcal{S}$ such that

$$B(\mathcal{S}) \subset \bigcup_{i=1}^{n} \mathcal{B}(w_i, \delta),$$

where $w_i = B(y_i)$, $\delta = \left(\frac{1-\alpha M}{c}\right)\epsilon$ and $\mathcal{B}(w_i, \delta)$ is an open ball in $\mathcal{X}$ centered at w_i of radius δ.

Therefore, for any $y \in \mathcal{S}$, we have a $y_k \in \mathcal{Y}$ such that

$$\|By - By_k\| < \left(\frac{1-\alpha M}{c}\right)\epsilon.$$

Also, we have

$$\begin{aligned}\|Ny_n - Ny\| &\leq \|AzBy - Az_kBy_k\| \\ &\leq \|AzBy - Az_kBy\| + \|Az_kBy - Az_kBy_k\| \\ &\leq \|Az - Az_k\|\|By\| + \|Az_k\|\|By_n - By\| \\ &\leq (\alpha M)\|z - z_k\| + \|Az_k\|\|By_n - By\| \\ &\leq \left(\frac{c}{1-\alpha M}\right)\|By_n - By\| \\ &\leq \epsilon.\end{aligned}$$

This is true for every $y \in \mathcal{S}$ and hence

$$N(\mathcal{S}) \subset \mathcal{B}(z_i, \epsilon),$$

where $z_i = N(y_i)$. As a result $N(\mathcal{S})$ is totally bounded. Since N is continuous, it is a compact operator on $\mathcal{S}$. Now an application of Schauder's fixed point yields that N has a fixed point in $\mathcal{S}$. Then, by the definition of N,

$$x = Nx = A(Nx)Bx = AxBx,$$

and so the operator equation $x = AxBx$ has a solution in $\mathcal{S}$. □

Corollary 3.2. *Let $\mathcal{S}$ be a closed, convex and bounded subset of a Banach algebra $(\mathcal{X}, \|.\|)$ and let $A : \mathcal{X} \to \mathcal{X}$, $B : \mathcal{S} \to \mathcal{X}$ be two operators such that*

(a) *A is multi-Lipschitzian with a with Lipschitz constant α,*
(b) *B is completely continuous, and*
(c) *$x = AxBy \Longrightarrow x \in \mathcal{S}$, for all $y \in \mathcal{S}$.*

Then the operator equation

$$AxBx = x$$

has a solution, whenever $M\alpha < 1$. *Here* $M = \|B(S)\|$.

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On the solutions and behavior of rational difference equations

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Abstract

In this paper we study existence of solutions and the periodicity character of the following difference equations

$$x_{n+1} = \frac{x_{n-2}x_{n-3}x_{n-4}}{x_n x_{n-1}\left(\pm 1 \pm x_{n-2}x_{n-3}x_{n-4}\right)},$$

where initial conditions are arbitrary nonzero real numbers such that the denominator is always nonzero. Also we investigate the behavior of the solutions of these equations.

Keywords: Difference equation, equilibrium point, periodic solution, stability.

AMS Classification: 39A10

1 Introduction and Preliminaries

Nonlinear difference equations have long interest in mathematics as well as other sciences. They play a key role in many applications such as the natural model of a discrete process. Recently, there have been many researches and interest in the field of nonlinear difference equations by several authors [1-22] and references therein. For example, Tollu et al. [12] investigated the solutions of two special types of Riccati difference equations

$$x_{n+1} = \frac{1}{1+x_n} \text{ and } y_{n+1} = \frac{1}{-1+y_n}$$

such that their solutions are associated with Fibonacci numbers. In addition, Cinar [14-15] also examined the solutions of the following difference equations

$$x_{n+1} = \frac{x_{n-1}}{1+ax_n x_{n-1}}, \quad x_{n+1} = \frac{x_{n-1}}{-1+ax_n x_{n-1}}.$$

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El-Metwally and Elsayed [3-4] obtained the solutions of the following difference equations

$$x_{n+1} = \frac{x_{n-1}x_{n-2}}{x_n\left(\pm 1 \pm x_{n-1}x_{n-2}\right)}, \quad x_{n+1} = \frac{x_n x_{n-3}}{x_{n-2}\left(\pm 1 \pm x_n x_{n-3}\right)}.$$

Ibrahim [13] got the solutions of the rational difference equation

$$x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1}\left(a + b x_n x_{n-2}\right)}.$$

Simsek et al. [19] investigated the solution of the following difference equation

$$x_{n+1} = \frac{x_{n-(5k+9)}}{1 + x_{n-4}x_{n-9}\ldots x_{n-(5k+4)}}.$$

Moreover, Karatas [22] studied the dynamics of the difference equation

$$x_{n+1} = \frac{A x_{n-m}}{B + C\prod_{i=0}^{2k+1} x_{n-i}}.$$

In this paper, we obtain the solutions of the following recursive sequences

$$x_{n+1} = \frac{x_{n-2}x_{n-3}x_{n-4}}{x_n x_{n-1}\left(\pm 1 \pm x_{n-2}x_{n-3}x_{n-4}\right)}, \ n = 0, 1, \ldots, \tag{1}$$

where initial conditions are arbitrary nonzero real numbers such that the denominator is always nonzero. Also, we analyzed behavior and periodic character of the solutions of the difference equation (1).

Now, we present some crucial necessities about the equilibrium point of a higher-order difference equation.

Let I be some interval of real numbers and let $f : I^{k+1} \to I$ be a continuously differentiable function. Then for every set of initial conditions $x_{-k},\ x_{-(k+1)}, \ldots, x_0 \in I$, the difference equation

$$x_{n+1} = f\left(x_n, x_{n-1}, \ldots, x_{n-k}\right), \quad n = 0, 1, \ldots, \tag{2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 1 *An equilibrium point for Eq. (2) is a point $\overline{x} \in I$ such that $\overline{x} = f\left(\overline{x}, \overline{x}, \ldots, \overline{x}\right)$.*

Definition 2 *A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.*

Definition 3 *(Stability).*

(i) The equilibrium point $\overline{x}$ of Eq. (2) is locally stable if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k},\ x_{-(k+1)}, \ldots, x_0 \in I$ with $|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + \ldots + |x_0 - \overline{x}| < \delta$, we have $|x_n - \overline{x}| < \varepsilon$ for all $n \geq -k$.

(ii) *The equilibrium point $\overline{x}$ of Eq. (2) is locally stable if $\overline{x}$ is a locally stable solution of Eq. (2) and there exists $\gamma > 0$, such that for all $x_{-k},\ x_{-(k+1)}, \ldots, x_0 \in I$ with $|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + \ldots + |x_0 - \overline{x}| < \gamma$, we have $\lim_{n\to\infty} x_n = \overline{x}$.*

(iii) *The equilibrium point $\overline{x}$ of Eq. (2) is a global attractor if for all $x_{-k},\ x_{-(k+1)}, \ldots, x_0 \in I$, we have $\lim_{n\to\infty} x_n = \overline{x}$.*

(iv) *The equilibrium point $\overline{x}$ of Eq. (2) is a global asymptotically stable if $\overline{x}$ is locally stable and $\overline{x}$ is also a global attractor of Eq. (2).*

(v) *The equilibrium point $\overline{x}$ of Eq. (2) is unstable if $\overline{x}$ is not locally stable.*

The linearized equation associated with Eq. (2) is

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f}{\partial x_{n-i}} (\overline{x}, \overline{x}, \ldots, \overline{x})\, y_{n-i},\ n = 0, 1, \ldots. \tag{3}$$

The characteristic equation associated with Eq. (2) is

$$\lambda^{k+1} - \sum_{i=0}^{k} \frac{\partial f}{\partial x_{n-i}} (\overline{x}, \overline{x}, \ldots, \overline{x})\, \lambda^{k-i} = 0. \tag{4}$$

Theorem 4 ([1]) *Assume that f is a C^1 function and let $\overline{x}$ be an equilibrium point of Eq. (2). Then the following statements are hold:*

(i) *If all roots of Eq.(4) lie in the open disk $|\lambda| < 1$, then $\overline{x}$ is locally asymptotically stable.*

(ii) *If at least one root of Eq.(4) has absolute value greater than one, then $\overline{x}$ is unstable.*

2 On the Difference Equation $x_{n+1} = \frac{x_{n-2}x_{n-3}x_{n-4}}{x_n x_{n-1}(1+x_{n-2}x_{n-3}x_{n-4})}$

In this section, we give the form of the solution of the following equation

$$x_{n+1} = \frac{x_{n-2}x_{n-3}x_{n-4}}{x_n x_{n-1}\left(1 + x_{n-2}x_{n-3}x_{n-4}\right)},\ n = 0, 1, 2, \ldots, \tag{5}$$

where the initial conditions are arbitrary positive real numbers.

Theorem 5 *Assume that $\{x_n\}$ is a solution of Eq. (5). Then for $n = 0, 1, 2, \ldots,$ the solution of Eq. (5) is given by following formulas:*

$$\begin{aligned}
x_{3n-4} &= \frac{d^n}{a^{n-1}} \prod_{i=0}^{n-1} \frac{(1 + iabc)}{(1 + ibcd)}, \\
x_{3n-3} &= \frac{e^n}{b^{n-1}} \prod_{i=0}^{n-1} \frac{(1 + ibcd)}{(1 + icde)}, \\
x_{3n-2} &= \frac{ca^n b^n}{d^n e^n} \prod_{i=0}^{n-1} \frac{(1 + icde)}{(1 + (i+1)\, abc)},
\end{aligned}$$

where $x_{-4} = a,\ x_{-3} = b,\ x_{-2} = c, x_{-1} = d,\ x_0 = e.$

Proof. For $n = 0$ the result is satisfied. Now suppose that $n > 0$ and our assumption is valid for $n-1$, that is,

$$\begin{aligned} x_{3n-7} &= \frac{d^{n-1}}{a^{n-2}} \prod_{i=0}^{n-2} \frac{(1+iabc)}{(1+ibcd)}, \\ x_{3n-6} &= \frac{e^{n-1}}{b^{n-2}} \prod_{i=0}^{n-2} \frac{(1+ibcd)}{(1+icde)}, \\ x_{3n-5} &= \frac{ca^{n-1}b^{n-1}}{d^{n-1}e^{n-1}} \prod_{i=0}^{n-2} \frac{(1+icde)}{(1+(i+1)\,abc)}. \end{aligned}$$

Now, we find from Eq. (5), that

$$\begin{aligned} x_{3n-4} &= \frac{x_{3n-7}x_{3n-8}x_{3n-9}}{x_{3n-5}x_{3n-6}\left(1+x_{3n-7}x_{3n-8}x_{3n-9}\right)} \\ &= \frac{bcd\left(\prod\limits_{i=0}^{n-2}\frac{(1+iabc)}{(1+ibcd)}\right)\left(\prod\limits_{i=0}^{n-3}\frac{(1+icde)}{(1+(i+1)abc)}\right)\left(\prod\limits_{i=0}^{n-3}\frac{(1+ibcd)}{(1+icde)}\right)}{\begin{array}{c}\frac{cba^{n-1}}{d^{n-1}}\left(\prod\limits_{i=0}^{n-2}\frac{(1+icde)}{(1+(i+1)abc)}\right)\left(\prod\limits_{i=0}^{n-2}\frac{(1+ibcd)}{(1+icde)}\right) \\ \left[1+bcd\left(\prod\limits_{i=0}^{n-2}\frac{(1+iabc)}{(1+ibcd)}\right)\left(\prod\limits_{i=0}^{n-3}\frac{(1+icde)}{(1+(i+1)abc)}\right)\left(\prod\limits_{i=0}^{n-3}\frac{(1+ibcd)}{(1+icde)}\right)\right]\end{array}} \\ &= \frac{\frac{bcd}{1+(n-2)bcd}}{\frac{cba^{n-1}}{d^{n-1}}\prod\limits_{i=0}^{n-2}\frac{(1+ibcd)}{(1+(i+1)abc)}\left(1+\frac{bcd}{1+(n-2)bcd}\right)} \\ &= \frac{d^{n-1}}{a^{n-1}}\left(\prod_{i=0}^{n-2}\frac{(1+(i+1)\,abc)}{(1+ibcd)}\right)\frac{1}{1+(n-1)\,bcd}. \end{aligned}$$

Thus, we obtain

$$x_{3n-4} = \frac{d^n}{a^{n-1}} \prod_{i=0}^{n-1} \frac{(1+iabc)}{(1+ibcd)}.$$

Similarly,

$$\begin{aligned} x_{3n-3} &= \frac{x_{3n-6}x_{3n-7}x_{3n-8}}{x_{3n-4}x_{3n-5}\left(1+x_{3n-6}x_{3n-7}x_{3n-8}\right)} \\ &= \frac{cde\left(\prod\limits_{i=0}^{n-2}\frac{(1+ibcd)}{(1+icde)}\right)\left(\prod\limits_{i=0}^{n-2}\frac{(1+iabc)}{(1+ibcd)}\right)\left(\prod\limits_{i=0}^{n-3}\frac{(1+icde)}{(1+(i+1)abc)}\right)}{\begin{array}{c}cd\frac{b^{n-1}}{e^{n-1}}\left(\prod\limits_{i=0}^{n-1}\frac{(1+iabc)}{(1+ibcd)}\right)\left(\prod\limits_{i=0}^{n-2}\frac{(1+icde)}{(1+(i+1)abc)}\right) \\ \left(1+cde\left(\prod\limits_{i=0}^{n-2}\frac{(1+ibcd)}{(1+icde)}\right)\left(\prod\limits_{i=0}^{n-2}\frac{(1+iabc)}{(1+ibcd)}\right)\left(\prod\limits_{i=0}^{n-3}\frac{(1+icde)}{(1+(i+1)abc)}\right)\right)\end{array}} \\ &= \frac{\frac{cde}{1+(n-2)cde}}{cd\frac{b^{n-1}}{e^{n-1}(1+(n-1)bcd)}\prod\limits_{i=0}^{n-2}\frac{(1+icde)}{(1+ibcd)}\left(1+\frac{cde}{1+(n-2)cde}\right)} \\ &= \left(\prod_{i=0}^{n-1}\frac{(1+ibcd)}{(1+icde)}\right)\frac{e^n}{b^{n-1}}. \end{aligned}$$

Hence, we have the following expression,

$$x_{3n-3} = \frac{e^n}{b^{n-1}} \prod_{i=0}^{n-1} \frac{(1+ibcd)}{(1+icde)}.$$

In the same way, one can easily obtain the other relations. Thus, the proof is completed. ∎

Theorem 6 *Eq.* (5) *has a unique equilibrium point which is the number zero and this equilibrium point is not locally asymptotically stable.*

Proof. For the equilibrium points of Eq. (5), we can write

$$\overline{x} = \frac{\overline{x}^3}{\overline{x}^2\left(1+\overline{x}^3\right)}.$$

Therefore, one can get,

$$\begin{aligned} \overline{x}^3\left(1+\overline{x}^3\right) &= \overline{x}^3, \\ \overline{x}^3\left(1+\overline{x}^3-1\right) &= 0, \end{aligned}$$

or,

$$\overline{x}^6 = 0.$$

Thus the equilibrium point of Eq. (5) is $\overline{x} = 0$. Assume that $f : (0,\infty)^5 \to (0,\infty)$ is a function defined by

$$f(x_1,x_2,x_3,x_4,x_5) = \frac{x_3x_4x_5}{x_1x_2(1+x_3x_4x_5)}.$$

So, it follows that

$$\begin{aligned} f_{x_1}(x_1,x_2,x_3,x_4,x_5) &= \frac{-x_3x_4x_5}{x_1^2x_2(1+x_3x_4x_5)}, \\ f_{x_2}(x_1,x_2,x_3,x_4,x_5) &= \frac{-x_3x_4x_5}{x_1x_2^2(1+x_3x_4x_5)}, \\ f_{x_3}(x_1,x_2,x_3,x_4,x_5) &= \frac{x_4x_5}{x_1x_2(1+x_3x_4x_5)^2}, \\ f_{x_4}(x_1,x_2,x_3,x_4,x_5) &= \frac{x_3x_5}{x_1x_2(1+x_3x_4x_5)^2}, \\ f_{x_5}(x_1,x_2,x_3,x_4,x_5) &= \frac{x_3x_4}{x_1x_2(1+x_3x_4x_5)^2}, \end{aligned}$$

One can easily see that

$$\begin{aligned} f_{x_1}(\overline{x},\overline{x},\overline{x},\overline{x},\overline{x}) &= f_{x_2}(\overline{x},\overline{x},\overline{x},\overline{x},\overline{x}) = -1, \\ f_{x_3}(\overline{x},\overline{x},\overline{x},\overline{x},\overline{x}) &= f_{x_4}(\overline{x},\overline{x},\overline{x},\overline{x},\overline{x}) = f_{x_5}(\overline{x},\overline{x},\overline{x},\overline{x},\overline{x}) = 1. \end{aligned}$$

The proof follows by using Theorem 4. ∎

3 On the Difference Equation $x_{n+1} = \frac{x_{n-2}x_{n-3}x_{n-4}}{x_n x_{n-1}(-1+x_{n-2}x_{n-3}x_{n-4})}$

In this section, we obtain the solution of the following equation

$$x_{n+1} = \frac{x_{n-2}x_{n-3}x_{n-4}}{x_n x_{n-1}\left(-1+x_{n-2}x_{n-3}x_{n-4}\right)},\ n = 0, 1, 2, \ldots, \tag{6}$$

where the initial conditions are arbitrary non zero real numbers with $x_0x_{-1}x_{-2} \neq 1,\ x_{-1}x_{-2}x_{-3} \neq 1,\ x_{-2}x_{-3}x_{-4} \neq 1.$

Theorem 7 *Let $\{x_n\}$ be a solution of Eq. (6). Then for $n = 0, 1, 2, \ldots,$ we have*

$$\begin{aligned}
x_{6n-4} &= \frac{d^{2n}\left(-1+abc\right)^n}{a^{2n-1}\left(-1+bcd\right)^n},\ x_{6n-3} = \frac{e^{2n}\left(-1+bcd\right)^n}{b^{2n-1}\left(-1+cde\right)^n}, \\
x_{6n-2} &= \frac{ca^{2n}b^{2n}}{d^{2n}e^{2n}}\frac{\left(-1+cde\right)^n}{\left(-1+abc\right)^n},\ x_{6n-1} = \frac{d^{2n+1}\left(-1+abc\right)^n}{a^{2n}\left(-1+bcd\right)^n}, \\
x_{6n} &= \frac{e^{2n+1}\left(-1+bcd\right)^n}{b^{2n}\left(-1+cde\right)^n},\ x_{6n+1} = \frac{ca^{2n+1}b^{2n+1}}{d^{2n+1}e^{2n+1}}\frac{\left(-1+cde\right)^n}{\left(-1+abc\right)^{n+1}},
\end{aligned}$$

where $x_{-4} = a,\ x_{-3} = b,\ x_{-2} = c, x_{-1} = d,\ x_0 = e.$

Proof. For $n = 0$ the result is trivial. Let $n > 0$ and let our assumption holds for $n-1$.That is,

$$\begin{aligned}
x_{6n-10} &= \frac{d^{2n-2}\left(-1+abc\right)^{n-1}}{a^{2n-3}\left(-1+bcd\right)^{n-1}},\ x_{6n-9} = \frac{e^{2n-2}\left(-1+bcd\right)^{n-1}}{b^{2n-3}\left(-1+cde\right)^{n-1}}, \\
x_{6n-8} &= \frac{ca^{2n-2}b^{2n-2}}{d^{2n-2}e^{2n-2}}\frac{\left(-1+cde\right)^{n-1}}{\left(-1+abc\right)^{n-1}},\ x_{6n-7} = \frac{d^{2n-1}\left(-1+abc\right)^{n-1}}{a^{2n-2}\left(-1+bcd\right)^{n-1}}, \\
x_{6n-6} &= \frac{e^{2n-1}\left(-1+bcd\right)^{n-1}}{b^{2n-2}\left(-1+cde\right)^{n-1}},\ x_{6n-5} = \frac{ca^{2n-1}b^{2n-1}}{d^{2n-1}e^{2n-1}}\frac{\left(-1+cde\right)^{n-1}}{\left(-1+abc\right)^{n}}.
\end{aligned}$$

By Eq. (6), it can be said that

$$\begin{aligned}
x_{6n-4} &= \frac{x_{6n-7}x_{6n-8}x_{6n-9}}{x_{6n-5}x_{6n-6}\left(-1+x_{6n-7}x_{6n-8}x_{6n-9}\right)} \\
&= \frac{bcd\left(\frac{(-1+abc)^{n-1}}{(-1+bcd)^{n-1}}\right)\left(\frac{(-1+cde)^{n-1}}{(-1+abc)^{n-1}}\right)\left(\frac{(-1+bcd)^{n-1}}{(-1+cde)^{n-1}}\right)}{\begin{array}{c}\frac{bca^{2n-1}}{d^{2n-1}}\left(\frac{(-1+cde)^{n-1}}{(-1+abc)^{n}}\right)\left(\frac{(-1+bcd)^{n-1}}{(-1+cde)^{n-1}}\right) \\ \left(-1+bcd\left(\frac{(-1+abc)^{n-1}}{(-1+bcd)^{n-1}}\right)\left(\frac{(-1+cde)^{n-1}}{(-1+abc)^{n-1}}\right)\left(\frac{(-1+bcd)^{n-1}}{(-1+cde)^{n-1}}\right)\right)\end{array}} \\
&= \frac{bcd}{\frac{bca^{2n-1}(-1+bcd)^{n-1}}{d^{2n-1}(-1+abc)^{n}}\left(-1+bcd\right)} \\
&= \frac{d^{2n}\left(-1+abc\right)^n}{a^{2n-1}\left(-1+bcd\right)^n}.
\end{aligned}$$

Likewise previous theorem, one can easily get following relation

$$\begin{aligned}
x_{6n-3} &= \frac{x_{6n-6}x_{6n-7}x_{6n-8}}{x_{6n-4}x_{6n-5}\left(-1+x_{6n-6}x_{6n-7}x_{6n-8}\right)} \\
&= \frac{cde\left(\frac{(-1+bcd)^{n-1}}{(-1+cde)^{n-1}}\right)\left(\frac{(-1+abc)^{n-1}}{(-1+bcd)^{n-1}}\right)\left(\frac{(-1+cde)^{n-1}}{(-1+abc)^{n-1}}\right)}{\frac{cdb^{2n-1}}{e^{2n-1}}\left(\frac{(-1+abc)^{n}}{(-1+bcd)^{n}}\right)\left(\frac{(-1+cde)^{n-1}}{(-1+abc)^{n}}\right)\left(-1+cde\left(\frac{(-1+bcd)^{n-1}}{(-1+cde)^{n-1}}\right)\left(\frac{(-1+abc)^{n-1}}{(-1+bcd)^{n-1}}\right)\left(\frac{(-1+cde)^{n-1}}{(-1+abc)^{n-1}}\right)\right)} \\
&= \frac{cde}{\frac{cdb^{2n-1}(-1+cde)^{n-1}}{e^{2n-1}(-1+bcd)^{n}}\left(-1+cde\right)} \\
&= \frac{e^{2n}\left(-1+bcd\right)^{n}}{b^{2n-1}\left(-1+cde\right)^{n}}.
\end{aligned}$$

The other relations can be proved by the same manner. ■

Theorem 8 *The following statements are valid:*

i) *Eq.* (6) *has two equilibrium points which are* $0,\ \sqrt[3]{2}$ *and these equilibrium points are not locally asymptotically stable.*

ii) *Eq.* (6) *has a periodic solution of period three iff* $a=d,\ b=e,\ abc=2$ *and will be taken the form* $\{a,b,c,a,b,c,\ldots\}$.

Proof.

i) For the equilibrium points of Eq. (6) ,it can be written

$$\overline{x}=\frac{\overline{x}^{3}}{\overline{x}^{2}\left(-1+\overline{x}^{3}\right)}.$$

Then we have

$$\begin{aligned}
\overline{x}^{3}\left(-1+\overline{x}^{3}\right) &= \overline{x}^{3}, \\
\overline{x}^{3}\left(-1+\overline{x}^{3}-1\right) &= 0,
\end{aligned}$$

or,

$$\overline{x}^{3}\left(\overline{x}^{3}-2\right)=0.$$

Thus the equilibrium point of Eq. (5) are $0,\ \sqrt[3]{2}$. Assume that $g:(0,\infty)^{5}\rightarrow(0,\infty)$ is a function defined by

$$g\left(x_1,x_2,x_3,x_4,x_5\right)=\frac{x_3x_4x_5}{x_1x_2\left(-1+x_3x_4x_5\right)}.$$

Therefore, it follows that

$$\begin{aligned} g_{x_1}(x_1,x_2,x_3,x_4,x_5) &= \frac{-x_3x_4x_5}{x_1^2x_2(-1+x_3x_4x_5)}, \\ g_{x_2}(x_1,x_2,x_3,x_4,x_5) &= \frac{-x_3x_4x_5}{x_1x_2^2(-1+x_3x_4x_5)}, \\ g_{x_3}(x_1,x_2,x_3,x_4,x_5) &= \frac{-x_4x_5}{x_1x_2(-1+x_3x_4x_5)^2}, \\ g_{x_4}(x_1,x_2,x_3,x_4,x_5) &= \frac{-x_3x_5}{x_1x_2(-1+x_3x_4x_5)^2}, \\ g_{x_5}(x_1,x_2,x_3,x_4,x_5) &= \frac{-x_3x_4}{x_1x_2(-1+x_3x_4x_5)^2}, \end{aligned}$$

we see that

$$\begin{aligned} g_{x_1}(\overline{x},\overline{x},\overline{x},\overline{x},\overline{x}) &= g_{x_2}(\overline{x},\overline{x},\overline{x},\overline{x},\overline{x}) = \pm 1, \\ g_{x_3}(\overline{x},\overline{x},\overline{x},\overline{x},\overline{x}) &= g_{x_4}(\overline{x},\overline{x},\overline{x},\overline{x},\overline{x}) = g_{x_5}(\overline{x},\overline{x},\overline{x},\overline{x},\overline{x}) = -1. \end{aligned}$$

The proof follows by using Theorem 4.

ii) By considering Theorem 7, the proof is easily seen.

■

4 On the Difference Equation $x_{n+1} = \frac{x_{n-2}x_{n-3}x_{n-4}}{x_nx_{n-1}(1-x_{n-2}x_{n-3}x_{n-4})}$

Now, we get the solution of the following equation

$$x_{n+1} = \frac{x_{n-2}x_{n-3}x_{n-4}}{x_nx_{n-1}(1-x_{n-2}x_{n-3}x_{n-4})}, \; n = 0,1,2,\ldots, \tag{7}$$

where the initial conditions are arbitrary positive real numbers with $x_0x_{-1}x_{-2} \neq 1,\; x_{-1}x_{-2}x_{-3} \neq 1,\; x_{-2}x_{-3}x_{-4} \neq 1$.

Theorem 9 *Let $\{x_n\}_{n=-4}^{\infty}$ be a solution of Eq. (7). Then for $n = 0,1,2,\ldots,$*

$$\begin{aligned} x_{3n-4} &= \frac{d^n}{a^{n-1}} \prod_{i=0}^{n-1} \frac{(-1+iabc)}{(-1+ibcd)}, \\ x_{3n-3} &= \frac{e^n}{b^{n-1}} \prod_{i=0}^{n-1} \frac{(-1+ibcd)}{(-1+icde)}, \\ x_{3n-2} &= -\frac{ca^nb^n}{d^ne^n} \prod_{i=0}^{n-1} \frac{(-1+icde)}{(-1+(i+1)abc)}, \end{aligned}$$

where $x_{-4} = a,\; x_{-3} = b,\; x_{-2} = c, x_{-1} = d,\; x_0 = e$.

Theorem 10 *Eq. (7) has a unique equilibrium point which is the number zero and this equilibrium point is not locally asymptotically stable.*

5 On the Difference Equation $x_{n+1}=\frac{x_{n-2}x_{n-3}x_{n-4}}{x_nx_{n-1}(-1-x_{n-2}x_{n-3}x_{n-4})}$

Finally, we consider the solution of the following equation

$$x_{n+1}=\frac{x_{n-2}x_{n-3}x_{n-4}}{x_nx_{n-1}\left(-1-x_{n-2}x_{n-3}x_{n-4}\right)},\ n=0,1,2,\ldots, \tag{8}$$

where the initial conditions are arbitrary non zero real numbers with $x_0x_{-1}x_{-2}\neq -1,\ x_{-1}x_{-2}x_{-3}\neq -1,\ x_{-2}x_{-3}x_{-4}\neq -1$.

Theorem 11 *Let $\{x_n\}_{n=-4}^{\infty}$ be a solution of Eq. (8). Then for $n=0,1,2,\ldots$,*

$$\begin{aligned}
x_{6n-4} &= \frac{d^{2n}\left(1+abc\right)^n}{a^{2n-1}\left(1+bcd\right)^n},\ x_{6n-3}=\frac{e^{2n}\left(1+bcd\right)^n}{b^{2n-1}\left(1+cde\right)^n},\\
x_{6n-2} &= \frac{ca^{2n}b^{2n}\left(1+cde\right)^n}{d^{2n}e^{2n}\left(1+abc\right)^n},\ x_{6n-1}=\frac{d^{2n+1}\left(1+abc\right)^n}{a^{2n}\left(1+bcd\right)^n},\\
x_{6n} &= \frac{e^{2n+1}\left(1+bcd\right)^n}{b^{2n}\left(1+cde\right)^n},\ x_{6n+1}=\frac{ca^{2n+1}b^{2n+1}\left(1+cde\right)^n}{d^{2n+1}e^{2n+1}\left(1+abc\right)^{n+1}},
\end{aligned}$$

where $x_{-4}=a,\ x_{-3}=b,\ x_{-2}=c, x_{-1}=d,\ x_0=e$.

Theorem 12 *The following expression are satisfied:*

i) *Eq. (8) has two equilibrium points which are $0,\ -\sqrt[3]{2}$ and these equilibrium points are not locally asymptotically stable.*

ii) *Eq. (8) has a periodic solution of period three iff $a=d,\ b=e,\ abc=-2$ and will be taken the form $\{a,b,c,a,b,c,\ldots\}$.*

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Difference methods for a fuzzy two-dimensional parabolic equation with variable coefficients and non-local boundary conditions†

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Abstract. Parabolic partial differential equations with non-local boundary conditions have important applications in different areas of subject and engineering. M. Dehghan, W.A. Day et al. have investigated its numerical approximation of solution in the case of constant coefficients. In this paper, numerical methods for solving fuzzy two-dimensional parabolic equations with variable coefficients and non-local boundary conditions is considered. Especially, the second-order forward Euler method, fully implicit scheme are studied with a numerical example.

Key words. Fuzzy two-dimensional parabolic equations; Non-local boundary conditions; Variable coefficients; Numerical approximation of solution; Difference methods

1 Introduction

Introduction to fuzzy partial differential equation is presented by Buckley and Feuring in [5]. The study of fuzzy partial differential equation forms a suitable setting for mathematical modeling of real-world problems in which uncertainties or vagueness pervade. Thinking about a physical problem which is transformed into a deterministic problem of partial differential equations we cannot usually be sure that this modeling is perfect. Especially, if the data (e.g. initial value) are not known precisely but only through some measurements the intervals which cover the data are determined. For example, mathematical models in science and engineering often contain parameters that are uncertain. These parameters are usually represented by random numbers, fields or processes. Therefore there appear

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problems of differential equations with uncertainty. However, when the stochastic characteristics of these parameters are not precisely known, an interval representation, or, more generally, a fuzzy representation may be more appropriate. Therefore, such uncertainty can be expressed in terms of intervals and ordinary differential equations are replaced with fuzzy partial differential equation which are more appropriate tool for modeling with uncertainties. It is also clear that uncertainty expressed by a fuzzy representation is more appropriate.

The concept of fuzzy sets which was originally introduced by Zadeh [21] led to the definition of the fuzzy number and its implementation in fuzzy control [7] and approximate reasoning problems [23, 22]. The fuzzy mapping function was introduced by Chang and Zadeh [7]. Later, Dubois and Prade [12] presented an elementary fuzzy calculus based on the extension principle [21]. Puri and Ralescu [20] suggested two definitions for fuzzy derivative of fuzzy functions. The first method was based on the H-difference notation and was further investigated by Kaleva [16]. The second method was derived from the embedding technique and was followed by Goetschel and Voxman [15] who gave it a more applicable representation. The concept of integration of fuzzy functions was first introduced by Dubois and Prade [12]. Alternative approaches were later suggested by Goetschel and Voxman [15], Kaleva [16] and others. While Goetschel and Voxman [15] preferred a Riemann integral type approach, Kaleva [16] chose to define the integral of fuzzy function, using the Lebesgue type concept for integration.

Knowledge about dynamical systems modeled by differential equations is often incomplete or vague. For example, for parametric quantities, functional relationships, or initial conditions, the well-known methods of solving fuzzy partial differential equation analytically or numerically can only be used for finding the selected system behavior, e.g., by fixing unknown parameters to some plausible values. As a new and powerful mathematical tool, fuzzy partial differential equations have been studied by several approaches [2, 1, 4]. Here, we are going to operationalize our approach, i.e., we are going to propose a method for computing approximate solution for a fuzzy two-dimensional parabolic equation with variable coefficients and non-local boundary conditions using numerical methods, which are widely applied over pre-assigned grid points to solve partial differential equations [9].

For the classical two-dimensional parabolic equation with variable coefficients and non-local boundary conditions, the authors of [6, 13, 18, 17] have investigated the existence, uniqueness and continuous dependence on data of the solution to this problem. In 2005, Dehghan [10] et.al studied the numerical approximations of classical two-dimensional parabolic equation with variable coefficients and non-local boundary conditions, they developed two-level finite difference procedures in this paper and the numerical results confirmed that the difference formulas obtained from the new discretization technique outlined in this paper do yield second order convergence for the solution of the problem. Here, we consider the fuzzy case of a two-dimensional parabolic equation with variable coefficients and non-local boundary conditions in the sense of Dehghan' [10]. Moreover, some finite difference methods for a fuzzy two-dimensional parabolic equation with variable coefficients and non-local boundary conditions are proposed in this paper.

The structure of this paper is as follows. In Section 2, we collect some necessary facts about the notions of continuity, differentiation and integration of fuzzy-number-valued functions. In Section 3, we introduce the fuzzy partial differential equation. The difference

methods are discussed in Section 4. The proposed algorithm is illustrated by solving an example in Section 5 and conclusions are drown in Section 6.

2 Preliminaries

Let $P_k(R^n)$ denote the family of all nonempty compact convex subset of R^n and define the addition and scalar multiplication in $P_k(R^n)$ as usual. Let A and B be two nonempty bounded subset of R^n. The distance between A and B is defined by the Hausdorff metric

$$d_H(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B}||a-b||, \sup_{b\in B}\inf_{a\in A}||b-a||\right\},$$

where $||\cdot||$ denotes the usual Euclidean norm in R^n [11]. Then $(P_k(R^n); d_H)$ is a metric space.

Denote by

$$E^n = \{u : R^n \to [0,1] | u \text{ satisfies (1)-(4) below}\}$$

is a fuzzy number space, where

(1) u is normal, i.e. there exists an $x_0 \in R^n$ such that $u(x_0) = 1$,

(2) u is fuzzy convex, i.e. $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$ for any $x, y \in R^n$ and $0 \leq \lambda \leq 1$,

(3) u is upper semi-continuous,

(4) $[u]_0 = cl\{x \in R^n | u(x) > 0\}$ is compact.

Here, $cl(X)$ denotes the closure of set X. For $0 < \alpha \leq 1$, the α-level set of u (or simply the α-cut) is defined by $[u]_\alpha = \{x \in R^n | u(x) \geq \alpha\}$. The core of u is the set of elements of R^n having membership grade 1, i.e., $[u]_1 = \{x | x \in R^n, u(x) = 1\}$. Then from above (1)-(4), it follows that the α-level set $[u]_\alpha \in P_k(R^n)$ for all $0 < \alpha \leq 1$. According to Zadeh's extension principle, we have addition and scalar multiplication in fuzzy number space E^n as follows:

$$[u+v]_\alpha = [u]_\alpha + [v]_\alpha = \{x+y | x \in [u]_\alpha, y \in [v]_\alpha\},$$

$$[ku]_\alpha = k[u]_\alpha = \{kx | x \in [u]_\alpha\}, [0]_\alpha = \{0\}.$$

where $u, v \in E^n$ and $0 < \alpha \leq 1$. Denote by

$$D(u,v) = \sup_{\alpha\in[0,1]} d_H([u]_\alpha, [v]_\alpha).$$

We recall some integrability and differentiability properties in [12, 20, 16, 19, 3] for fuzzy set-valued mappings.

Let $x, y \in E^n$. If there exist a $z \in E^n$ such that $x = y + z$, then z is called H-difference of x and y. That is denoted $x - y$. For brevity, we always assume that it satisfies the H-difference when dealing with the operation of subtraction of fuzzy numbers throughout this paper.

Definition 2.1 A map $u : J_a \times J_b \times J_c \to E^n$ is called level-wise continuous at $(x_0, y_0, t_0) \in J_a \times J_b \times J_c$ if the multi-valued map $u_\alpha(x,y,t) = [u(x,y,t)]_\alpha$ is continuous at $(x,y,t) = (x_0, y_0, t_0)$ with respect to the Hausdorff metric d_H for all $\alpha \in [0,1]$. A map $u : J_a \times J_b \times J_c \to E^n$ is called integrably bounded if there exists an integrable function $h \in L^1(J_a \times J_b \times J_c, R^n)$

such that $||y|| \leq h(x,y,t)$ for all $y \in u_0(x,y,t)$.
Definition 2.2 Let $u : J_a \times J_b \times J_c \to E^n$. The integral of u over $J_a \times J_b \times J_c$, denoted a $\int_0^a \int_0^b \int_0^c u(x,y,t)dxdydt$, is defined by

$$[\int_0^a \int_0^b \int_0^c u(x,y,t)dxdydt]_\alpha = \int_0^a \int_0^b \int_0^c u_\alpha(x,y,t)dxdydt$$
$$= \Big\{ \int_0^a \int_0^b \int_0^c v(x,y,t)dxdydt | v : J_a \times J_b \times J_c \to R^n \text{ is a measurable selection for } u_\alpha \Big\}$$

for all $\alpha \in (0,1]$. A strongly measurable and integrably bounded map $u : J_a \times J_b \times J_c \to E^n$ is said to be integrable over $J_a \times J_b \times J_c$, if $\int_0^a \int_0^b \int_0^c u(x,y,t)dxdydt \in E^n$.

If $u : J_a \times J_b \times J_c \to E^n$ is measurable and integrably bounded, then u is integrable.
Definition 2.3 [1] Let $u : J_a \times J_b \times J_c \to E^n$ and $(x_0, y_0, t_0) \in J_a \times J_b \times J_c$. u is said to be strongly generalized differentiable at (x_0, y_0, t_0) respect to x, if there exists an element $\frac{\partial u(x_0,y_0,t_0)}{\partial x} \in E^n$, such that for all $(h,0,0) > 0$ sufficiently small, $\exists u(x_0+h, y_0, t_0) - u(x_0, y_0, t_0), u(x_0, y_0, t_0) - u(x_0 - h, y_0, t_0)$ and the limits

$$\frac{\partial u(x_0,y_0,t_0)}{\partial x} = \lim_{h \to 0^+} \frac{u(x_0+h,y_0,t_0) - u(x_0,y_0,t_0)}{h} = \lim_{h \to 0^+} \frac{u(x_0-h,y_0,t_0) - u(x_0,y_0,t_0)}{h}.$$

Here the limit is taken in the metric space (E^n, D). The fuzzy partial derivatives of u with respect to y and t at the point $(x_0, y_0, t_0) \in J_a \times J_b \times J_c$ are defined similarly.

3 Fuzzy partial differential equation

Now, we consider the numerical solutions of the two-dimensional fuzzy parabolic equations with variable coefficients and non-local boundary conditions, i.e.

$$\frac{\partial \widetilde{u}}{\partial t} = a(x,y,t)\frac{\partial^2 \widetilde{u}}{\partial x^2} + b(x,y,t)\frac{\partial^2 \widetilde{u}}{\partial y^2}, 0 < x, y < 1, 0 < t < T, \tag{3.1}$$

$$\widetilde{u}(x,y,0) = \widetilde{f}(x,y), 0 \leq x, y \leq 1, \tag{3.2}$$

$$\widetilde{u}(x,0,t) = \int_0^1 \int_0^1 k(x,0,\eta,\gamma)\widetilde{u}(\eta,\gamma,t)d\eta d\gamma . 0 \leq x \leq 1, \tag{3.3}$$

$$\widetilde{u}(x,1,t) = \int_0^1 \int_0^1 k(x,1,\eta,\gamma)\widetilde{u}(\eta,\gamma,t)d\eta d\gamma . 0 \leq x \leq 1, \tag{3.4}$$

$$\widetilde{u}(0,y,t) = \int_0^1 \int_0^1 k(0,y,\eta,\gamma)\widetilde{u}(\eta,\gamma,t)d\eta d\gamma . 0 \leq y \leq 1, \tag{3.5}$$

$$\widetilde{u}(1,y,t) = \int_0^1 \int_0^1 k(1,y,\eta,\gamma)\widetilde{u}(\eta,\gamma,t)d\eta d\gamma . 0 \leq y \leq 1. \tag{3.6}$$

where the weight functions $a, b : [0,1] \times [0,1] \times [0,T] \to R$ are supposed to be continuous and positive. $\widetilde{f}$, $\widetilde{u}$, $\frac{\partial \widetilde{u}}{\partial t}$, $\frac{\partial^2 \widetilde{u}}{\partial x^2}$, $\frac{\partial^2 \widetilde{u}}{\partial y^2}$ are fuzzy-number-valued functions and their α-cut sets are as follows:

$$[\widetilde{f}(x,y)]_\alpha = [\underline{f}(x,y,\alpha), \overline{f}(x,y,\alpha)],$$

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$$\begin{aligned}
[\widetilde{u}(x,y,t)]_\alpha &= [\underline{u}(x,y,t,\alpha),\overline{u}(x,y,t,\alpha)],\\
[\frac{\partial^2\widetilde{u}}{\partial x^2}(x,y,t)]_\alpha &= [\frac{\partial^2\underline{u}}{\partial x^2}(x,y,t,\alpha),\frac{\partial^2\overline{u}}{\partial x^2}(x,y,t,\alpha)],\\
[\frac{\partial^2\widetilde{u}}{\partial y^2}(x,y,t)]_\alpha &= [\frac{\partial^2\underline{u}}{\partial y^2}(x,y,t,\alpha),\frac{\partial^2\overline{u}}{\partial y^2}(x,y,t,\alpha)],\\
[\frac{\partial\widetilde{u}}{\partial t}(x,y,t)]_\alpha &= [\frac{\partial\underline{u}}{\partial t}(x,y,t,\alpha),\frac{\partial\overline{u}}{\partial t}(x,y,t,\alpha)].
\end{aligned}$$

For some constant $0<\rho<1$ the kernel $k(x,y,\xi,\eta)$ satisfies

$$0<\int_0^1\int_0^1|k(x,y,\eta,\gamma)|\le\rho<1, 0\le x,y\le 1. \tag{3.7}$$

Problem (3.1)-(3.6) is non-standard because of the unusual boundary conditions (3.3)-(3.6).

We assume that the $\underline{u}(x,y,t,\alpha)$ and $\overline{u}(x,y,t,\alpha)$ have continuous partial derivatives so that $\frac{\partial^2\underline{u}}{\partial x^2}(x,y,t,\alpha)+\frac{\partial^2\underline{u}}{\partial y^2}(x,y,t,\alpha)$ and $\frac{\partial^2\overline{u}}{\partial x^2}(x,y,t,\alpha)+\frac{\partial^2\overline{u}}{\partial y^2}(x,y,t,\alpha)$ are continuous for all $(x,y,t)\in[0,1]\times[0,1]\times[0,T]$, and all α. Define

$$\begin{aligned}
\Gamma(x,y,t,\alpha) &= [\Gamma^-(x,y,t,\alpha),\Gamma^+(x,y,t,\alpha)]\\
&= [a(x,y,t)\frac{\partial^2\underline{u}}{\partial x^2}(x,y,t,\alpha)+b(x,y,t)\frac{\partial^2\underline{u}}{\partial y^2}(x,y,t,\alpha),a(x,y,t)\frac{\partial^2\overline{u}}{\partial x^2}(x,y,t,\alpha)\\
&\quad +b(x,y,t)\frac{\partial^2\overline{u}}{\partial y^2}(x,y,t,\alpha)].
\end{aligned}$$

for all $(x,y,t)\in[0,1]\times[0,1]\times[0,T]$, and all α. If for each fixed $(x,y,t)\in[0,1]\times[0,1]\times[0,T]$, $\Gamma(x,y,t,\alpha)$ defines the α-cut of a fuzzy number, then we will say that $\widetilde{u}(x,y,t)$ is differentiable. Sufficient conditions for $\Gamma(x,y,t,\alpha)$ to define α-cuts of a fuzzy number are:

1. $\Gamma^-(x,y,t,\alpha)$ is an increasing function of α for each $(x,y,t)\in[0,1]\times[0,1]\times[0,T]$;
2. $\Gamma^+(x,y,t,\alpha)$ is an decreasing function of α for each $(x,y,t)\in[0,1]\times[0,1]\times[0,T]$; and
3. $\Gamma^-(x,y,t,1)\le\Gamma^+(x,y,t,1)$ for all $(x,y,t)\in[0,1]\times[0,1]\times[0,T]$.

Hence we get from (3.1):

$$\begin{aligned}
[\frac{\partial\underline{u}}{\partial t}(x,y,t,\alpha),\frac{\partial\overline{u}}{\partial t}(x,y,t,\alpha)] &= a(x,y,t)[\frac{\partial^2\underline{u}}{\partial x^2}(x,y,t,\alpha),\frac{\partial^2\overline{u}}{\partial x^2}(x,y,t,\alpha)]\\
&\quad +b(x,y,t)[\frac{\partial^2\underline{u}}{\partial y^2}(x,y,t,\alpha),\frac{\partial^2\overline{u}}{\partial y^2}(x,y,t,\alpha)].
\end{aligned}$$

Consider the system of partial differential equations

$$\frac{\partial\underline{u}}{\partial t}(x,y,t,\alpha)=a(x,y,t)\frac{\partial^2\underline{u}}{\partial x^2}(x,y,t,\alpha)+b(x,y,t)\frac{\partial^2\underline{u}}{\partial y^2}(x,y,t,\alpha), \tag{3.8}$$

$$\frac{\partial\overline{u}}{\partial t}(x,y,t,\alpha)]=a(x,y,t)\frac{\partial^2\overline{u}}{\partial x^2}(x,y,t,\alpha)+b(x,y,t)\frac{\partial^2\overline{u}}{\partial y^2}(x,y,t,\alpha), \tag{3.9}$$

We append to equations (3.8) and (3.9) any boundary conditions and initial values conditions, for example, if they are $\widetilde{u}(0,y,t) = \widetilde{c}_1, \widetilde{u}(1,y,t) = \widetilde{c}_2, \widetilde{u}(x,0,t) = \widetilde{c}_3, \widetilde{u}(x,1,t) = \widetilde{c}_4$, and $\widetilde{u}(x,y,0) = \widetilde{f}$, then we add

$$\underline{u}(0,y,t,\alpha) = \underline{c}_1(\alpha),\ \underline{u}(1,y,t,\alpha) = \underline{c}_2(\alpha), \underline{u}(x,0,t,\alpha) = \underline{c}_3(\alpha),$$

$$\underline{u}(x,1,t,\alpha) = \underline{c}_4(\alpha), \underline{u}(x,y,0) = \underline{f}(x,y) \tag{3.10}$$

to equation (3.8) and

$$\overline{u}(0,y,t,\alpha) = \overline{c}_1(\alpha),\ \overline{u}(1,y,t,\alpha) = \overline{c}_2(\alpha), \overline{u}(x,0,t,\alpha) = \overline{c}_3(\alpha),$$

$$\overline{u}(x,1,t,\alpha) = \overline{c}_4(\alpha), \overline{u}(x,y,0) = \overline{f}(x,y) \tag{3.11}$$

to equation (3.9) where $[\widetilde{c}_i]_\alpha = [\underline{c}_i(\alpha), \overline{c}_i(\alpha)], i = 1,2,3,4$. Let $\underline{u}(x,y,t,\alpha)$ and $\overline{u}(x,y,t,\alpha)$ solves equations (3.8) and (3.9), plus the boundary equations, respectively. If $[\widetilde{u}(x,y,t)]_\alpha = [\underline{u}(x,y,t,\alpha), \overline{u}(x,y,t,\alpha)]$ determined a fuzzy number, for all $(x,y,t) \in [0,1]\times[0,1]\times[0,T]$, then $\widetilde{u}(x,y,t)$ is the solution for (3.1), see [5].

4 Difference methods

At the beginning of this section we introduce notations that we will use through out the whole paper.

The domain $[0,1]^2 \times [0,T]$ will be divided into an $M^2 \times N$ mesh with spatial step size $h = 1/M$ in both x and y directions and the time step size $k = T/N$ respectively.

Grid points (x_i, y_j, t_n) are given by

$$x_i = ih, ; i = 0,1,2,\cdots, M, \tag{4.1}$$

$$y_j = jh, ; j = 0,1,2,\cdots, M, \tag{4.2}$$

$$t_n = nk, ; n = 0,1,2,\cdots, N, \tag{4.3}$$

in which M is an even integer. We use $\underline{u}^n_{i,j}(\overline{u}^n_{i,j})$ to denote the finite difference approximations of $\underline{u}(ih,jh,nk)(\overline{u}(ih,jh,nk))$. The notation $\underline{u}^{n+1/2}_{i,j}(\overline{u}^{n+1/2}_{i,j})$ refers to values of $\underline{u}_{i,j}(\overline{u}_{i,j})$ computed at the intermediate stage, that is, at time $(t_n + k/2)$, i.e.

$$\underline{u}^{n+\frac{1}{2}}_{ij} = \frac{1}{2}(\underline{u}^n_{ij} + \underline{u}^{n+1}_{ij}), \tag{4.4}$$

$$\overline{u}^{n+\frac{1}{2}}_{ij} = \frac{1}{2}(\overline{u}^n_{ij} + \overline{u}^{n+1}_{ij}). \tag{4.5}$$

The numerical methods suggested here are based on second-order forward Euler method and fully explicit finite difference scheme. Firstly, the second-order forward Euler method is studied.

Let assume that

$$\int_0^1 \int_0^1 k(x,y,\eta,\gamma)[\widetilde{u}(\eta,\gamma,t)]_\alpha d\eta d\gamma$$

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$$= \int_0^1 \int_0^1 k(x,y,\eta,\gamma)[\underline{u}(\eta,\gamma,t,\alpha),\overline{u}(\eta,\gamma,t,\alpha)]d\eta d\gamma$$
$$\simeq \sum_{m=0}^{M}\sum_{l=0}^{M} w_{m,l}k(x,y,x_m,y_l)[\underline{u}^n_{m,l}(\alpha),\overline{u}^n_{m,l}(\alpha)]\Delta x\Delta y, \tag{4.6}$$

where $w_{m,l} \geq 0$ are weights and $\underline{u}^n_{m,l}(\alpha) = u(x_m,y_l,t_n,\alpha), \overline{u}^n_{m,l}(\alpha) = u(x_m,y_l,t_n,\alpha)$ with $x_m = m\Delta x = mh, y_l = l\Delta y = lh, t_n = nk, m,l = 0,1,2,\cdots,M, n = 0,1,2,\cdots,N$.

If we use the trapezoidal numerical integration rule [14] then

$$w_{m,l} = 1, m,l = 1,2,\cdots,M-1,$$
$$w_{m,l} = \frac{1}{4}, m,l \in \{0,M\}, \tag{4.7}$$
$$w_{m,l} = \tfrac{1}{2}, \text{otherwise.}$$

4.1 The second-order forward Euler method (SFEM)

The second-order forward Euler method uses the following approximations for $\underline{u}_{xx}(\alpha)$,$\overline{u}_{xx}(\alpha)$ and $\underline{u}_{yy}(\alpha), \overline{u}_{yy}(\alpha)$ and $\underline{u}_t(\alpha)$,$\overline{u}_t(\alpha)$.

$$\frac{\partial^2\underline{u}}{\partial x^2}|^n_{i,j}(\alpha) \simeq \frac{\underline{u}^n_{i+1,j}(\alpha) - 2\underline{u}^n_{ij}(\alpha) + \underline{u}^n_{i-1,j}(\alpha)}{h^2}, \tag{4.1.1}$$
$$\frac{\partial^2\overline{u}}{\partial x^2}|^n_{i,j}(\alpha) \simeq \frac{\overline{u}^n_{i+1,j}(\alpha) - 2\overline{u}^n_{ij}(\alpha) + \overline{u}^n_{i-1,j}(\alpha)}{h^2}, \tag{4.1.2}$$
$$\frac{\partial^2\underline{u}}{\partial x^2}(x_i,y_j,t_n,\alpha) = \frac{\partial^2\underline{u}}{\partial x^2}|^n_{i,j}(\alpha) - \frac{h^2}{24}\big(\frac{\partial^4\underline{u}}{\partial x^4}(\xi_i,y_j,t_n,\alpha) + \frac{\partial^4\underline{u}}{\partial x^4}(\eta_i,y_j,t_n,\alpha)\big), \tag{4.1.3}$$
$$\frac{\partial^2\overline{u}}{\partial x^2}(x_i,y_j,t_n,\alpha) = \frac{\partial^2\overline{u}}{\partial x^2}|^n_{i,j}(\alpha) - \frac{h^2}{24}\big(\frac{\partial^4\overline{u}}{\partial x^4}(\xi_i^{'},y_j,t_n,\alpha) + \frac{\partial^4\overline{u}}{\partial x^4}(\eta_i^{'},y_j,t_n,\alpha)\big), \tag{4.1.4}$$

$\xi_i, \xi_i^{'} \in (x_{i-1},x_i), \eta_i, \eta_i^{'} \in (x_i,x_{i+1})$, provided that $\frac{\partial^2\widetilde{u}}{\partial x^2}(x,y,t) \in E$, it means we only need to check if $\frac{\partial}{\partial\alpha}\big(\frac{\partial^2\underline{u}}{\partial x^2}(x,y,t,\alpha)\big) > 0$ and $\frac{\partial}{\partial\alpha}\big(\frac{\partial^2\underline{u}}{\partial x^2}(x,y,t,\alpha)\big) < 0$. Since the $\frac{\partial^2\underline{u}}{\partial x^2}$ and $\frac{\partial^2\overline{u}}{\partial x^2}$ are continuous and $\frac{\partial^2\underline{u}}{\partial x^2}(x,y,t,1) = \frac{\partial^2\overline{u}}{\partial x^2}(x,y,t,1)$. Then following equations defines the α-cuts of fuzzy numbers,

$$\big[\frac{\partial^2\widetilde{u}}{\partial x^2}(x,y,t)\big]_\alpha = \big[\frac{\partial^2\underline{u}}{\partial x^2}(x,y,t,\alpha), \frac{\partial^2\overline{u}}{\partial x^2}(x,y,t,\alpha)\big], \alpha \in [0,1], \tag{4.1.5}$$

and also

$$[\widetilde{u}^n_{i,j}]_\alpha = [\underline{u}^n_{i,j}(\alpha), \overline{u}^n_{i,j}(\alpha)], \tag{4.1.6}$$

where

$$\frac{\partial^2\underline{u}}{\partial x^2}(x,y,t,\alpha) = \min\{f | f \in \big[\frac{\partial^2\widetilde{u}}{\partial x^2}(x,y,t)\big]_\alpha\}, \tag{4.1.7}$$
$$\frac{\partial^2\overline{u}}{\partial x^2}(x,y,t,\alpha) = \max\{f | f \in \big[\frac{\partial^2\widetilde{u}}{\partial x^2}(x,y,t)\big]_\alpha\}, \tag{4.1.8}$$
$$\underline{u}^n_{i,j}(\alpha) = \min\{g | g \in [\widetilde{u}^n_{i,j}]_\alpha\}, \tag{4.1.9}$$

$$\overline{u}_{i,j}^{n}(\alpha) = \max\{g | g \in [\widetilde{u}_{i,j}^{n}]_{\alpha}\}. \tag{4.1.10}$$

Also for $\frac{\partial^2 \widetilde{u}}{\partial y^2}(x, y, t)$ we have the following formulas:

$$\frac{\partial^2 \underline{u}}{\partial y^2}|_{i,j}^{n}(\alpha) \simeq \frac{\underline{u}_{i,j+1}^{n}(\alpha) - 2\underline{u}_{ij}^{n}(\alpha) + \underline{u}_{i,j-1}^{n}(\alpha)}{h^2}, \tag{4.1.11}$$

$$\frac{\partial^2 \overline{u}}{\partial y^2}|_{i,j}^{n}(\alpha) \simeq \frac{\overline{u}_{i,j+1}^{n}(\alpha) - 2\overline{u}_{ij}^{n}(\alpha) + \overline{u}_{i,j-1}^{n}(\alpha)}{h^2}, \tag{4.1.12}$$

$$\frac{\partial^2 \underline{u}}{\partial y^2}(x_i, y_j, t_n, \alpha) = \frac{\partial^2 \underline{u}}{\partial y^2}|_{i,j}^{n}(\alpha) - \frac{h^2}{24}\big(\frac{\partial^4 \underline{u}}{\partial y^4}(x_i, \xi_j, t_n, \alpha) + \frac{\partial^4 \underline{u}}{\partial y^4}(x_i, \eta_j, t_n, \alpha)\big), \tag{4.1.13}$$

$$\frac{\partial^2 \overline{u}}{\partial y^2}(x_i, y_j, t_n, \alpha) = \frac{\partial^2 \overline{u}}{\partial y^2}|_{i,j}^{n}(\alpha) - \frac{h^2}{24}\big(\frac{\partial^4 \overline{u}}{\partial y^4}(x_i, \xi_j^{'}, t_n, \alpha) + \frac{\partial^4 \overline{u}}{\partial y^4}(x_i, \eta_j^{'}, t_n, \alpha)\big), \tag{4.1.14}$$

where $\xi_j, \xi_j^{'} \in (y_{j-1}, y_j), \eta_j, \eta_j^{'} \in (y_j, y_{j+1})$, provided that $\frac{\partial^2 \widetilde{u}}{\partial y^2}(x, y, t) \in E$.
For $\frac{\partial \widetilde{u}}{\partial t}(x, y, t)$ we have

$$\frac{\partial \underline{u}}{\partial t}|_{i,j}^{n}(\alpha) \simeq \frac{u_{i,j}^{n+1}(\alpha) - u_{ij}^{n}(\alpha)}{k}, \tag{4.1.15}$$

$$\frac{\partial \overline{u}}{\partial t}|_{i,j}^{n}(\alpha) \simeq \frac{u_{i,j}^{n+1}(\alpha) - u_{ij}^{n}(\alpha)}{k}. \tag{4.1.16}$$

$$\frac{\partial \underline{u}}{\partial t}(x_i, y_j, t_n, \alpha) = \frac{\partial \underline{u}}{\partial t}|_{i,j}^{n}(\alpha) - \frac{k}{2} \cdot \frac{\partial^2 \underline{u}}{\partial t^2}(x_i, y_j, \xi_n, \alpha), \tag{4.1.17}$$

$$\frac{\partial \overline{u}}{\partial t}(x_i, y_j, t_n, \alpha) = \frac{\partial \overline{u}}{\partial t}|_{i,j}^{n}(\alpha) - \frac{k}{2} \cdot \frac{\partial^2 \overline{u}}{\partial t^2}(x_i, y_j, \xi_n^{'}, \alpha), \tag{4.1.18}$$

where $\xi_n, \xi_n^{'} \in (t_n, t_{n+1})$, provided that $\frac{\partial \widetilde{u}}{\partial t}(x, y, t) \in E$.
The second-order forward Euler method for solving our two-dimensional problem leads to

$$\underline{u}_{i,j}^{n+1}(\alpha) = a_{ij}^{n} s \underline{u}_{i-1,j}^{n}(\alpha) + a_{ij}^{n} s \underline{u}_{i+1,j}^{n}(\alpha) + (1 - 2a_{ij}^{n} s - 2b_{ij}^{n} s)\underline{u}_{i,j}^{n}(\alpha) + b_{ij}^{n} s \underline{u}_{i,j-1}^{n}(\alpha) + b_{ij}^{n} s \underline{u}_{i,j+1}^{n}(\alpha), \tag{4.1.19}$$

$$\overline{u}_{i,j}^{n+1}(\alpha) = a_{ij}^{n} s \overline{u}_{i-1,j}^{n}(\alpha) + a_{ij}^{n} s \overline{u}_{i+1,j}^{n}(\alpha) + (1 - 2a_{ij}^{n} s - 2b_{ij}^{n} s)\overline{u}_{i,j}^{n}(\alpha) + b_{ij}^{n} s \overline{u}_{i,j-1}^{n}(\alpha) + b_{ij}^{n} s \overline{u}_{i,j+1}^{n}(\alpha), \tag{4.1.20}$$

where $a_{i,j}^{n}$ and $b_{i,j}^{n}$ denote the finite difference approximations of $a(ih, jh, nk)$ and (ih, jh, nk), respectively. $0 \leq n \leq N - 1$ for $i, j = 1, 2, \cdots, M - 1$, and

$$s = \frac{k}{h^2}.$$

By applying "frozen" coefficient method of Ref.[8], we can get the range of stability for this scheme is

$$0 \leq \lambda \leq \frac{1}{4},$$

where

$$\lambda = \lambda^{'} * s, \lambda^{'} = \max_{i,j,n}\{a_{ij}^{n}, b_{ij}^{n}\}.$$

It can be seen that the local truncation error for this equations is $\tau_{i,j}^{n} = O(h^2 + h^2 + k)$.

4.2 The fully explicit finite difference scheme (FEFDS)

The fully explicit finite difference scheme uses a forward-difference approximation for the time derivative and the weighted approximations for the spatial derivatives. For (3.1) we have the following approximations:

$$\frac{\underline{u}_{ij}^{n+1}-\underline{u}_{ij}^{n}}{k} \simeq a_{i,j-1}^{n}\frac{\underline{u}_{i+1,j-1}^{n}-2\underline{u}_{i,j-1}^{n}+\underline{u}_{i-1,j-1}^{n}}{h^2}+b_{i-1,j}^{n}\frac{\underline{u}_{i-1,j+1}^{n}-2\underline{u}_{i-1,j}^{n}+\underline{u}_{i-1,j-1}^{n}}{h^2}, \tag{4.2.1}$$

$$\frac{\underline{u}_{ij}^{n+1}-\underline{u}_{ij}^{n}}{k} \simeq a_{i,j}^{n}\frac{\underline{u}_{i+1,j}^{n}-2\underline{u}_{i,j}^{n}+\underline{u}_{i-1,j}^{n}}{h^2}+b_{i,j}^{n}\frac{\underline{u}_{i,j+1}^{n}-2\underline{u}_{i,j}^{n}+\underline{u}_{i,j-1}^{n}}{h^2}, \tag{4.2.2}$$

$$\frac{\underline{u}_{ij}^{n+1}-\underline{u}_{ij}^{n}}{k} \simeq a_{i,j+1}^{n}\frac{\underline{u}_{i+1,j+1}^{n}-2\underline{u}_{i,j+1}^{n}+\underline{u}_{i-1,j+1}^{n}}{h^2}+b_{i+1,j}^{n}\frac{\underline{u}_{i+1,j+1}^{n}-2\underline{u}_{i+1,j}^{n}+\underline{u}_{i+1,j-1}^{n}}{h^2}. \tag{4.2.3}$$

By $\frac{s}{2}\times(4.2.1)+\frac{s}{2}\times(4.2.3)+(1-s)\times(4.2.2)$, we have

$$\begin{aligned}\frac{\underline{u}_{ij}^{n+1}-\underline{u}_{ij}^{n}}{\tau} &= \frac{s}{2}\Big(a_{i,j-1}^{n}\frac{\underline{u}_{i+1,j-1}^{n}-2\underline{u}_{i,j-1}^{n}+\underline{u}_{i-1,j-1}^{n}}{h^2}+b_{i-1,j}^{n}\frac{\underline{u}_{i-1,j+1}^{n}-2\underline{u}_{i-1,j}^{n}+\underline{u}_{i-1,j-1}^{n}}{h^2}\Big)\\ &+\frac{s}{2}\Big(a_{i,j+1}^{n}\frac{\underline{u}_{i+1,j+1}^{n}-2\underline{u}_{i,j+1}^{n}+\underline{u}_{i-1,j+1}^{n}}{h^2}+b_{i+1,j}^{n}\frac{\underline{u}_{i+1,j+1}^{n}-2\underline{u}_{i+1,j}^{n}+\underline{u}_{i+1,j-1}^{n}}{h^2}\Big)\\ &+(1-s)\Big(a_{i,j}^{n}\frac{\underline{u}_{i+1,j}^{n}-2\underline{u}_{i,j}^{n}+\underline{u}_{i-1,j}^{n}}{h^2}+b_{i,j}^{n}\frac{\underline{u}_{i+1,j}^{n}-2\underline{u}_{i,j}^{n}+\underline{u}_{i-1,j}^{n}}{h^2}\Big).\end{aligned} \tag{4.2.4}$$

These approximations produce the following finite difference equation:

$$\begin{aligned}\underline{u}_{i,j}^{n+1} &= \frac{s^2}{2}\Big[\big(a_{i,j-1}^{n}+b_{i+1,j}^{n}\big)\underline{u}_{i+1,j-1}^{n}+\big(a_{i,j-1}^{n}+b_{i-1,j}^{n}\big)\underline{u}_{i-1,j-1}^{n}+\big(a_{i,j+1}^{n}+b_{i+1,j}^{n}\big)\underline{u}_{i+1,j+1}^{n}\\ &+\big(a_{i,j+1}^{n}+b_{i-1,j}^{n}\big)\underline{u}_{i-1,j+1}^{n}\Big]+\big(sb_{ij}^{n}-s^2b_{ij}^{n}-s^2a_{i,j-1}^{n}\big)\underline{u}_{i,j-1}^{n}+\big(sb_{ij}^{n}-s^2b_{ij}^{n}-s^2a_{i,j+1}^{n}\big)\underline{u}_{i,j+1}^{n}\\ &+\big(sa_{ij}^{n}-s^2a_{ij}^{n}-s^2b_{i-1,j}^{n}\big)\underline{u}_{i-1,j}^{n}+\big(sa_{ij}^{n}-s^2a_{ij}^{n}-s^2b_{i+1,j}^{n}\big)\underline{u}_{i+1,j}^{n}+\big[1-2(a_{ij}^{n}+b_{ij}^{n})s(1-s)\big]\underline{u}_{ij}^{n}.\end{aligned} \tag{4.2.5}$$

Moreover, we can get the following finite difference equation for $\overline{u}_{i,j}^{n+1}$ as follows:

$$\begin{aligned}\overline{u}_{i,j}^{n+1} &= \frac{s^2}{2}\Big[\big(a_{i,j-1}^{n}+b_{i+1,j}^{n}\big)\overline{u}_{i+1,j-1}^{n}+\big(a_{i,j-1}^{n}+b_{i-1,j}^{n}\big)\overline{u}_{i-1,j-1}^{n}+\big(a_{i,j+1}^{n}+b_{i+1,j}^{n}\big)\overline{u}_{i+1,j+1}^{n}\\ &+\big(a_{i,j+1}^{n}+b_{i-1,j}^{n}\big)\overline{u}_{i-1,j+1}^{n}\Big]+\big(sb_{ij}^{n}-s^2b_{ij}^{n}-s^2a_{i,j-1}^{n}\big)\overline{u}_{i,j-1}^{n}+\big(sb_{ij}^{n}-s^2b_{ij}^{n}-s^2a_{i,j+1}^{n}\big)\overline{u}_{i,j+1}^{n}\\ &+\big(sa_{ij}^{n}-s^2a_{ij}^{n}-s^2b_{i-1,j}^{n}\big)\overline{u}_{i-1,j}^{n}+\big(sa_{ij}^{n}-s^2a_{ij}^{n}-s^2b_{i+1,j}^{n}\big)\overline{u}_{i+1,j}^{n}+\big[1-2(a_{ij}^{n}+b_{ij}^{n})s(1-s)\big]\overline{u}_{ij}^{n}.\end{aligned} \tag{4.2.6}$$

provided that $\frac{\partial\widetilde{u}}{\partial t}(x,y,t)\in E$.

5 Computational example

We consider a problem to computationally test the present finite difference methods and compare their performance. We demonstrate the effectiveness of our proposed schemes by

computing approximate solutions of a two-dimensional test problem. Consider (3.1)-(3.6) with

$$[\widetilde{f}(x,y)]_\alpha = [(0.75+0.25\alpha)xe^y, (1.25-0.25\alpha)xe^y], \tag{5.1}$$

$$a(x,y,t) = xy, \tag{5.2}$$

$$b(x,y,t) = 1, \tag{5.3}$$

which is easily seen to have exact solutions for (3.8) and (3.9) are

$$\underline{u}(x,y,t,\alpha) = (0.75+0.25\alpha)xe^{y+t}, \quad \overline{u}(x,y,t,\alpha) = (1.25-0.25\alpha)xe^{y+t}.$$

Without loss of generality, in what follows, we will examine our method with k given by:

$$k(x,y,\eta,\gamma) = \frac{1}{4\pi}. \tag{5.4}$$

We will use equations (4.1.19) and (4.1.20) to approximate the exact solutions with $T = 1.00, h = 0.02, s = \frac{1}{8}$ and $\alpha = 0.2$.

Table 5.1: Relative values at various points for $\underline{u}(x,y,t,\alpha)$

x	y	t	α	SFEM	Exact
0.1	0.1	5.0×10^{-4}	0.2	0.0057	0.0885
0.2	0.2	5.0×10^{-4}	0.2	0.0166	0.1955
0.3	0.3	5.0×10^{-4}	0.2	0.0386	0.3241
0.4	0.4	5.0×10^{-4}	0.2	0.0791	0.4776
0.5	0.5	5.0×10^{-4}	0.2	0.1490	0.6598
0.6	0.6	5.0×10^{-4}	0.2	0.2654	0.8751
0.7	0.7	5.0×10^{-4}	0.2	0.4573	1.1283
0.8	0.8	5.0×10^{-4}	0.2	0.7764	1.4251
0.9	0.9	5.0×10^{-4}	0.2	1.3113	1.7718

Table 5.2: Relative values at various points for $\overline{u}(x,y,t,\alpha)$

x	y	t	α	SFEM	Exact
0.1	0.1	5.0×10^{-4}	0.2	0.0085	0.1327
0.2	0.2	5.0×10^{-4}	0.2	0.0250	0.2933
0.3	0.3	5.0×10^{-4}	0.2	0.0579	0.4862
0.4	0.4	5.0×10^{-4}	0.2	0.1186	0.7164
0.5	0.5	5.0×10^{-4}	0.2	0.2235	0.9897
0.6	0.6	5.0×10^{-4}	0.2	0.3981	1.3126
0.7	0.7	5.0×10^{-4}	0.2	0.6860	1.6924
0.8	0.8	5.0×10^{-4}	0.2	1.1645	2.1376
0.9	0.9	5.0×10^{-4}	0.2	1.9669	2.6577

The numerical results of the exact solutions and approximate solutions of $\underline{u}(x,y,t,\alpha)$ are shown with Fig 1. The first picture characterizes the exact solutions of $\underline{u}(x,y,t,\alpha)$ at

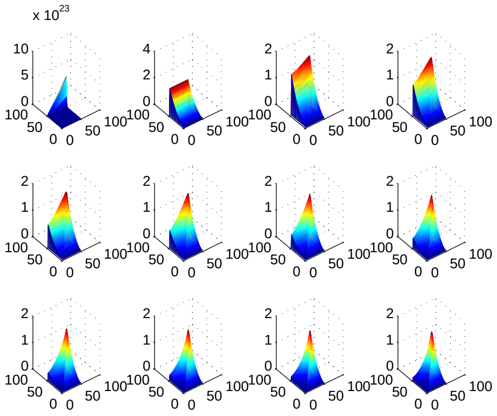

Figure 1: Exact solutions and approximate solutions for $\underline{u}$ with $t = 5 * 10^{-4}, \alpha = 0.2$, $x, y \in [0, 1]$, the first picture is exact solutions, and the last picture is approximate solutions with $n = 11$.

$(x, y, 5 * 10^{-4}, 0.2)$, $x, y \in [0, 1]$; and the others are approximate solutions for $\underline{u}^n_{i,j}$ from $n = 1$ to $n = 11$. For example, the last picture describes the approximate solutions for $\underline{u}^n_{i,j}$ with $n = 11$. The numerical results of the exact solutions and approximate solutions of $\overline{u}(x, y, t, \alpha)$ are shown with Fig 2. The first picture characterizes the exact solutions of $\overline{u}(x, y, t, \alpha)$ at $(x, y, 5 * 10^{-4}, 0.2)$, $x, y \in [0, 1]$; and the others are approximate solutions for $\overline{u}^n_{i,j}$ from $n = 1$ to $n = 11$. For example, the last picture describes the approximate solutions for $\overline{u}^n_{i,j}$ with $n = 11$.

The numerical results of the absolute errors for $\underline{u}(x, y, t, \alpha)$ are shown with Fig 3. The first picture describes the exact solution of $\overline{u}(x, y, t, \alpha)$ at $(x, y, 5*10^{-4}, 0.2)$, $x, y \in [0, 1]$; and the others are absolute errors for $\underline{u}(x, y, t, \alpha)$ from $n = 1$ to $n = 11$. For example, the last picture describes the absolute errors between $\underline{u}^n_{i,j}$ and $\underline{u}(x, y, t, \alpha)$ with $n = 11, t = 5 * 10^{-4}$ for any $x, y \in [0, 1]$, $i, j = 1, 2, \cdots, 50$.

The numerical results of the absolute errors for $\underline{u}(x, y, t, \alpha)$ are shown with Fig 3. The first picture describes the exact solution of $\overline{u}(x, y, t, \alpha)$ at $(x, y, 5*10^{-4}, 0.2)$, $x, y \in [0, 1]$; and the others are absolute errors for $\underline{u}(x, y, t, \alpha)$ from $n = 1$ to $n = 11$. For example, the last picture describes the absolute errors between $\underline{u}^n_{i,j}$ and $\underline{u}(x, y, t, \alpha)$ with $n = 11, t = 5 * 10^{-4}$ for any $x, y \in [0, 1]$, $i, j = 1, 2, \cdots, 50$.

From Fig. 1 and Fig. 3, the approximation effect for SFEM of $\underline{u}(x, y, t, \alpha)$ is better (also see Table 5.1). Similarly, From Fig. 2 and Fig. 4, the approximation effect for SFEM of $\overline{u}(x, y, t, \alpha)$ is also better (also see Table 5.2).

6 Conclusions and future work

Two kinds of two-level finite difference procedures have been developed in this paper for the numerical solution of a fuzzy parabolic partial differential equation with non-standard boundary conditions on four boundaries. Numerical results confirmed that the difference formulas obtained from the new discretization technique outlined in this paper do yield second order convergence for the solution of our problem. This paper has outlined a new

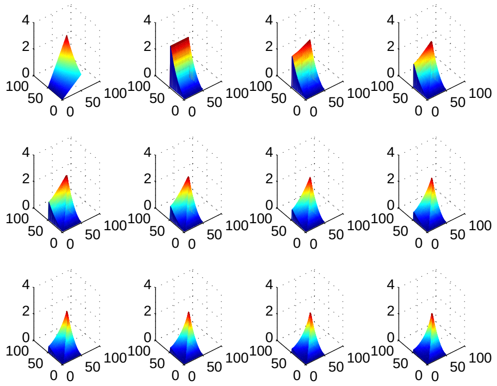

Figure 2: Exact solutions and approximate solutions for $\overline{u}$ with $t = 5 * 10^{-4}, \alpha = 0.2$, $x, y \in [0, 1]$, the first picture is exact solutions, and the last picture is approximate solutions with $n = 11$.

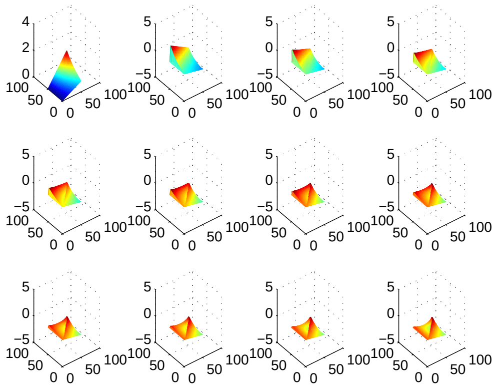

Figure 3: Exact solutions and absolute errors for $\underline{u}$ with $t = 5 * 10^{-4}, \alpha = 0.2$, $x, y \in [0, 1]$, the first picture is exact solutions, and the last picture is absolute errors with $n = 11$.

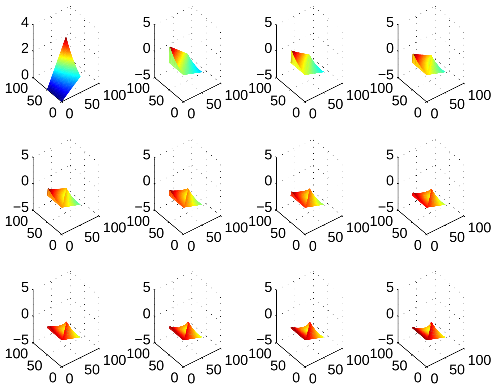

Figure 4: Exact solutions and absolute errors for $\overline{u}$ with $t = 5 * 10^{-4}, \alpha = 0.2$, $x, y \in [0, 1]$, the first picture is exact solutions, and the last picture is absolute errors with $n = 11$.

approach for the study of the two-dimensional fuzzy parabolic partial differential equations with non-classical boundary conditions. Finally, an example is given to illustrate the second-order forward Euler method. Furthermore, some graphics of exact solutions, approximate solutions and absolute errors are given to illustrate the results which are obtained from Table 5.1 and Table 5.2. It is worth noting that our approaches could extend to the similar three-dimensional problem.

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2. **G.G.Lorentz, (title of book in italics) *Bernstein Polynomials* (2nd ed.), Chelsea,New York,1986.**

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Decision making based on intuitionistic fuzzy soft sets and its algorithm *

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Abstract: This paper discusses decision making based on intuitionistic fuzzy soft sets by means of grey relational analysis and D-S theory of evidence. An algorithm on this decision making is presented and an example is employed to show the algorithm is efficient for solving decision problems.

Keywords: Intuitionistic fuzzy soft sets; Decision making; Grey relational analysis; D-S theory of evidence; Algorithm.

1 Introduction

In 1999, Molodtsov [13] initiated soft set as a new mathematical tool for dealing with vagueness and uncertainties. Compared with some traditional tools for dealing with uncertainties, such as probability theory, fuzzy set theory [22], rough set theory [17], soft set theory has the advantage of freeing from the inadequacy of the parametrization tools of those theories.

With the rapid development of soft set theory, there has been some progress on the practical applications, especially the use of soft sets in decision making. Roy et al. [8] discussed score value as the evaluation basis to find an optimal choice object in fuzzy soft sets. But Kong et al. [6] argued that the Roy's method was incorrect by using a counter example to discuss two evaluation bases of choice value and score value, and they proposed a revised algorithm. Later Feng et al. [9] applied level soft sets to discuss fuzzy soft sets based decision making. Based on Feng' works, Basu et al. [2] further investigated the previous methods to fuzzy soft sets in decision making and introduced the mean potentiality approach, which was showed more efficient and more accurate

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than the previous methods. Jiang et al. [11] presented an adjustable approach to intuitionistic fuzzy soft sets based decision making by generalizing Feng's approach to fuzzy soft sets based decision making [9]. Zhang proposed a rough set approach to intuitionistic fuzzy soft set based decision making [26].

All of the above methods for soft sets in decision making are mainly based on the level soft set to obtain useful information such as choice values and score values. However, the existing methods have their limitations. For example, it is very difficult for decision maker to select a suitable level soft set to reduce subjectivity and uncertainty (see [26]). Moreover, there has been rather little work completed for intuitionistic fuzzy soft set based decision making. Then it is necessary to pay attention to this issue.

Grey relational analysis, initiated by Deng [4], is an important method to reflect uncertainty in grey system theory, which is utilized for generalizing estimates under small samples and uncertain conditions. It has been successfully applied in solving decision-making problems [5, 19, 20, 25]. D-S theory of evidence, proposed by Dempster [3] and Shafer [18], is a powerful method for combining accumulative evidence of changing prior opinions in the light of new evidences [18]. Compared to probability theory, this theory captures more information to support decision making by identifying the uncertain and unknown evidence. It provides a mechanism to derive solutions from various vague evidences without knowing much prior information. Therefore, combining both theories enables the decision makers to take advantage of both methods' merits and make evaluation experts to deal with uncertainty and risk confidently. Thus, this not only allows us to avoid selecting a suitable level soft set, but also helps reducing humanistic and subjective in nature to raise the choices decision level. The hybrid model is effective and practical under uncertainty [19, 21].

The purpose of this paper is to investigate decision making based on intuitionistic fuzzy soft sets by combining grey relational analysis and D-S theory of evidence.

2 Preliminaries

Throughout this paper, U denotes an initial universe, E denotes the set of all possible parameters, 2^U denotes the family of all subsets of U. We only consider the case where U and E are both nonempty finite sets.

2.1 Intuitionistic fuzzy soft sets

Definition 2.1 ([1]). *An intuitionistic fuzzy set $\widetilde{X}$ over U is an object having the form $\widetilde{X} = \{(x, \mu_{\widetilde{X}}(x), \nu_{\widetilde{X}}(x)) | x \in U\}$ $(e \in A)$, where $\mu_{\widetilde{X}} : U \to [0,1]$ and $\nu_{\widetilde{X}} : U \to [0,1]$ satisfy $0 \leqslant \mu_{\widetilde{X}}(x) + \nu_{\widetilde{X}}(x) \leqslant 1$ for all $x \in U$.*

$\mu_{\widetilde{X}}(x)$ and $\nu_{\widetilde{X}}(x)$ are called the membership degree and non-membership degree of the element $x \in U$ to $\widetilde{X}$.

The set of all intuitionistic fuzzy subsets of U is denoted by $IF(U)$.

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Definition 2.2 ([13]). *Let $A \subseteq E$. A pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \to 2^U$.*

Definition 2.3 ([15]). *Let $A \subseteq E$. A pair (F, A) is called an intuitionistic fuzzy soft set over U, where F is a mapping given by $F : A \to IF(U)$.*

In other words, an intuitionistic fuzzy soft set over U is a parameterized family of intuitionistic fuzzy subsets of U. For any $e \in A$, $F(e)$ is referred as the set of e-approximate elements of (F, A) and can be written as:

$$F(e) = \{(x, \mu_{F(e)}(x), \nu_{F(e)}(x))|x \in U\} \ (e \in A),$$

where $\mu_{F(e)} : U \to [0, 1]$, $\nu_{F(e)} : U \to [0, 1]$ satisfy $0 \leqslant \mu_{F(e)}(x) + \nu_{F(e)}(x) \leqslant 1$.

Definition 2.4 ([15]). *Let (F, A) be an intuitionistic fuzzy soft set over U. Let $x \in U$ and $e \in A$. $(\mu_{F(e)}(x), \nu_{F(e)}(x))$ is called e-intuitionistic fuzzy number of x. $\mu_{F(e)}(x)$ and $\nu_{F(e)}(x)$ are called the membership degree and non-membership degree that x holds e, respectively. $\pi_{F(e)}(x) = 1 - \mu_{F(e)}(x) - \nu_{F(e)}(x)$ is called the hesitating degree of x holds e.*

Example 2.5. *Let $U = \{h_1, h_2, h_3, h_4, h_5\}$ be a set of houses and let $A = \{e_1, e_2, e_3, e_4, \} \subseteq E$ be a set of status of houses where e_j $(j = 1, 2, 3, 4)$ stand for the parameters "beautiful", "modern", "cheap" and "in the green surroundings", respectively.*

Now, we consider an intuitionistic fuzzy soft set (F, A) over U, which describes "the attractiveness of the houses" to this decision maker and its tabular representation is shown in Table 1. For example, the characteristic of the house h_1 under the parameter e_1 is $(0.3, 0.4)$. The values of 0.3 and 0.4 are the degrees of membership and non-membership of the house h_1 with respect to the parameter e_1, respectively. In other words, house h_1 is expensive on the degree of 0.3 and it is not expensive on the degree of 0.4.

Table 1: Tabular representation of the intuitionistic soft set (F, A)

	e_1	e_2	e_3	e_4
h_1	(0.3, 0.4)	(0.7, 0.1)	(0.5, 0.1)	(0.2, 0.4)
h_2	(0.4, 0.4)	(0.2, 0.7)	(0.4, 0.5)	(0.8, 0.1)
h_3	(0.3, 0.4)	(1.0, 0.0)	(0.7, 0.1)	(0.9, 0.1)
h_4	(0.4, 0.3)	(0.5, 0.3)	(0.6, 0.2)	(0.2, 0.7)
h_5	(0.3, 0.5)	(0.7, 0.3)	(0.6, 0.2)	(0.2, 0.6)

2.2 D-S theory of evidence

D-S theory of evidence is a new important reasoning method under uncertainty. It has an advantage to deal with subjective judgments and to synthesize the uncertainty knowledge [24].

A frame of discernment, denoted Θ, is a finite nonempty set of mutually exclusive and exhaustive hypotheses, denoted $\{A_1, A_2, \cdots, A_n\}$ and $A_i \cap A_j = \emptyset$. 2^Θ denotes the set of all subsets of Θ.

Definition 2.6 ([18]). *Let Θ be a frame of discernment. A basic probability assignment function (or Mass function) on Θ is defined a mapping $m : 2^{\Theta} \rightarrow [0, 1]$, m satisfies*

$$m(\emptyset) = 0 \ , \ \sum_{A \subseteq \Theta} m(A) = 1 \ for \ any \ A \in 2^{\Theta}.$$

For any $A \subseteq \Theta$, A is called as focal elements if $m(A) > 0$, $m(A)$ represents the belief measuser that one is willing to commit exactly to A, given a certain piece of evidence.

Definition 2.7 ([18]). *Let Θ be the frame of discernment and $m : 2^{\Theta} \rightarrow [0, 1]$ be a Mass function. Then a belief function on Θ is defined a mapping $Bel : 2^{\Theta} \rightarrow [0, 1]$, Bel satisfies*

$$Bel(\emptyset) = 0, \ Bel(\Theta) = 1, \ Bel(A) = \sum_{B \subseteq A} m(B) \ for \ any \ A \subseteq \Theta.$$

$Bel(A)$ can be interpreted as a global belief measure that the hypothesis A is true, and represents the imprecision and uncertainty in the decision-making process. In the case of single hypothesis, $Bel(A) = m(A)$.

Definition 2.8 ([18]). *Let Θ be the frame of discernment. Suppose there are two Mass functions are m_1 and m_2 over Θ, induced by two independent items of evidences $A_1, A_2, \cdots, A_s$ and $B_1, B_2, \cdots, B_t$, respectively. D-S rule of evidence combination is defined and denoted as follows:*

$$m(A) = m_1 \oplus m_2(A) = \begin{cases} \frac{1}{1-K} \sum\limits_{A_i \cap B_j = A} m_1(A_i) m_2(B_j), \ \forall A \subseteq \Theta, A \neq \emptyset, \\ 0, \ A = \emptyset, \end{cases}$$

where $K = \sum\limits_{A_i \cap B_j = \emptyset} m_1(A_i) m_2(B_j) < 1.$

K is called the conflict probability and reflects the extent of the conflict between the evidences. Coefficient $\frac{1}{1-K}$ is called normalized factor, its role is to avoid the probability of assigning non-0 to empty set $\emptyset$ in the combination.

D-S rule of evidence combination can be generalized to multiple Mass functions, the belief measure resulting from the combination of multiply evidences A_i is as follows:

$$m_1 \oplus m_2 \cdots \oplus m_n(A) = \frac{1}{1-K} \sum_{\bigcap_{i=1}^{n} A_i = A, A_i \subset \Theta} m_1(A_1) m_2(A_2) \cdots m_n(A_n),$$

where $K = \sum\limits_{\bigcap_{i=1}^{n} A_i = \emptyset, A_i \subset \Theta} m_1(A_1) m_2(A_2) \cdots m_n(A_n) < 1.$

D-S rule of evidence combination can increase belief measure of hypotheses and reduce the uncertain degree to improve reliability.

Example 2.9. *Let $\Theta = \{A_1, A_2\}$ be the frame of discernment. Suppose there are two Mass functions m_1 and m_2 over Θ, induced by the independent items of evidences A_1, A_2, given by*

$m_1(A_1) = 0.3, \quad m_1(A_2) = 0.4, \quad m_1(\Theta) = 0.3,$
$m_2(A_1) = 0.4, \quad m_2(A_2) = 0.3, \quad m_2(\Theta) = 0.3.$

Combining the two evidences by D-S rule of evidence combination leads to:

$m(A_1) = m_1 \oplus m_2(A_1) = \frac{m_1(A_1)m_2(A_1)+m_1(A_1)m_2(\Theta)+m_1(\Theta)m_2(A_1)}{1-K} = 0.44,$
$m(A_2) = m_1 \oplus m_2(A_2) = \frac{m_1(A_2)m_2(A_2)+m_1(A_2)m_2(\Theta)+m_1(\Theta)m_2(A_2)}{1-K} = 0.44,$
$m(\Theta) = m_1 \oplus m_2(\Theta) = \frac{m_1(\Theta)m_2(\Theta)}{1-K} = 0.12,$

where $K = m_1(A_1)m_2(A_2) + m_1(A_2)m_2(A_1) = 0.25$.

3 Decision making based on intuitionistic fuzzy soft sets

Now we discuss decision making based on intuitionistic fuzzy soft sets by means of grey relational analysis and D-S theory of evidence. It is divided three phases: First, grey relational analysis is applied to calculate the grey mean relational degree and the uncertain degree of each parameter is obtained. Second, the corresponding Mass function with respect to each parameter is constructed by the uncertain degree of each parameter. Third, we apply D-S rule of evidence combination to aggregate individual alternatives into a collective alternative, by which the candidate alternatives are ranked and the best alternative is obtained.

In the following, we consider the decision-making problem with m mutually exclusive alternatives x_i and n evaluation parameters (or indexes) e_j. μ_{ij} denotes the membership degree that x_i holds e_j. ν_{ij} denotes the non-membership degree that x_i opposes e_j. Put

$$\Theta = \{x_1, x_2, \cdots, x_m\} \text{ and } A = \{e_1, e_2, \cdots, e_n\}.$$

Define $F : A \to IF(\Theta)$ by $F(e_j)(x_i) = (\mu_{F(e_j)}(x_i), \nu_{F(e_j)}(x_i))$. Then (F, A) is an intuitionistic fuzzy soft set over Θ and $IFSM = (F(e_j)(x_i))_{m \times n}$ is called an intuitionistic fuzzy soft decision matrix induced by (F, A). Here, we see the set of parameters as a item of evidences information.

The key to solve decision problems by using D-S theory of evidence is how to obtain the uncertain degree of evidences.

Definition 3.1. *Let $x_i \in U$, $e_j \in A$ and let $a_{ij} = (\mu_{F(e_j)}(x_i), \nu_{F(e_j)}(x_i))$ be e_j-intuitionistic fuzzy number of x_i. Then score function of a_{ij} is defined and denoted as $s(a_{ij}) = \mu_{F(e_j)}(x_i) - \nu_{F(e_j)}(x_i)$.*

It is obvious that $0 \leqslant s_{ij} \leqslant 1$. $s(a_{ij})$ presents the global degree that the alternative x_i holds the parameter e_j. If $s(a_{ij})$ is very much larger, it implies x_i holds e_j more heavily. Especially, $s(a_{ij}) = -1$ indicates that x_i wholly opposes e_j. $s(a_{ij}) = 1$ indicates that x_i wholly holds e_j. $s(a_{ij}) = 0$ represents the same degree of support and opposition.

To obtain Mass functions of each alternative with respect to each parameter, we consider score function values may be negative, so we should normalize the score function values by the following formula:

$$d_{ij} = \frac{s(a_{ij}) - min_{1\leqslant i\leqslant m} s(a_{ij})}{max_{1\leqslant i\leqslant m} s(a_{ij}) - min_{1\leqslant i\leqslant m} S(a_{ij})}, \quad 1 \leqslant i \leqslant m, \ 1 \leqslant j \leqslant n.$$

Hence, we can get normalized matrix of score function values $D = (d_{ij})_{m\times n}$.

Next, inspired by the paper [12], we define the grey mean relational degree and the uncertain degree of the parameter as follows.

Definition 3.2. *Let $\Theta = \{x_1, x_2, \cdots, x_m\}$, $A = \{e_1, e_2, \cdots, e_n\}$ and let (F, A) be an intuitionistic fuzzy soft set on Θ. Suppose that $D = (d_{ij})_{m\times n}$ is normalized matrix of score function values. For any i, j, denote*

$$\widetilde{d_i} = \frac{1}{n}\sum_{j=1}^{n} d_{ij}, \quad \triangle d_{ij} = |d_{ij} - \widetilde{d_i}|,$$

$$r_{ij} = \frac{min_{1\leqslant i\leqslant m} \ \triangle d_{ij} + \rho \ max_{1\leqslant i\leqslant m} \ \triangle d_{ij}}{\triangle d_{ij} + \rho \ max_{1\leqslant i\leqslant m} \ \triangle d_{ij}}, \ \rho \in (0, 1),$$

$$DOI(e_j) = \frac{1}{m}(\sum_{i=1}^{m}(r_{ij})^q)^{\frac{1}{q}} \ (j = 1, 2, \cdots, n).$$

r_{ij} is called the grey mean relational degree between d_{ij} and $\widetilde{d_i}$. $DOI(e_j)$ is called q order uncertain degree of the parameter e_j.

ρ aims to expand or compress the range of the grey relational coefficient. In this paper, we pick $q = 2$, $\rho = 0.5$ to obtain strong distinguishing effectiveness. We call $DOI(e_j)$ the uncertain degree of e_j for short.

It is worthy to notice that the method to obtain the uncertain degree varies from different situation in Definition 3.2. General speaking, since a index (or parameter) is specially more matching the mean of the index system than other indexes, it contains more satisfying information for decision making and the uncertain degree of the index information is lower. Then, in this paper we just consider grey mean relational degree between d_{ij} and $\widetilde{d_i}$.

Definition 3.3 ([26])**.** *Let $X = (x_1, x_2, \cdots, x_m)$ be a finite difference information sequence, where there exists $x_{i_k} \neq 0$ for $k = 1, 2, \cdots, m$ and $1 \leqslant i_k \leqslant m$. Then the information structure image sequence $Y = (y_1, y_2, \cdots, y_m)$ is given by $y_i = \frac{x_i}{\sum\limits_{i=1}^{m} x_i}$.*

In the normalized matrix of score function values $D = (d_{ij})_{m\times n}$, the information structure image sequence with respect to a parameter e_j is denoted by $d_j = \{\widetilde{d_{1j}}, \widetilde{d_{2j}}, \widetilde{d_{3j}}, \cdots, \widetilde{d_{mj}}\}$, where $\widetilde{d_{ij}} = \frac{d_{ij}}{\sum\limits_{i=1}^{m} d_{ij}}$. Then we obtain an information

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structure image matric $\widetilde{D} = (\widetilde{d_{ij}})_{m\times n}$ induced by d_j $(j = 1, 2, \cdots, n)$.

D-S theory of evidence is a powerful method for combining accumulative evidence of changing prior opinions in the light of new evidences [18]. The primary procedure of combining the known evidences or information with other evidences is to construct suitable Mass functions of evidences.

Now, by the uncertain degree of each parameter, we can obtain Mass function of each alternative with respect to each parameter.

Theorem 3.4. *Let* $\Theta = \{x_1, x_2, \cdots, x_m\}$, $A = \{e_1, e_2, \cdots, e_n\}$ *and let* (F, A) *be an intuitionistic fuzzy soft set on* Θ. *Suppose that* $D = (d_{ij})_{m\times n}$ *is the normalized matrix of score function values. Denote* $\widetilde{d_{ij}} = \frac{d_{ij}}{\sum\limits_{i=1}^{m} d_{ij}}$. *For any* i, j, *we define functions* $m_{e_j}(j = 1, 2, \cdots, n)$ *with respect to the parameter* e_j, *it satisfies:*

$$m_{e_j}(x_i) = \widetilde{d_{ij}}\ (1 - DOI(e_j)), \quad m_{e_j}(\Theta) = 1 - \sum_{i=1}^{m} m_j(i).$$

Then $m_{e_j}(j = 1, 2, \cdots, n)$ *are Mass functions.*

In a normalized matrix of score function values $D = (d_{ij})_{m\times n}$, denote $m_{e_j}(x_i)$, $m_{e_j}(\Theta)$ by $m_j(i)$ and $m_j(m+1)$, respectively. $m_j(i)$ implies the belief measure that holds the alternative x_i with the parameter e_j and $m_j(m+1)$ implies the belief measure of the whole uncertainty with parameter e_j.

Next, using D-S rule of evidence combination to compose m_j $(j = 1, 2, \cdots, n)$, we get the belief measure of each alternative with all the parameters, by which the candidate alternatives are ranked and thus the best alternative is obtained.

4 Algorithms and examples

4.1 Algorithm

Based on the above analysis, the detailed step-wise procedure as an algorithm is given as follows:

Input: An intuitionistic fuzzy soft set (F, E).

Output: The optimal decision-making results.

Step 1. Input an intuitionistic fuzzy soft set (F, E) and construct an intuitionistic fuzzy soft decision matrix induced by (F, E).

Step 2. Compute the normalized matrix of score function values ($D = (d_{ij})_{m\times n}$).

Step 3. Compute the mean of all the score function values ($\widetilde{d_i}$) with respect to each alternative.

Step 4. Compute the difference information between d_{ij} and $\widetilde{d_i}$.

Step 5. Compute the gray mean relational degree between d_{ij} and $\widetilde{d_i}$.

Step 6. Compute the uncertain degree $DOI(e_j)$ of each parameter e_j.

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Step 7. Compute the information structure image sequence $\widetilde{d_{ij}}$ with respect to each parameter e_j by Definition 3.3.

Step 8. Compute Mass function values of the alternative x_i and Θ with respect to the parameter e_j by Theorem 3.4.

Step 9. Compute belief measure of each alternative x_i by combining these Mass functions $m_{e_j}(j = 1, 2, \cdots, n)$ respectively by Definition 2.8.

Step 10. The optimal decision is to select x_k if $c_k = max_i \{Bel(x_i)\}$. Optimal choices have more than one alternative if there are more alternatives corresponding to the maximum.

4.2 An example

Suppose that a company wants to fill a position. There are ten candidates who fill in a form in order to apply formally for the position. Suppose that the set of candidates $U = \{x_1, x_2, ..., x_{10}\}$ which are characterized by a set of parameters $A = \{e_1, e_2, ..., e_8\}$. For $i = 1, 2, ..., 8$, the parameters e_i stand for experience, computer knowledge, training, young age, higher education, marriage status, good health and skilled foreign languages, respectively. There is a decision maker from the department of human resources, providing his/her assessment of each candidate on each parameter as an intuitionistic fuzzy soft set (F, A). Its tabular representation is shown in Table 2.

Table 2: Tabular representation of the intuitionistic soft set (F, A)

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
x_1	(0.4, 0.4)	(0.7, 0.1)	(0.6, 0.1)	(0.2, 0.7)	(0.9, 0.1)	(0.9, 0.1)	(0.7, 0.1)	(0.9, 0.1)
x_2	(0.3, 0.4)	(0.2, 0.7)	(0.4, 0.5)	(0.8, 0.1)	(0.5, 0.4)	(0.4, 0.4)	(0.3, 0.6)	(0.5, 0.5)
x_3	(0.3, 0.4)	(1.0, 0.0)	(0.7, 0.1)	(0.9, 0.1)	(0.3, 0.3)	(0.6, 0.3)	(0.6, 0.3)	(0.5, 0.3)
x_4	(0.4, 0.3)	(0.5, 0.3)	(0.6, 0.2)	(0.2, 0.7)	(0.6, 0.2)	(0.8, 0.2)	(0.2, 0.7)	(0.7, 0.2)
x_5	(0.4, 0.2)	(0.8, 0.1)	(0.8, 0.0)	(0.2, 0.5)	(0.7, 0.1)	(0.7, 0.2)	(0.7, 0.0)	(0.8, 0.1)
x_6	(0.8, 0.1)	(0.7, 0.2)	(0.7, 0.2)	(0.5, 0.3)	(0.5, 0.4)	(0.4, 0.5)	(0.6, 0.4)	(0.5, 0.3)
x_7	(0.0, 1.0)	(0.3, 0.5)	(0.2, 0.6)	(0.7, 0.2)	(0.6, 0.4)	(0.6, 0.4)	(0.5, 0.4)	(0.6, 0.2)
x_8	(0.5, 0.4)	(0.2, 0.6)	(0.4, 0.5)	(0.8, 0.1)	(0.8, 0.1)	(0.7, 0.2)	(0.8, 0.1)	(0.8, 0.1)
x_9	(0.4, 0.5)	(0.6, 0.2)	(0.5, 0.1)	(0.1, 0.6)	(0.7, 0.0)	(0.6, 0.3)	(0.7, 0.2)	(0.9, 0.0)
x_{10}	(0.1, 0.4)	(1.0, 0.0)	(0.8, 0.2)	(0.7, 0.3)	(0.5, 0.5)	(0.5, 0.3)	(0.6, 0.3)	(0.4, 0.5)

Now, we suppose that the ten mutually exclusive and exhaustive candidates consist a frame of discernment, denoted $\Theta = \{x_1, x_2, ..., x_{10}\}$. And we consider the set of parameters $A = \{e_1, e_2, ..., e_8\}$ as a set of evidences.

Step 1. Construct an intuitionistic fuzzy soft decision matrix induced by (F, A) as follows:

$$\begin{pmatrix} (0.4,0.4) & (0.7,0.1) & (0.6,0.1) & (0.2,0.7) & (0.9,0.1) & (0.9,0.1) & (0.7,0.1) & (0.9,0.1) \\ (0.3,0.4) & (0.2,0.7) & (0.4,0.5) & (0.8,0.1) & (0.5,0.4) & (0.4,0.4) & (0.3,0.6) & (0.5,0.5) \\ (0.3,0.4) & (1.0,0.0) & (0.7,0.1) & (0.9,0.1) & (0.3,0.3) & (0.6,0.3) & (0.6,0.3) & (0.5,0.3) \\ (0.4,0.3) & (0.5,0.3) & (0.6,0.2) & (0.2,0.7) & (0.6,0.2) & (0.8,0.2) & (0.2,0.7) & (0.7,0.2) \\ (0.4,0.2) & (0.8,0.1) & (0.8,0.0) & (0.2,0.5) & (0.7,0.1) & (0.7,0.2) & (0.7,0.0) & (0.8,0.1) \\ (0.8,0.1) & (0.7,0.2) & (0.7,0.2) & (0.5,0.3) & (0.5,0.4) & (0.4,0.5) & (0.6,0.4) & (0.5,0.3) \\ (0.0,1.0) & (0.3,0.5) & (0.2,0.6) & (0.7,0.2) & (0.6,0.4) & (0.6,0.4) & (0.5,0.4) & (0.6,0.2) \\ (0.5,0.4) & (0.2,0.6) & (0.4,0.5) & (0.8,0.1) & (0.8,0.1) & (0.7,0.2) & (0.8,0.1) & (0.8,0.1) \\ (0.4,0.5) & (0.6,0.2) & (0.5,0.1) & (0.1,0.6) & (0.7,0.0) & (0.6,0.3) & (0.7,0.2) & (0.9,0.0) \\ (0.1,0.4) & (1.0,0.0) & (0.8,0.2) & (0.7,0.3) & (0.5,0.5) & (0.5,0.3) & (0.6,0.3) & (0.4,0.5) \end{pmatrix} \tag{1}$$

8

Step 2. Compute the normalized matrix of score function values as follows:

$$D=(d_{ij})_{10\times 8}=\begin{pmatrix} 0.5882 & 0.7333 & 0.7500 & 0.0000 & 1.0000 & 1.0000 & 0.9867 & 0.9000 \\ 0.5294 & 0 & 0.2500 & 0.9231 & 0.1250 & 0.1111 & 0.1467 & 0.1000 \\ 0.5294 & 1.0000 & 0.8333 & 1.0000 & 0 & 0.4444 & 0.9467 & 0.3000 \\ 0.6471 & 0.4667 & 0.6667 & 0.0000 & 0.5000 & 0.7778 & 0 & 0.6000 \\ 0.7059 & 0.8000 & 1.0000 & 0.1538 & 0.7500 & 0.6667 & 1.0000 & 0.8000 \\ 1.0000 & 0.6667 & 0.7500 & 0.5385 & 0.1250 & 0 & 0.9333 & 0.3000 \\ 0 & 0.2000 & 0 & 0.7692 & 0.2500 & 0.3333 & 0.9200 & 0.5000 \\ 0.6471 & 0.0667 & 0.2500 & 0.9231 & 0.8750 & 0.6667 & 1.0000 & 0.8000 \\ 0.5294 & 0.6000 & 0.6667 & 0 & 0.8750 & 0.4444 & 0.9733 & 1.0000 \\ 0.4118 & 1.0000 & 0.8333 & 0.6923 & 0 & 0.3333 & 0.9467 & 0 \end{pmatrix} \tag{2}$$

Step 3. Compute the mean of all parameters with respect to each candidate x_i as follows:

$$\widetilde{d_1}=0.5958,\ \widetilde{d_2}=0.2185,\ \widetilde{d_3}=0.5054, \widetilde{d_4}=0.3658,\ \widetilde{d_5}=0.5876,$$

$$\widetilde{d_6}=0.4313,\ \widetilde{d_7}=0.2973,\ \widetilde{d_8}=0.5228,\ \widetilde{d_9}=0.5089,\ \widetilde{d_{10}}=0.4217.$$

Step 4. Compute the difference information between d_{ij} and $\widetilde{d_i}$, and construct the difference matrix as follows:

$$\triangle D=\begin{pmatrix} 0.0076 & 0.1375 & 0.1542 & 0.5958 & 0.4042 & 0.4042 & 0.3908 & 0.3042 \\ 0.3109 & 0.2185 & 0.0315 & 0.7046 & 0.0935 & 0.1074 & 0.0719 & 0.1185 \\ 0.0240 & 0.4946 & 0.3279 & 0.4946 & 0.5054 & 0.0609 & 0.4413 & 0.2054 \\ 0.2812 & 0.1008 & 0.3008 & 0.3658 & 0.1342 & 0.4120 & 0.3658 & 0.2342 \\ 0.1182 & 0.2124 & 0.4124 & 0.4338 & 0.1624 & 0.0790 & 0.4124 & 0.2124 \\ 0.5687 & 0.2353 & 0.3187 & 0.1071 & 0.3063 & 0.4313 & 0.5020 & 0.1313 \\ 0.2973 & 0.0973 & 0.2973 & 0.4720 & 0.0473 & 0.0361 & 0.6227 & 0.2027 \\ 0.1242 & 0.4562 & 0.2728 & 0.4002 & 0.3522 & 0.1438 & 0.4772 & 0.2772 \\ 0.0205 & 0.0911 & 0.1578 & 0.5089 & 0.3661 & 0.0644 & 0.4644 & 0.4911 \\ 0.0100 & 0.5783 & 0.4116 & 0.2706 & 0.4217 & 0.0884 & 0.5249 & 0.4217 \end{pmatrix} \tag{3}$$

Step 5. Compute the gray mean relational degree between d_{ij} and $\widetilde{d_i}$ based on $\triangle D$ as follows:

$$(r_{ij})_{10\times 8}=\begin{pmatrix} 1.0000 & 0.8913 & 0.6595 & 0.4845 & 0.4566 & 0.4061 & 0.5457 & 0.6623 \\ 0.4904 & 0.7490 & 1.0000 & 0.4347 & 0.8664 & 0.7792 & 1.0000 & 1.0000 \\ 0.9467 & 0.4852 & 0.4449 & 0.5424 & 0.3957 & 0.9101 & 0.5092 & 0.8074 \\ 0.5161 & 0.9750 & 0.4687 & 0.6397 & 0.7753 & 0.4011 & 0.5659 & 0.7589 \\ 0.7251 & 0.7582 & 0.3842 & 0.5844 & 0.7227 & 0.8543 & 0.5295 & 0.7951 \\ 0.3422 & 0.7250 & 0.4528 & 1.0000 & 0.5365 & 0.3891 & 0.4712 & 0.9660 \\ 0.5019 & 0.9841 & 0.4721 & 0.5573 & 1.0000 & 1.0000 & 0.4103 & 0.8121 \\ 0.7145 & 0.5102 & 0.4961 & 0.6105 & 0.4959 & 0.7003 & 0.4860 & 0.6965 \\ 0.9576 & 1.0000 & 0.6530 & 0.5335 & 0.4847 & 0.8987 & 0.4940 & 0.4942 \\ 0.9919 & 0.4384 & 0.3847 & 0.7376 & 0.4447 & 0.8279 & 0.4582 & 0.5456 \end{pmatrix} \tag{4}$$

Step 6. Compute the uncertain degree of each parameter e_j by Definition 3.2 as follows:

$$DOI(e_1)=0.2389,\ DOI(e_2)=0.2462,\ DOI(e_3)=0.1802,\ DOI(e_4)=0.1995,$$

$$DOI(e_5)=0.2051,\ DOI(e_6)=0.2372,\ DOI(e_7)=0.1800,\ DOI(e_8)=0.2433.$$

Step 7. Compute the information structure image sequence with respect to each parameter and construct the matrix as follows:

$$\widetilde{D} = (\widetilde{d_{ij}})_{3\times 5} = \begin{pmatrix} 0.1053 & 0.1325 & 0.1250 & 0.0000 & 0.2222 & 0.2093 & 0.1256 & 0.1698 \\ 0.0947 & 0 & 0.0417 & 0.1846 & 0.0278 & 0.0233 & 0.0187 & 0.0189 \\ 0.0947 & 0.1807 & 0.1389 & 0.2000 & 0 & 0.0930 & 0.1205 & 0.0566 \\ 0.1158 & 0.0843 & 0.1111 & 0.0000 & 0.1111 & 0.1628 & 0 & 0.1132 \\ 0.1263 & 0.1446 & 0.1667 & 0.0308 & 0.1667 & 0.1395 & 0.1273 & 0.1509 \\ 0.1789 & 0.1205 & 0.1250 & 0.1077 & 0.0278 & 0 & 0.1188 & 0.0566 \\ 0 & 0.0361 & 0 & 0.1538 & 0.0556 & 0.0698 & 0.1171 & 0.0943 \\ 0.1158 & 0.0120 & 0.0417 & 0.1846 & 0.1944 & 0.1395 & 0.1273 & 0.1509 \\ 0.0947 & 0.1084 & 0.1111 & 0 & 0.1944 & 0.0930 & 0.1239 & 0.1887 \\ 0.0737 & 0.1807 & 0.1389 & 0.1385 & 0 & 0.0698 & 0.1205 & 0 \end{pmatrix} \tag{5}$$

Step 8. Let $2^{\Theta} = \{\{x_1\}, \{x_2\}, ..., \{x_{10}\}, \Theta\}$. Compute Mass function values of the candidate x_i and Θ with respect to the parameter e_j by Theorem 3.4:

$$(m_j(i))_{10\times 8} = \begin{pmatrix} 0.0721 & 0 & 0.0342 & 0.1478 & 0.0221 & 0.0177 & 0.0153 & 0.0143 \\ 0.0721 & 0.1362 & 0.1139 & 0.1601 & 0 & 0.0710 & 0.0989 & 0.0428 \\ 0.0881 & 0.0636 & 0.0911 & 0.0000 & 0.0883 & 0.1242 & 0 & 0.0857 \\ 0.0961 & 0.1090 & 0.1366 & 0.0246 & 0.1325 & 0.1064 & 0.1044 & 0.1142 \\ 0.1362 & 0.0908 & 0.1025 & 0.0862 & 0.0221 & 0 & 0.0975 & 0.0428 \\ 0 & 0.0272 & 0 & 0.1232 & 0.0442 & 0.0532 & 0.0961 & 0.0714 \\ 0.0881 & 0.0091 & 0.0342 & 0.1478 & 0.1546 & 0.1064 & 0.1044 & 0.1142 \\ 0.0721 & 0.0817 & 0.0911 & 0 & 0.1546 & 0.0710 & 0.1016 & 0.1428 \\ 0.0561 & 0.1362 & 0.1139 & 0.1108 & 0 & 0.0532 & 0.0989 & 0 \end{pmatrix} \tag{6}$$

and

$$m_1(11) = 0.2389, \quad m_2(11) = 0.2462, \quad m_3(11) = 0.1802, \quad m_4(11) = 0.1995,$$
$$m_5(11) = 0.2051, \quad m_6(11) = 0.2372, \quad m_7(11) = 0.1800, \quad m_8(11) = 0.2433.$$
$$\frac{1}{8}\sum_{j=1}^{8} m_j(11) = 0.2163.$$

Step 9. By $Bel(\{x_i\}) = m_1 \oplus m_2 \oplus m_3 \cdots \oplus m_8(\{x_i\})$, we combine these Mass functions and compute each belief measure of each candidate x_i respectively as follows:

$$Bel(\{x_1\}) = 0.1799,\ Bel(\{x_2\}) = 0.0230,\ Bel(\{x_3\}) = 0.1119,\ Bel(\{x_4\}) = 0.0568,$$
$$Bel(\{x_5\}) = 0.1807,\ Bel(\{x_6\}) = 0.0744,\ Bel(\{x_7\}) = 0.0368,\ Bel(\{x_8\}) = 0.1398,$$
$$Bel(\{x_9\}) = 0.1178,\ Bel(\{x_{10}\}) = 0.0704,\ Bel(\{\Theta\}) = 0.0086.$$

Then the final rang order is $x_5 \succ x_1 \succ x_8 \succ x_9 \succ x_3 \succ x_6 \succ x_10 \succ x_4 \succ x_7 \succ x_2$.

Step 10. x_5 is the optimal candidate for the position for $max_i\ \{Bel(x_i)\} = 0.1807$.

From the above results, the belief measure of the uncertainty with respect to the whole candidates Θ is declined from 0.2163 to 0.0086, after applying grey relational analysis to construct the corresponding Mass functions for different evidences and then using the rule of evidence combination to compose these information. This implies the above algorithm is effective and practical under uncertainties. It not only allows us to avoid selecting the suitable level soft set, but also helps reducing humanistic and subjective in nature to raise the choices decision level. Moreover, it broadens the application field of the grey system theory and D-S theory of evidence.

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On quicker convergence towards Euler's constant

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Abstract

In this paper, we deduced the following new asymptotic series

$$H_n - \ln n \sim \gamma + \frac{1}{2(n+1)} + \frac{5}{12n(n+1)\left(1 + \frac{1/5}{n} + \frac{1/50}{n^2} - \frac{1/50}{n^3} + \frac{59/52500}{n^4} + \frac{437/37500}{n^5} - ...\right)}$$

which faster converge to the Euler's constant with the increase in the terms considered, where H_n is the harmonic number. Also, we presented the following double inequality

$$\frac{\frac{1}{2} + \frac{5}{12n\left(1+\frac{1/5}{n}+\frac{1/50}{n^2}\right)}}{n+1} < H_n - \ln n - \gamma < \frac{\frac{1}{2} + \frac{5}{12n\left(1+\frac{1/5}{n}\right)}}{n+1}; \qquad n = 1, 2, 3, ... ,$$

which improved some known inequalities of the sequence $H_n - \ln n - \gamma$.

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1 Introduction.

Euler's constant was first introduced by the Swiss genius Leonhard Euler (1707-1783) in 1734 as

$$\gamma = \lim_{n\to\infty} \left(H_n - \ln n\right), \tag{1}$$

where $H_n = \sum_{k=1}^{n} \frac{1}{k}$ is the harmonic number. It is also known as the Euler-Mascheroni constant. More generally, for $\alpha > -n$ [11]

$$\gamma = \lim_{n\to\infty} \left(H_n - \ln(n+\alpha)\right). \tag{2}$$

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Also, γ is related to the Gamma function $\Gamma(x)$ by the relation [2]

$$\gamma = -\Gamma'(1) \tag{3}$$

and there are many integral representations for γ are recorded in [5].

The constant γ is one of the important constants in mathematics and it has many applications in analysis, special functions, number theory, probability and physics. For an interesting discussion of this constant and its many connections to various fields of Mathematics, see J. Havil [11].

The sequence $\gamma_n = H_n - \ln n$ converges toward its limit γ very slowly like $\frac{1}{n}$. So, direct use of formula (1) to compute Euler constant is of poor interest. In fact, using the harmonic number notation H_n, we have the estimation

$$H_n - \ln n - \gamma \sim \frac{1}{2n}.$$

This estimation is the first term of an asymptotic expansion which can be used to compute effectively γ. The Euler-Maclaurin summation can be used to have a complete asymptotic expansion of the harmonic numbers

$$H_n - \ln n - \gamma \sim \frac{1}{2n} - \sum_{k\geq 1} \frac{B_{2k}}{2k\; n^{2k}},$$

where the B_{2k} are the Bernoulli numbers. Since B_{2k} grows like $\frac{2(2k)!}{(2\pi)^{2k}}$ the asymptotic expansion should be stopped at a given k [10].

Many lower and upper estimates of the sequence $H_n - \ln n - \gamma$ have been obtained for $n \in N$ in the literature. Here some examples:

$$\text{Young [19]}: \quad \frac{1}{2(n+1)} < H_n - \ln n - \gamma < \frac{1}{2n} \tag{4}$$

$$\text{Tóth [18]}: \quad \frac{1}{2n+2/5} < H_n - \ln n - \gamma < \frac{1}{2n+1/3} \tag{5}$$

$$\text{Alzer [1]}: \quad \frac{1}{2n+\frac{2\gamma-1}{1-\gamma}} < H_n - \ln n - \gamma < \frac{1}{2n+1/3} \tag{6}$$

also,

$$\text{DeTemple [7], [6]}: \quad \frac{1}{24(n+1)^2} < H_n - \ln(n+1/2) - \gamma < \frac{1}{24n^2}, \tag{7}$$

where $H_n - \ln(n+1/2) - \gamma$ converges like n^{-2}. Also, D. W. DeTemple and S. H. Wang [6] established an estimate for $\gamma - H_n + \ln(n+1)$ where Bernoulli's numbers are involved.

Recently, C. Mortici [15] open a new direction to accelerate the sequence $H_n - \ln n - \gamma$ by considering the sequence

$$M_n = H_n - \ln n - \gamma + \ln \frac{P(n)}{Q(n)}, \tag{8}$$

where $P(n)$ and $Q(n)$ are polynomials of the same degree, having the leading coefficient equal to one. Precisely, He introduced the two sequences

$$s_n = H_n - \ln n - \gamma + \ln \frac{n - 1/12}{n + 5/12} \tag{9}$$

and

$$t_n = H_n - \ln n - \gamma + \ln \frac{n^2 + \frac{33}{140}n + \frac{37}{1680}}{n^2 + \frac{103}{140}n + \frac{61}{336}} \tag{10}$$

whose speeds of convergence increase to n^{-3}, respective n^{-5}. The papers [4], [14], [3], [8], [9], [16] and [17] presented some important improvements of the speed of convergence of the sequence $H_n - \ln n - \gamma$.

2 Main results.

In view of the inequality (4), we define the sequence

$$\lambda_n = H_n - \ln n - \gamma + \frac{\frac{1}{b} + \frac{1}{an}}{n+1}, \tag{11}$$

where a and b are real parameters, which provide the fastest sequence. In what follows, our study is based on the following result; which represents a powerful tool for constructing some asymptotic expansions, or to accelerate some convergences.

Lemma 2.1. *If $(w_n)_{n\geq 1}$ is convergent to zero and there exists the limit*

$$\lim_{n\longrightarrow\infty} n^k (w_n - w_{n+1}) = l \in \mathbb{R} \tag{12}$$

with $k > 1$, then there exists the limit:

$$\lim_{n\longrightarrow\infty} n^{k-1} w_n = \frac{l}{k-1}.$$

This Lemma was first used by C. Mortici for constructing asymptotic expansions, or to accelerate some convergences [12], [13]. By using Lemma (2.1), clearly the sequence $(w_n)_{n\geq 1}$ converges more quickly when the value of k satisfying (12) is larger.

Now, using the relation (11), we get

$$\lambda_n - \lambda_{n+1} = \frac{-1}{n+1} - \ln\left(\frac{n}{n+1}\right) + \frac{\frac{1}{b} + \frac{1}{an}}{n+1} - \frac{\frac{1}{b} + \frac{1}{a(n+1)}}{n+2}$$

and

$$\lambda_n - \lambda_{n+1} = \frac{1}{2n^2} - \frac{2}{3n^3} + \frac{3}{4n^4} - \frac{4}{5n^5} + \frac{2b + an}{abn(n+1)(n+2)} + O(n^{-6}).$$

Then

$$\lim_{n\to\infty} n^2 \left(\lambda_n - \lambda_{n+1}\right) = \frac{1}{b} + \frac{1}{2}$$

and so we choose $b = -2$ and at this value we get

$$\lim_{n\to\infty} n^3 (\lambda_n - \lambda_{n+1}) = \frac{5}{6} + \frac{2}{a}.$$

Then we choose $a = -12/5$ and we can conclude the following

Lemma 2.2. *The sequence*

$$\lambda_n = H_n - \ln n - \gamma - \frac{\frac{1}{2} + \frac{5}{12n}}{n+1} \tag{13}$$

has a rate of convergence equal to n^{-3}, *where*

$$\lim_{n\to\infty} n^4 (\lambda_n - \lambda_{n+1}) = -\frac{1}{4}.$$

Our second step will by considering the following sequence

$$\mu_n = H_n - \ln n - \gamma - \frac{\frac{1}{2} + \frac{5}{12n(1+\frac{h}{n})}}{n+1}, \tag{14}$$

where h is a real parameter. Then

$$\mu_n - \mu_{n+1} = \frac{1}{2n^2} - \frac{2}{3n^3} + \frac{3}{4n^4} - \frac{4}{5n^5} - \frac{10 + 11h + 6h^2 + 16n + 12nh + 6n^2}{12(n+1)(n+2)(n+h)(n+h+1)} + O(n^{-6})$$

and hence

$$\lim_{n\to\infty} n^4 (\mu_n - \mu_{n+1}) = \frac{1}{4}(5h - 1).$$

So we choose $h = 1/5$ and we we can conclude the following

Lemma 2.3. *the sequence*

$$\mu_n = H_n - \ln n - \gamma - \frac{\frac{1}{2} + \frac{5}{12n\left(1+\frac{1/5}{n}\right)}}{n+1} \tag{15}$$

has a rate of convergence equal to n^{-4}, *where*

$$\lim_{n\to\infty} n^5 (\mu_n - \mu_{n+1}) = -\frac{1}{30}.$$

This procedure will give us an easy technique to construct an asymptotic expansion of the harmonic numbers

$$H_n - \ln n - \gamma \sim \frac{1}{2(n+1)} + \frac{5}{12n(n+1)\left(1 + \frac{1/5}{n} + \frac{1/50}{n^2} - \frac{1/50}{n^3} + \frac{59/52500}{n^4} + \frac{437/37500}{n^5} - \ldots\right)}, \tag{16}$$

which increases the speed of the convergence with the increase in the terms considered. For example the sequence

$$\theta_n = \frac{1}{2(n+1)} + \frac{5}{12n(n+1)\left(1 + \frac{1/5}{n} + \frac{1/50}{n^2} - \frac{1/50}{n^3} + \frac{59/52500}{n^4} + \frac{437/37500}{n^5} - \frac{4547/875000}{n^6} - \frac{11633/875000}{n^7}\right)} \tag{17}$$

has a rate of convergence equal to n^{-10}, where

$$\lim_{n\to\infty} n^{11} (\theta_n - \theta_{n+1}) = \frac{394566341}{7276500000}.$$

Of course the sequence (17) give us a superiority over the two sequences (9) and (10).

3 New double inequality of $H_n - \ln n - \gamma$.

Let $f(n) = \mu_n - \mu_{n+1}$, then

$$f(x) = \frac{-1}{x+1} - \ln\left(\frac{x}{x+1}\right) - \frac{\frac{1}{2} + \frac{5}{12x\left(1+\frac{1/5}{x}\right)}}{x+1} + \frac{\frac{1}{2} + \frac{5}{12(x+1)\left(1+\frac{1/5}{(x+1)}\right)}}{x+2}; \qquad x > 0$$

and

$$f'(x) = \frac{625x^3 + 1375x^2 - 20x - 864}{6x(1+x)^2(2+x)^2(1+5x)^2(6+5x)^2} > 0; \qquad x > 0.$$

Also,

$$\lim_{x\to\infty} f(x) = 0.$$

Now for $x > 0$, the function $f(x)$ is monotonically increasing and tends to 0 as x tends to ∞, then

$$f(x) < 0, \qquad x > 0.$$

Hence

$$\mu_n < \mu_{n+1}, \qquad n = 1, 2, 3, \ldots .$$

Now μ_n is increasing sequence and $\lim_{n\to\infty} \mu_n = 0$, then we get

$$\mu_n < 0, \qquad n = 1, 2, 3, \ldots$$

and hence

$$H_n - \ln n - \gamma < \frac{\frac{1}{2} + \frac{5}{12n\left(1+\frac{1/5}{n}\right)}}{n+1}, \qquad n = 1, 2, 3, \ldots .$$

Similarly, if we let $\nu_n = H_n - \ln n - \gamma - \frac{\frac{1}{2} + \frac{5}{12n\left(1+\frac{1/5}{n}+\frac{1/50}{n^2}\right)}}{n+1}$ and $g(n) = \nu_n - \nu_{n+1}$, we can prove that for $x > 0$, the function $g(x)$ is monotonically decreasing and tends to 0 as x tends to ∞, then

$$g(x) > 0, \qquad x > 0.$$

Hence

$$\nu_n > \nu_{n+1}, \qquad n = 1, 2, 3, \ldots .$$

Now ν_n is decreasing sequence and $\lim_{n\to\infty} \nu_n = 0$, then we get

$$\nu_n > 0, \qquad n = 1, 2, 3, \ldots$$

and hence

$$H_n - \ln n - \gamma > \frac{\frac{1}{2} + \frac{5}{12n\left(1+\frac{1/5}{n}+\frac{1/50}{n^2}\right)}}{n+1}, \qquad n = 1, 2, 3, \ldots .$$

Then we obtain the following result:

Lemma 3.1. *The sequence $H_n - \ln n - \gamma$ satisfies the double inequality*

$$\frac{\frac{1}{2} + \frac{5}{12n\left(1+\frac{1/5}{n}+\frac{1/50}{n^2}\right)}}{n+1} < H_n - \ln n - \gamma < \frac{\frac{1}{2} + \frac{5}{12n\left(1+\frac{1/5}{n}\right)}}{n+1}, \qquad n = 1, 2, 3, \ldots . \tag{18}$$

The inequality (18) give us a superiority over the inequality (4), since

$$\frac{\frac{1}{2}+\frac{5}{12n\left(1+\frac{1/5}{n}\right)}}{n+1}<\frac{1}{2n} \quad and \quad \frac{1}{2(n+1)}<\frac{\frac{1}{2}+\frac{5}{12n\left(1+\frac{1/5}{n}+\frac{1/50}{n^2}\right)}}{n+1}.$$

Also, it improved the two inequalities (5) and (6), since

$$\frac{\frac{1}{2}+\frac{5}{12n\left(1+\frac{1/5}{n}\right)}}{n+1}<\frac{1}{2n+1/3}$$

and

$$\frac{1}{2n+2/5}<\frac{1}{2n+\frac{2\gamma-1}{1-\gamma}}<\frac{\frac{1}{2}+\frac{5}{12n\left(1+\frac{1/5}{n}+\frac{1/50}{n^2}\right)}}{n+1} \quad for\ n>1.$$

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On Chain Recurrent Points of Graph Maps

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Abstract. A continuous map f from a graph G to it self is called a graph. In [4],Block and Franke proved that if $f:S^1\to S^1$ is a circle map $CR(f)=P(f)$ if and only if $P(f)$ is a nonempty closed set and for every $x\in S^1-P(f)$, some element of $\omega(x,f)$ has a generalized attracting neighborhood. In this paper, we show that if $f:G\to G$ is a graph map, and if $P(f)$ is a nonempty closed set and for every $x\in G-P(f)$, there is a point in $\omega(x,f)$ which has a generalized attracting neighborhood containing no circle, then $CR(f)=P(f)$. An example is constructed to show that the inverse is not true.

§ 1 Introduction

Let (X,d) be a metric space. For any $Y\subset X$, denote by $Int_X(Y)$, $\partial_X(Y)$ and $Clos_X(Y)$ the interior, the boundary, the closure of Y in X ,respectively. If there is no confusion, we also write $\overline{Y}$ for $Clos_X(Y)$ .For any $y\in Y\subset X$ and $r>0$, write $B(y,r)=\{x\in X:d(x,y)<r\}$ and $B(Y,r)=\{x\in X:d(x,Y)<r\}$. For a finite set Y, denote $|Y|$ the number of the elements in Y.

Denote by $C^0(X)$ the set of all continuous maps from X to it self. Let $\mathbb{N}$ be the set of all positive integers, and let $Z_+=\mathbb{N}\cup\{0\}$ .For any $n\in\mathbb{N}$, write $\mathbb{N}_n=\{1,2,\cdots,n\}$ and $Z_n=\{0,1,\cdots,n-1\}$. For any $f\in C^0(X)$ and $x\in X$, the orbit of x under f, denoted by $o(x,f)$, is the set $\{f^n(x):n=0,1,2,\cdots\}$, where $f^0=id_x$, $f^1=f$, and $f^n=f\circ f^{n-1}$ $(n\ge 2)$ is the $n-fold$ composition of f. Let $\mathbb{N}$ be the set of all positive integers. For any $n\in\mathbb{N}$, write $\mathbb{N}_n=\{1,2,\cdots,n\}$. Write $\mathbb{Z}_+=\mathbb{N}\cup\{0\}$.

A point $x\in X$ is called a ***periodic point*** of f with period n if $f^n(x)=x$ and $f^i(x)\ne x$ for $1\le i<n$. $x\in X$ is called a ***fixed point*** of f if $f(x)=x$. The set $\omega(x,f)\equiv\bigcap_{m=0}^{\infty}\overline{O(f^m(x),f)}$ is called the $\omega-$***limit set*** of a point $x\in X$ under f. Write $\omega(f)=\bigcup_{x\in X}\omega(x,f)$, called the $\omega-$limit set of f. $x\in X$ is called a recurrent point if $x\in\omega(x,f)$. x is called an ***almost recurrent point*** if for any neighborhood U of x in X there exists an $m\in\mathbb{N}$ such that $\{f^{n+i}(x):i\in\mathbb{N}_m\}\cap U\ne\varnothing$ for every $n\in\mathbb{Z}_+$. For any $x,y\in X$ and $\varepsilon>0$, a sequence $(x_0,x_1,\cdots,x_n)$ of points in X with $n\ge 1$ is called an $\varepsilon-$***pseudo orbit*** or an $\varepsilon-$ chain of f from x to y if $x_0=x,\ x_n=y$ and $d(x_i,f(x_{i-1}))<\varepsilon$ for all $i\in\mathbb{N}_n$. If there is an $\varepsilon-$pseudo orbit from x to y for every $\varepsilon>0$, then we say x can be chained to y .A point $x\in X$ is called a chain recurrent point of f

if x chain to x. A subset Y of X is called positively chain invariant if for every $y \in Y$ and $x \notin Y$, y cannot be chained to x. Denote by $Fix(f)$, $P(f)$, $AP(f)$, $R(f)$ and $CR(f)$ the set of fixed points, periodic points, almost periodic points, recurrent points, chain recurrent points of f,respectively. From the definition, it is easy to see that $Fix(f) \subset P(f) \subset AP(f) \subset R(f) \subset \omega(f) \subset CR(f)$.

Let x be a periodic point with period n, A ***generalized attracting neighborhood*** of $O(x, f)$ is an open neighborhood V_x of x with $\overline{V_x} \neq X$ and $f^n(\overline{V_x}) \subset V_x$.

A non-degenerate metric space X is called an arc(resp. an open arc, a circle) if it is homeomorphic to the interval $[0,1]$(resp. the open interval $(0,1)$, the unit circle S^1). By a graph G we mean a connected Hausdorff space which is the union of finitely many subspaces G_i, each of them is a arc and $|G_i \cap G_j| \leq 1$. By a tree T we mean a graph which contains no circle, that is a uniquely arcwise connected graph.

Let G be a graph, let $x \in G$ and U be an open (in G) neighborhood of x such that $\overline{U}$ is a tree. The number of connected components of $U \setminus \{x\}$ is called the valence of x and is denoted by $Val_G(x)$ (or simply $V(x)$ if there is no confusion). If $Val_G(x) = 1$, x is called an endpoint of G; if $Val_G(x) > 2$, x is called a branch point of G.We use $End(G)$ and $Br(G)$ to denote the set of endpoints and the set of branch points of G respectively. Let $V(G) = End(G) \bigcup Br(G)$. A finite set $D(G) \supset V(G)$ is a set of vertices of G such that the closure of each connected component of $G - D(G)$ is homeomorphic to $[0,1]$ and if I, J are two different elements in $E(G) \equiv \{\overline{L} : L$ is a connected component of G-D(G)$\}$, then $|I \cap J| \leq 1$. The element of $E(G)$ is said to be the edge of G.For some edge I of G and any $a, b \in I$, we use $[a,b]_I$ (or simply $[a,b]$ if there is no confusion) to denote the smallest connected closed subset of I containing $\{a,b\}$.Define $(x, y] = [x, y] - \{x\}$ and $(x, y) = (x, y] - \{y\}$.

In [3],Block and Franke proved that if $f : I \to I$ is an interval map and $P(f)$ is a closed set then $CR(f) = P(f)$. In [4], Block and Franke proved that if $f : S^1 \to S^1$ is a circle map $CR(f) = P(f)$ if and only if $P(f)$ is a nonempty closed set and for every $x \in S^1 - P(f)$, some element of $\omega(x, f)$ has a generalized attracting neighborhood. In [8], Mai and Sao proved that if $f : G \to G$ is a graph map and $P(f)$ is a closed set, then $\omega(f) = R(f)$. Our main result is the following theorems.

Theorem 2.1. Let G be a graph, $f : G \to G$ be a continuous map. If $P(f)$ is a nonempty closed set and for every $x \in G - P(f)$, there is a point in $\omega(x, f)$ which has a generalized attracting neighborhood containing no circle, then $CR(f) = P(f)$.

Theorem 2.2. Let G be a graph, $f : G \to G$ be a continuous map. If $P(f)$ is closed and for every $y \in P(f)$, y has a generalized attracting neighborhood containing no circle, then $CR(f) = AP(f)$.

§ 2 Chain recurrent point of graph maps with closed periodic point set

In this section, the properties of chain recurrent point set of graph maps with closed periodic point set are discussed.

Lemma 2.1.([4])Let X be a compact metric space, $f: X \to X$ be a continuous map. If Y is an open subset such that $f(\overline{Y}) \subset Y$ then $\overline{Y}$ is positively chain invariant and $CR(f) \cap \overline{Y} = CR(f|\overline{Y})$.

Lemma 2.2. Let G be a graph, $f: G \to G$ be a continuous map and $y \in P(f)$. If y has a generalized attracting neighborhood then so does each point in the orbit of y.

Proof. Let n denote the period of y and $y_1 = f^{n-1}(y)$. It suffices to show that y_1 has a generalized attracting neighborhood. Let V be a connected a generalized attracting neighborhood of y and V_1 denote the component of $f^{-1}(V)$ which contains y_1. Then V_1 is a open set and $f(\overline{V_1}) \subset \overline{V}$.

Case 1. $\overline{V_1} = G$ .In this case, $f(G) \subset \overline{V} \subsetneq G$. Then x has a generalized attracting neighborhood for each $x \in P(f)$.

Case 2. $\overline{V_1} \neq G$ .Let $A = \overline{V_1} - V_1$ .Then $A \neq \varnothing$ .For any $a \in A$,we claim that $f(a) \notin V$ .If $f(a) \in V$, there is a connected open set U_1 with $a \in U_1$ such that $f(U_1) \subset V$ .Therefore, $U_1 \subset V_1$,which contradicts with $a \in \overline{V_1} - V_1$ .Thus $f(A) \cap V = \varnothing$ .It follows that $f(A) \cap f^n(V) = \varnothing$ and $A \cap f^{n-1}(V) = \varnothing$. Thus, $f^n(\overline{V_1}) = f^{n-1}(f(\overline{V_1})) \subset f^{n-1}(\overline{V}) \subset V_1$

So V_1 is a generalized attracting neighborhood of y_1. ■

In [5], Li and Ye proved the following lemma. They got a necessary and sufficient condition of tree maps f with $CR(f) = P(f)$.

Lemma 2.3. Let T be a tree, $f: T \to T$ be a continuous map. Then $CR(f) = P(f)$ if and only if $P(f)$ is closed.

Theorem 2.1. Let G be a graph, $f: G \to G$ be a continuous map. If $P(f)$ is a nonempty closed set and for every $x \in G - P(f)$, there is a point in $\omega(x, f)$ which has a generalized attracting neighborhood containing no circle, then $CR(f) = P(f)$.

Proof. Suppose that $P(f)$ is a nonempty closed set and for every $x \in G - P(f)$, there is a stable periodic point in $\omega(x, f)$. Let $x \in G - P(f)$. We will show that $x \notin CR(f)$.

Let y be a element of $\omega(x, f)$ which has a generalized attracting neighborhood and denote by $\{y_k = f^k(y) : k = 0,1,\cdots,n-1\}$ the orbit of y, where. By Lemma 2.2, each y_k has a generalized attracting neighborhood V_k containing no circle. Without loss of generality, suppose that V_k is connected for $k = 0,1,\cdots,n-1$ .Let $V = \bigcup_{k=0}^{n-1} V_k$. Then V is a neighborhood $\{y_0, y_1, \cdots, y_{n-1}\}$ and

$f^n(\overline{V}) \subset V$. Thus, by Lemma 2.1, $\overline{V}$ is positively chain invariant under f^n.Choose $\varepsilon_1 > 0$ such that $B(y_k, \varepsilon_1) \subset V_k$ for $k = 0,1,\cdots,n-1$. Choose $\delta_1 > 0$ such that if $d(u,v) < \delta_1$ then $d(f^i(u), f^i(v)) < \varepsilon_1$ for $i = 0,1,\cdots,n$. Since $y \in \omega(x, f)$, there is some point $f^m(x) \in B(y, \delta_1)$. It follows that $f^{m+k}(x) \in V_k$ for $k = 1,2,\cdots,n-1$. For any $\varepsilon > 0$, choose $\delta > 0$ such that $d(u,v) < \delta$ then $d(f^i(u), f^i(v)) < \dfrac{\varepsilon}{n}$ for $i = 0,1,\cdots,n$. Assume that $x \in CR(f)$, then $f^{m+n-1}(x)$ can be chained to x. There is a $\delta-$chain $\left(x_0 = f^{m+n-1}(x), x_1, x_2, \cdots, x_t = x\right)$. Choose $k \in \{0,1,\cdots,n-1\}$ such that $n-k-1+t = \ln$ for some $l \in \mathbb{N}$. Since $\left(f^{m+k}(x),\ f^{m+k+1}(x), \cdots, x_0 = f^{m+n-1}(x), x_1, x_2, \cdots, x_t = x\right)$ is a $\delta-$chain. Denote $y_0 = f^{m+k}(x)$, $y_1 = f^{m+k+1}(x), \cdots, y_{n-k-1} = f^{m+n-1}(x)$ and $y_{n-k} = x_1, \cdots, y_{n-k-1+t} = x$. Then $(y_0, y_1, \cdots, y_{n-k-1+t})$ is a $\delta-$chain from $f^{m+k}(x)$ to x. Since $d(f(y_j), y_{j+1}) < \delta$ for $j = 0,1,\cdots,nl-1$, we have

$$d(f^n(y_i), y_{i+n}) \le d(f^n(y_i), f^{n-1}(y_{i+1})) + d(f^{n-1}(y_{i+1}), f^{n-2}(y_{i+2})) + \cdots + d(f(y_{i+n-1}), y_{i+n}) < \varepsilon$$

for $i = 0,1,\cdots,(l-1)n$. Thus $(y_0, y_n, y_{2n}, \cdots, y_{nl})$ is an $\varepsilon-$chain from $f^{m+k}(x)$ to x. Since $\overline{V}$ is positively chain invariant under f^n, we have $x \in V$. Without loss of generality, suppose that $x \in \overline{V_0}$.

It follows by Lemma 2.1 that $x \in CR(f) \cap \overline{V_0} = CR(f^n) \cap \overline{V_0} = CR(f^n |\overline{V_0})$. Obviously, $f^n |\overline{V_0}$ is a tree map. Thus, by Lemma 2.3, $x \in P(f^n |\overline{V_0}) \subset P(f)$, which contradicts with the assumption. It follows that $CR(f) = P(f)$. ■

One can construct example where $f: G \to G$ is a graph map, $CR(f) = P(f)$, $P(f) \ne \varnothing$ is closed and there is no periodic point in $\omega(x, f)$ has a generalized attracting neighborhood containing no circle for some $x \in G - P(f)$.

Example 2.1. Let $S_i = \left\{(x,y) : (x + \frac{1}{2} \times (-1)^i)^2 + y^2 = \frac{1}{4}\right\}$, for i=1,2 and $G = S_1 \cup S_2$. Map $f: G \to G$ is defined as following:

(1) $f((0,0)) = (0,0)$, $f((1,0)) = (1,0)$, $f((-1,0)) = (-1,0)$;

(2) $f((x,y)) = (\sqrt{x}, \dfrac{|y|}{y}\sqrt{\sqrt{x} - x})$ if $(x,y) \in S_1$ and $y \ne 0$;

(3) $f((x,y)) = (-x^2, \dfrac{|y|}{y}\sqrt{x^2 - x^4})$ if $(x,y) \in S_2$ and $y \ne 0$;

It is easy to check that $CR(f) = P(f) = \{(-1,0), (0,0), (1,0)\}$ and $(0,0) = \omega(s)$ for every $s = (x,y) \in S_2$ with $-1 < x < 0$. Obviously, V is not a generalized attracting neighborhood of $(0,0)$ for each open neighborhood V of $(0,0)$ with $\overline{V} \ne G$. Thus, the inverse of theorem 2.1 doesn't hold. ■

Lemma 2.4.([8]) Let G be a graph, $f: G \to G$ be a continuous map. If

$P(f)$ is closed, then $\omega(f)=R(f)$.

Lemma 2.5.([6]) Let G be a graph, $f:G\to G$ be a continuous map. Then $\overline{R(f)}=AP(f)\cup\overline{P(f)}$

Theorem 2.2. Let G be a graph, $f:G\to G$ be a continuous map. If $P(f)$ is closed and for every $y\in P(f)$, y has a generalized attracting neighborhood containing no circle, then $CR(f)=AP(f)$.

Proof. Suppose that $P(f)$ is a closed set. By Lemma 2.4 and 2.5, we have $\overline{\omega(f)}=\overline{R(f)}=AP(f)\cup\overline{P(f)}=AP(f)\cup P(f)=AP(f)\subset R(f)\subset\omega(f)$.

Thus, $AP(f)=R(f)=\omega(f)$. Let $x\in G-AP(f)$. We will show that $x\notin CR(f)$.

Let y be a element of $\omega(x,f)$. Then $y\in AP(f)$.

Case 1. $y\in P(f)$. In this case, it is easy to prove $x\notin CR(f)$ by the same process as in the proof of Theorem 2.1.

Case 2. $y\notin P(f)$. In this case, it follows by Theorem 4.6 in [6] that $\overline{O(y,f)}=\bigcup_{i=1}^{k}C_i$, where C_i is a circle, $f(C_i)=C_{i+1\,(\mathrm{mod}\,k)}$ and $f^k|C_i$ is topologically semi-conjugate to an irrational rotation of the unit circle S_1 for each $i\in\mathbb{N}_k$. Therefore, there is an open set U such that $O(y,f)\subset U$ and $f(U)=\overline{O(y,f)}$. Since $y\in\omega(x,f)$, there is integer $m\in\mathbb{N}$ such that $f^m(x)\in U$. It follows that

$$f^{m+1}(x)\in\bigcup_{i=1}^{k}C_i \text{ .Let } \varepsilon=\frac{1}{2}\max\left\{d(\bigcup_{i=1}^{k}C_i,x),d(\bigcup_{i=1}^{k}C_i,G-U)\right\}.$$

If $x\in CR(f)$, then $f^{m+1}(x)$ can be chained to x. Thus, there is an $\varepsilon-$ chain $\left(x_0=f^{m+1}(x),x_1,x_2,\cdots,x_n=x\right)$. Obviously, $f(x_0)\in\bigcup_{i=1}^{k}C_i$, $d(x_0,x)>\varepsilon$ and $d(f(x_0),x)\ge 2\varepsilon$. Thus, $d(x_1,x)>\varepsilon$ and $f(x_1)\in\bigcup_{i=1}^{k}C_i$ and $d(f(x_1),x)\ge 2\varepsilon$.

Suppose that $d(x_j,x)>\varepsilon$, $f(x_j)\in\bigcup_{i=1}^{k}C_i$ and $d(f(x_j),x)\ge 2\varepsilon$. Then

$$f(x_{j+1})\in\bigcup_{i=1}^{k}C_i \text{ and } d(f(x_{j+1}),x)\ge 2\varepsilon.$$

It follows that $d(f(x_{n-1}),x)>\varepsilon$ which contradicts the fact that $\left(x_0=f^{m+1}(x),x_1,x_2,\cdots,x_n=x\right)$ is an $\varepsilon-$chain. Hence, $x\notin CR(f)$.

It follows by case 1 and 2 that $CR(f)=AP(f)$. ■

One can construct example to show that the condition that the generalized attracting neighborhood contains no circle in Theorem 2.1 and 2.2 can not be omitted.

Example 2.2. Let $S_1=\left\{(x,y):(x-\frac{1}{2})^2+y^2=\frac{1}{4}\right\}$, $A=\left\{(x,0):-1\le x\le 0\right\}$, $B=\left\{(x,0):1\le x\le 2\right\}$, and $G=S_1\cup A\cup B$. Map $f:G\to G$ is defined as following:
(1) $f((0,0))=(0,0)$, $f((1,0))=(1,0)$;

(2) $f((x,y)) = (\sqrt{x}, \sqrt{\sqrt{x} - x})$ if $(x,y) \in S_1$ and $y > 0$;

(3) $f((x,y)) = (x^2, -\sqrt{x^2 - x^4})$ if $(x,y) \in S_1$ and $y < 0$;

(4) $f(A) = \{(0,0)\}$, $f(B) = \{(1,0)\}$

Obviously, $(0,0) \in \omega(s)$ for every $s = (x,y) \in A$, $V = \left\{(x,y) : x > -\frac{1}{2}\right\} \cap G$ is a generalized attracting neighborhood of $(0,0)$ which contains a circle S_1. It is easy to check that $CR(f) = S_1$, $\omega(f) = P(f) = \{(0,0),(1,0)\}$. ■

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Some properties of pseudo almost periodic sequences*

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Abstract

In this paper, some new properties of pseudo almost periodic sequences are established. Especially, we obtain two interesting characterization theorems for pseudo almost periodic sequences.

Keywords: pseudo almost periodic, almost periodic, ergodic zero set.
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1 Introduction

In 1990s, Zhang (see [8]) introduced the concept of pseudo almost periodic function, which is a generalization of the classical almost periodic function. Since then, there has been of great interest for many authors to study pseudo almost periodic functions and it applications in differential equations.

On the other hand, recently, many authors investigated almost periodic type solutions for various difference equations (see, e.g., [1, 3–6] and references therein). Therefore, it is very necessary to make further study on the properties of almost periodic type sequences. In this paper, we aim to make some study on pseudo almost periodic sequences and obtain some new results.

Throughout the rest of this paper, we denote by X a real Banach space, by $\mathbb{R}$ the set of real numbers, by $\mathbb{Z}$ the set of integers, by $\mathbb{N}$ the set of positive integers. Next, we first recall some basic notations about almost periodic type sequences. For more details, we refer the reader to [2, 8].

Definition 1. *A set $P \subset \mathbb{Z}$ is called relatively dense in $\mathbb{Z}$ if there exists a number $l > 0$ such that $\forall a \in \mathbb{R}$, $(a, a + l) \bigcap P \neq \varnothing$.*

Definition 2. *A set $P \subset \mathbb{Z}$ is called relatively dense in $\mathbb{Z}$ if there exists a number $l \in \mathbb{N}$ such that $\forall a \in \mathbb{Z}$, $[a, a + l] \bigcap P \neq \varnothing$.*

Remark 3. It is not difficult to show that the above two definitions are equivalent.

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Definition 4. *A sequence $g : \mathbb{Z} \to X$ is called almost periodic if for every $\epsilon > 0$,*

$$P_\epsilon = \{\tau \in \mathbb{Z} : \sup_{n\in\mathbb{Z}} \|f(n+\tau) - f(n)\| < \epsilon\}$$

is relatively dense in $\mathbb{Z}$. We denote the set of all such sequences by $AP(\mathbb{Z}, X)$ or $AP(\mathbb{Z})$ if there is no confusion.

Next, we denote $C_0(\mathbb{Z}, X)$ be the set of all sequences $f : \mathbb{Z} \to X$ satisfying $\lim\limits_{n\to+\infty} f(n) = 0$, and

$$PAP_0(\mathbb{Z}, X) := \left\{ f : \mathbb{Z} \to X \text{ is bounded} : \lim_{n\to+\infty} \frac{1}{2n} \sum_{k=-n}^{n} \|f(k)\| = 0 \right\}.$$

Definition 5. *A function $f : \mathbb{Z} \to X$ is called asymptotically almost periodic if it can be expressed as $f = g + h$, where $g \in AP(\mathbb{Z}, X)$ and $h \in C_0(\mathbb{Z}, X)$. The set of such functions will be denoted by $AAP(\mathbb{Z}, X)$.*

Definition 6. *A bounded function $f : \mathbb{Z} \to X$ is called pseudo almost periodic if it can be expressed as $f = g + h$, where $g \in AP(\mathbb{Z}, X)$ and $h \in PAP_0(\mathbb{Z}, X)$. The set of such functions will be denoted by $PAP(\mathbb{Z}, X)$.*

Definition 7. *A set $E \subset \mathbb{Z}$ is said to be an ergodic zero set in $\mathbb{Z}$ if*

$$\lim_{n\to+\infty} \frac{card(E \bigcap [-n, n])}{2n} = 0.$$

Lemma 8. *Let $f : \mathbb{Z} \to X$ be bounded. Then $f \in PAP_0(\mathbb{Z}, X)$ if and only if for every $\epsilon > 0$, $E(f, \epsilon) = \{k \in \mathbb{Z} : \|f(k)\| \geq \epsilon\}$ is an ergodic zero set in $\mathbb{Z}$.*

Proof. The proof is similar to [4, Lemma 2.9]. So we omit the details. □

2 Main results

In this section, we will establish two characterization theorems for pseudo almost periodic sequences. Before establish our main results, we first prove some lemmas.

Lemma 9. *Let $E \subset \mathbb{Z}$ be an ergodic zero set. Then, for every $L > 0$, there exists an interval $(u, v) \subset \mathbb{R}$ with $v - u > L$ such that $E \bigcap (u, v) = \emptyset$.*

Proof. We prove it by contradiction. If the conclusion is not true, then there exists $L_0 > 0$ such that for every $a \in \mathbb{R}$, $E \bigcap (a, a + L_0) \neq \emptyset$, which yields that

$$\liminf_{n\to+\infty} \frac{card(E \bigcap [-n, n])}{2n} \geq \lim_{r\to+\infty} \frac{[\frac{2n}{L_0}]}{2n} = \frac{1}{L_0} > 0,$$

which contradicts with the fact that E is an ergodic zero set. □

Lemma 10. *Let $P \subset \mathbb{Z}$ be a relatively dense set, and $E \subset \mathbb{Z}$ be an ergodic zero set. Then for any given interval $[c, d]$, there exist $(u, v) \subset \mathbb{R}$ and $\tau \in P$ such that $E \bigcap (u, v) = \emptyset$ and $[c, d] + \tau \subset (u, v)$.*

Proof. Since P is relatively dense, there exists a number $l > 0$ such that $\forall a \in \mathbb{R}$, $(a, a+l) \bigcap P \neq \varnothing$. Letting $L = l+d-c$, by Lemma 9, there exists an interval (u, v) with $v-u > L$ such that $E \bigcap (u, v) = \emptyset$. Taking $\tau \in (u-c, u-c+l) \bigcap P$, for all $t \in [c, d]$, we have

$$u < c + \tau \leq t + \tau \leq d + \tau \leq d + u - c + l = u + L < v,$$

which means that $[c, d] + \tau \subset (u, v)$. □

Lemma 11. *Let $P \subset \mathbb{Z}$ be a relatively dense set, $E \subset \mathbb{Z}$ be an ergodic zero set, and $k_i \in \mathbb{Z}$, $i = 1, 2, \ldots, n$, where $n \in \mathbb{N}$ is a fixed constant. Then, there exits $\tau \in P$ such that $k_i + \tau \notin E$, $i = 1, 2, \ldots, n$.*

Proof. Without loss for generality, we can suppose that $k_1 < k_2 < \cdots < k_n$. It follows from Lemma 10 that there exists an interval (u, v) and $\tau \in P$ such that $E \bigcap (u, v) = \emptyset$ and $[k_1, k_n] + \tau \subset (u, v)$. Thus $E \bigcap ([k_1, k_n] + \tau) = \emptyset$, which means that $k_i + \tau \notin E$, $i = 1, 2, \ldots, n$. □

Now we are ready to present one of our main results, which extends the corresponding result in [7] to pseudo almost periodic sequences.

Theorem 12. *Let $f : \mathbb{Z} \to X$ be a bounded sequence. Then a necessary and sufficient condition for $f \in PAP(\mathbb{Z}, X)$ is that for every $\epsilon > 0$, there exist a relatively dense set $P_\epsilon \subset \mathbb{Z}$ and an ergodic zero set $E_\epsilon \subset \mathbb{Z}$ such that*

$$\|f(k+\tau) - f(k)\| < \epsilon, \tag{1}$$

for all $k \in \mathbb{Z}$ and $\tau \in P_\epsilon$ with $k, k+\tau \notin E_\epsilon$.

Proof. "Necessity".

Suppose that $f \in PAP(\mathbb{Z}, X)$. Then, there exist $g \in AP(\mathbb{Z}, X)$ and $h \in PAP_0(\mathbb{Z}, X)$ such that $f = g + h$. For every $\epsilon > 0$, since g is almost periodic, there is a relatively dense set $P_\epsilon \subset \mathbb{Z}$ such that $\|g(k+\tau) - g(k)\| < \epsilon$ for all $k \in \mathbb{Z}$ and $\tau \in P_\epsilon$. We denote

$$E_\epsilon = \{k \in \mathbb{Z} : \|h(k)\| \geq \epsilon\}.$$

Noting $h \in PAP_0(\mathbb{Z}, X)$, by Lemma 8, E_ϵ is an ergodic zero set. Then, for all $k \in \mathbb{Z}$ and $\tau \in P_\epsilon$ with $k, k+\tau \notin E_\epsilon$, we have

$$\begin{aligned} \|f(k+\tau) - f(k)\| &= \|f(k+\tau) - g(k+\tau)\| + \|g(k+\tau) - g(k)\| + \|g(k) - f(k)\| \\ &= \|h(k+\tau)\| + \|g(k+\tau) - g(k)\| + \|h(k)\| \\ &< 3\epsilon. \end{aligned}$$

"Sufficiency".

By (1), for every $n \in \mathbb{N}$, there exist a relatively dense set $P_n \subset \mathbb{Z}$ and an ergodic zero set $E_n \subset \mathbb{Z}$ such that

$$\|f(k+\tau) - f(k)\| < \frac{1}{n}, \tag{2}$$

for all $k \in \mathbb{Z}$ and $\tau \in P_n$ with $k, k+\tau \notin E_n$.

For every $n \in \mathbb{N}$ and every $k \in \mathbb{Z}$, by Lemma 11, there exists $\tau_n^k \in P_n$ such that $k + \tau_n^k \notin E_n$. Now, we denote

$$f_n(k) = f(k + \tau_n^k), \quad n \in \mathbb{N},\ k \in \mathbb{Z}.$$

Next, we divide the remaining proof by five steps.

Step 1. For every $n \in \mathbb{N}$, $\tau \in P_n$ and $k \in \mathbb{Z}$, there holds

$$\|f_n(k+\tau) - f_n(k)\| < \frac{5}{n}.$$

Again by Lemma 11, we can choose $\tau' \in P_n$ such that

$$k + \tau + \tau_n^{k+\tau} + \tau', k + \tau + \tau', k + \tau', k + \tau' + \tau_n^k \notin E_n.$$

Then, by (2), we have

$$\begin{aligned}
& \|f_n(k+\tau) - f_n(k)\| \\
= \;& \|f(k+\tau+\tau_n^{k+\tau}) - f(k+\tau_n^k)\| \\
\leq \;& \|f(k+\tau+\tau_n^{k+\tau}) - f(k+\tau+\tau_n^{k+\tau}+\tau')\| + \|f(k+\tau+\tau_n^{k+\tau}+\tau') - f(k+\tau+\tau')\| \\
& + \|f(k+\tau+\tau') - f(k+\tau')\| + \|f(k+\tau') - f(k+\tau'+\tau_n^k)\| + \|f(k+\tau'+\tau_n^k) - f(k+\tau_n^k)\| \\
< \;& \frac{5}{n}.
\end{aligned}$$

Step 2. For every $m, n \in \mathbb{N}$ with $m \geq n$ and $k \in \mathbb{Z}$, there holds

$$\|f_m(k) - f_n(k)\| < \frac{4}{n}.$$

Again by (2), we get

$$\begin{aligned}
& \|f_m(k) - f_n(k)\| \\
= \;& \|f(k+\tau_m^k) - f(k+\tau_n^k)\| \\
\leq \;& \|f(k+\tau_m^k) - f(k+\tau_m^k+\tau'')\| + \|f(k+\tau_m^k+\tau'') - f(k+\tau'')\| \\
& + \|f(k+\tau'') - f(k+\tau''+\tau_n^k)\| + \|f(k+\tau''+\tau_n^k) - f(k+\tau_n^k)\| \\
< \;& \frac{1}{n} + \frac{1}{m} + \frac{1}{n} + \frac{1}{n} < \frac{4}{n},
\end{aligned}$$

where $\tau'' \in P_n$ satisfying

$$k + \tau_m^k + \tau'', k + \tau'', k + \tau'' + \tau_n^k \notin E_n \bigcup E_m.$$

Step 3. Let

$$g(k) = \lim_{n \to +\infty} f_n(k), \quad k \in \mathbb{Z}.$$

It follows from Step 2 that g is well-defined, and

$$\|f_n(k) - g(k)\| < \frac{4}{n}, \quad n \in \mathbb{N},\ k \in \mathbb{Z}. \tag{3}$$

Step 4. $g \in AP(\mathbb{Z}, X)$.

In fact, by Step 1 and (3), we have

$$\begin{aligned}\|g(k+\tau)-g(k)\| &\leq \|g(k+\tau)-f_n(k+\tau)\|+\|f_n(k+\tau)-f_n(k)\|+\|f_n(k)-g(k)\| \\ &< \frac{4}{n}+\frac{5}{n}+\frac{4}{n}=\frac{13}{n},\end{aligned}$$

for all $n \in \mathbb{N}$, $\tau \in P_n$ and $k \in \mathbb{Z}$.

Step 5. $f-g \in PAP_0(\mathbb{Z}, X)$.

It follows from (3) that

$$\begin{aligned}\|f(k)-g(k)\| &\leq \|f(k)-f_n(k)\|+\|f_n(k)-g(k)\| \\ &< \|f(k)-f(k+\tau_n^k)\|+\frac{4}{n}<\frac{5}{n},\end{aligned}$$

for every $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ with $k \notin E_n$, which means that for every $n \in \mathbb{N}$,

$$\{k \in \mathbb{Z} : \|f(k)-g(k)\| \geq \frac{5}{n}\} \subset E_n.$$

Noting that every E_n is an ergodic zero set, we conclude that $\{k \in \mathbb{Z} : \|f(k)-g(k)\| \geq \frac{5}{n}\}$ is an ergodic zero set for every $n \in \mathbb{N}$. Then, by Lemma 8, $f-g \in PAP_0(\mathbb{Z}, X)$. This completes the proof. □

Next, we will present another characterization theorem for pseudo almost periodic sequences. Before establishing our main result, we first prove the following interesting lemma:

Lemma 13. *Let $f : \mathbb{Z} \to X$ be a bounded sequence. Then a necessary and sufficient condition for $f \in PAP_0(\mathbb{Z}, X)$ is that there exists an ergodic zero set $E \subset \mathbb{Z}$ such that*

$$\lim_{\substack{k\to+\infty \\ k\in\mathbb{Z}\setminus E}} f(k)=0.$$

Proof. "Sufficiency"

Since $\lim_{\substack{k\to+\infty \\ k\in\mathbb{Z}\setminus E}} f(k)=0$, we know that for every $\epsilon > 0$,

$$card\{k \in \mathbb{Z}\setminus E : \|f(k)\| \geq \epsilon\} < +\infty,$$

which yields that for every $\epsilon > 0$,

$$\lim_{n\to+\infty} \frac{card\{k \in [-n,n] : \|f(k)\| \geq \epsilon\}}{2n}=0$$

since E is an ergodic zero set and

$$card\{k \in [-n,n] : \|f(k)\| \geq \epsilon\} \leq card([-n,n]\cap E)+card\{k \in \mathbb{Z}\setminus E : \|f(k)\| \geq \epsilon\}.$$

Thus, by Lemma 8, $f \in PAP_0(\mathbb{Z}, X)$.

"Necessity". Suppose that $f \in PAP_0(\mathbb{Z}, X)$. We denote

$$E_k=\{n \in \mathbb{Z} : \|f(n)\| \geq \frac{1}{k}\}, \quad k \in \mathbb{N}.$$

Then, every E_k is an ergodic zero set. For $k = 1$, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$,

$$\frac{card([-n,n] \cap E_2)}{2n} \leq 1,$$

and

$$\frac{card([-N_1,N_1] \cap E_1)}{2n} \leq \frac{card([-n,n] \cap E_1)}{2n} \leq 1.$$

For $k = 2$, there exists $N_2 \in \mathbb{N}$ ($N_2 > N_1$) such that for all $n \geq N_2$,

$$\frac{card([-n,n] \cap E_3)}{2n} \leq \frac{1}{2},$$

and

$$\frac{card([-N_1,N_1] \cap E_1) + card([-N_2,N_2] \cap E_2)}{2n} \leq \frac{2N_1 + 1 + card([-n,n] \cap E_2)}{2n} \leq \frac{1}{2}.$$

Continuing by this way, we can get a sequence $\{N_k\}_{k=1}^{\infty} \subset \mathbb{N}$ satisfying that for every $k \in \mathbb{N}$ and $n \geq N_k$, there hold

$$\frac{card([-n,n] \cap E_{k+1})}{2n} \leq \frac{1}{k}, \tag{4}$$

and

$$\frac{\sum\limits_{i=1}^{k} card([-N_i,N_i] \cap E_i)}{2n} \leq \frac{\sum\limits_{i=1}^{k-1} (2N_i + 1) + card([-n,n] \cap E_k)}{2n} \leq \frac{1}{k}. \tag{5}$$

Now, we denote $E = \bigcup_{k=1}^{\infty} \widetilde{E_k}$, where

$$\widetilde{E_1} = [-N_1, N_1] \cap E_1, \quad \widetilde{E_k} = ([-N_k, N_k] \backslash [-N_{k-1}, N_{k-1}]) \cap E_k, \ k = 2, \ldots.$$

We claim that E is an ergodic zero set. In fact, for every $m \in \mathbb{N}$ and $n \in [N_m, N_{m+1}]$, by (4) and (5), we have

$$\begin{aligned}
\frac{card([-n,n] \cap E)}{2n} &= \frac{card([-n,n] \bigcap (\bigcup_{k=1}^{\infty} \widetilde{E_k}))}{2n} \\
&\leq \frac{\sum_{k=1}^{m} card \widetilde{E_k} + card([-n,n] \cap \widetilde{E_{m+1}})}{2n} \\
&\leq \frac{\sum_{k=1}^{m} card([-N_k,N_k] \cap E_k) + card([-n,n] \cap E_{m+1})}{2n} \\
&\leq \frac{2}{m}.
\end{aligned}$$

Noting that $N_m \to +\infty$ as $m \to +\infty$, we conclude that

$$\lim_{n \to +\infty} \frac{card([-n,n] \cap E)}{2n} = 0.$$

It remains to show that $\lim\limits_{\substack{k \to +\infty \\ k \in \mathbb{Z} \backslash E}} f(k) = 0$. For every $k \in \mathbb{N}$ and $n \in \mathbb{Z} \backslash E$ with $n > N_k$, there exists $m \geq k$ such that

$$n \in [-N_{m+1}, N_{m+1}] \backslash [-N_m, N_m].$$

Since $n \notin E$, $n \notin \widetilde{E_{m+1}}$, and thus $n \notin E_{m+1}$, which means that

$$\|f(n)\| < \frac{1}{m+1} < \frac{1}{k}.$$

This completes the proof. $\square$

Theorem 14. *Let $f : \mathbb{Z} \to X$ be a bounded sequence. Then a necessary and sufficient condition for $f \in PAP(\mathbb{Z}, X)$ is that there exist an ergodic zero set $E \subset \mathbb{Z}$ and $\widetilde{f} \in AAP(\mathbb{Z}, X)$ such that $\widetilde{f}|_{\mathbb{Z}\backslash E} = f$.*

Proof. Let $f \in PAP(\mathbb{Z}, X)$. Then there exist $g \in AP(\mathbb{Z}, X)$ and $h \in PAP_0(\mathbb{Z}, X)$ such that $f = g + h$. By using Lemma 13, there is an ergodic zero set $E \subset \mathbb{Z}$ such that

$$\lim_{\substack{k\to+\infty \\ k\in\mathbb{Z}\backslash E}} h(k) = 0.$$

Let

$$\widetilde{h}(k) = \begin{cases} h(k), & k \in \mathbb{Z}\backslash E, \\ \ \ 0\ \ , & k \in E, \end{cases}$$

and $\widetilde{f} = g + \widetilde{h}$. Then $\widetilde{f} \in AAP(\mathbb{Z}, X)$ and $\widetilde{f}|_{\mathbb{Z}\backslash E} = f$.

On the other hand, if there exist an ergodic zero set $E \subset \mathbb{Z}$ and $\widetilde{f} \in AAP(\mathbb{Z}, X)$ such that $\widetilde{f}|_{\mathbb{Z}\backslash E} = f$. Let $\widetilde{f} = g + h$, where $g \in AP(\mathbb{Z}, X)$ and $h \in C_0(\mathbb{Z}, X)$. Noting $(f - g)|_{\mathbb{Z}\backslash E} = h$, by a similar proof to the "Sufficiency" part of Lemma 13, we can obtain $f - g \in PAP_0(\mathbb{Z}, X)$, and thus $f \in PAP(\mathbb{Z}, X)$. $\square$

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ISHIKAWA ITERATIVE SCHEME FOR LIPSCHITZIAN PSEUDOCONTRACTIONS

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Abstract. In this paper, we establish the strong convergence for the Ishikawa iterative scheme associated with Lipschitz pseudocontractive mappings in real Banach spaces.

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Key words and phrases: Ishikawa iterative scheme, Lipschitz mappings, pseudocontractive mappings, Banach spaces.

1. Introduction and Preliminaries

Let K be a nonempty convex subset of a real Banach space E. Let J denote the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 \text{ and } \|f^*\| = \|x\|\}$$

for all $x \in E$, where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. We shall denote the single-valued duality mapping by j.

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Let $T : K \to K$ be a mapping.

Definition 1.1. The mapping T is said to be *Lipschitzian* if there exists $L > 0$ such that

$$\|Tx - Ty\| \leq L \|x - y\|$$

for all $x, y \in K$.

Definition 1.2. The mapping T is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in K$.

Definition 1.3. The mapping T is said to be *pseudocontractive* if

$$\|x - y\| \leq \|x - y + t((I - T)x - (I - T)y\|$$

for all $x, y \in K$ and $t > 0$.

As a consequence of a result of Kato [8], it follows that T is *pseudocontractive* if and only if for all $x, y \in K$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \leq \|x - y\|^2 .$$

Definition 1.4. The mapping T is called *accretive* if

$$\|x - y\| \leq \|x - y + s(Tx - Ty)\|$$

for all $x, y \in K$ and $s > 0$.

Remark 1.5. It is well known that every nonexpansive mapping is pseudocontractive. Indeed if T is nonexpansive mapping, then for all x, $y \in K$, there exists $j(x - y) \in J(x - y)$ such that

$$\begin{aligned}\langle Tx - Ty, j(x - y)\rangle &\leq \|Tx - Ty\| \|x - y\| \\ &\leq \|x - y\|^2 .\end{aligned}$$

Rhoades in [15] showed that the class of pseudocontractive mappings properly contains the class of nonexpansive mappings.

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The class of pseudocontractions is, perhaps, the most important generalization of the class of nonexpansive mappings because of its strong relationship with the class of accretive mappings. A mapping $T : E \to E$ is accretive if and only if $I - T$ is pseudocontractive.

Let K be a nonempty convex subset of a normed space E.

(I) For arbitrary $x_1 \in K$, the sequence $\{x_n\}$ defined by

$$\begin{cases} x_{n+1} = (1 - a_n)x_n + a_nTy_n, \\ y_n = (1 - b_n)x_n + b_nTx_n, \quad n \geq 1, \end{cases}$$

where $\{b_n\}$ and $\{b_n\}$ are sequences in $[0, 1]$ is known as the Ishikawa iteration scheme [7].

If $b_n = 0$ for $n \geq 1$, then the Ishikawa iteration scheme becomes the Mann iteration scheme becomes [10].

In the last few years or so, numerous papers have been published on the iterative approximation of fixed points of Lipschitz *strongly* pseudocontractive mappings using the *Ishikawa iteration scheme* (see for example, [7]). Results which had been known only in *Hilbert spaces* and only for *Lipschitz mappings* have been extended to more general Banach spaces (see for example [2]-[5], [16], [18] and the references cited therein).

In 1974, Ishikawa [7] proved the following result.

Theorem 1.6. *Let K is a compact convex subset of a Hilbert space H and $T : K \to K$ be a Lipschitz pseudocontractive mapping.*

Fot arbitrary $x_1 \in K$, let $\{x_n\}$ be a sequence defined iteratively by

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTy_n, \\ y_n = (1 - \beta_n)x_n + \beta_nTx_n, \end{cases} \tag{I}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences satisfying

(i) $0 \leq \alpha_n \leq \beta_n \leq 1$;

(ii) $\lim_{n\to\infty} \beta_n = 0$;

(iii) $\sum_{n\geq1} \alpha_n\beta_n = \infty$.

Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Rhoades [13, Theorem 8], using the special case of (I) for which $\beta_n = 0$, and different conditions on α_n has established a similar result for strictly pseudocontractive mappings. In [14], Rhoades further pointed out that no such results can be proved for pseudocontractive mappings.

In [2], Chidume extended the results of Schu [16] from Hilbert spaces to the much more general class of real Banach spaces and approximate the fixed points of strongly pseudocontractive mappings.

In this paper, we establish the strong convergence for the Ishikawa iterative scheme associated with Lipschitz pseudocontractive mappings in real Banach spaces. We also generalize the results of Schu [16] from Hilbert spaces to more general Banach spaces and improve the results presented in [3]-[5], [7], [17], [18].

2. Main Results

We will need the following results.

Lemma 2.1. ([11]) *Let $\{r_n\}$, $\{s_n\}$ and $\{t_n\}$ be three nonnegative sequences satisfying*

$$r_{n+1} \leq (1 + s_n)r_n + t_n, \quad \forall n \in \mathbb{N}.$$

If $\sum_{n\geq 1} s_n < \infty$ and $\sum_{n\geq 1} t_n < \infty$, then $\lim_{n\to\infty} r_n$ exists.

Lemma 2.2. ([1]) *Let $J : E \to 2^E$ be the normalized duality mapping. Then for all $x, y \in E$ and $j(x + y) \in J(x + y)$, we have*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle.$$

Lemma 2.3. ([12]) *If there exists $n_0 \in \mathbb{N}$ such that*

$$\rho_{n+1} \leq (1 - \delta_n^2)\rho_n + b_n, \quad \forall n \geq n_0,$$

then

$$\lim_{n\to\infty} \rho_n = 0,$$

where $\delta_n \in [0, 1)$, $\sum_{n\geq 1} \delta_n^2 = \infty$ and $b_n = \mathrm{o}(\delta_n)$.

The following is our main result.

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Theorem 2.4. *Let K be a nonempty closed convex subset of a real Banach space E, $S : K \to K$ be nonexpansive and $T : K \to K$ be Lipschitz pseudo-contractive mappings such that $F(S) \cap F(T) = \{x \in K : Sx = x = Tx\} \neq \emptyset$ and*

$$\|x - Sy\| \leq \|Sx - Sy\|, \quad \|x - Ty\| \leq \|Tx - Ty\| \tag{C}$$

for all $x, y \in K$. Let $\{\alpha_n\}_{n\geq 1}$ and $\{\beta_n\}_{n\geq 1}$ be sequences in $[0,1]$ satisfying

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n\geq 1} \alpha_n^2 = \infty$;

(iii) $\sum_{n\geq 1} \beta_n < \infty$.

For arbitrary $x_1 \in K$, let $\{x_n\}_{n\geq 1}$ be a sequence defined iteratively by

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n S y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \quad n \geq 1. \end{cases} \tag{2.1}$$

Then the sequence $\{x_n\}_{n\geq 1}$ converges strongly to the common fixed point p of S and T.

Proof. For pseudocontractive mappings, the existence of a fixed point follows from Deimling [6].

By using condition (C), we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1-\alpha_n)x_n + \alpha_n S y_n - p\| \\ &= \|x_n - p + \alpha_n(Sy_n - x_n)\| \\ &\leq \|x_n - p\| + \alpha_n\|Sy_n - x_n\| \\ &\leq \|x_n - p\| + \alpha_n\|Sy_n - Sx_n\| \\ &\leq \|x_n - p\| + \alpha_n\|y_n - x_n\| \\ &= \|x_n - p\| + \alpha_n\beta_n \|x_n - Tx_n\| \\ &\leq (1 + (1+L)\,\alpha_n\beta_n)\,\|x_n - p\|, \\ &\leq (1 + (1+L)\,\beta_n)\,\|x_n - p\| \end{aligned}$$

and by using condition (iii) and Lemma 2.1, we can conclude that the sequence $\{x_n - p\}_{n\geq 1}$ is bounded. Since T is Lipschitzian, so $\{Tx_n - p\}_{n\geq 1}$ is also bounded. Let $M_1 = \sup_{n\geq 1}\|x_n - p\| + \sup_{n\geq 1}\|Tx_n - p\|$.

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Also by (iii), we have

$$\begin{aligned}
\|x_n - y_n\| &= \|x_n - (1-\beta_n)x_n + \beta_n T x_n\| \\
&= \beta_n \|x_n - T x_n\| \\
&\leq M_1 \beta_n \\
&\to 0
\end{aligned}$$

as $n \to \infty$. This show that $\{x_n - y_n\}_{n\geq 1}$ is bounded, so let $M_2 = \sup_{n\geq 1} \|x_n - y_n\| + M_1$. Further

$$\begin{aligned}
\|y_n - p\| &\leq \|y_n - x_n\| + \|x_n - p\| \\
&\leq M_2,
\end{aligned}$$

which implies that $\{y_n - p\}_{n\geq 1}$ is bounded. Therefore $\{Sy_n - p\}_{n\geq 1}$ is also bounded.

Set

$$M_3 = \sup_{n\geq 1} \|y_n - p\| + \sup_{n\geq 1} \|Sy_n - p\|.$$

Denote

$$M = M_1 + M_2 + M_3.$$

Obviously $M < \infty$.

Now from Lemma 2.2, we obtain for all $n \geq 1$,

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|(1-\alpha_n)x_n + \alpha_n S y_n - p\|^2 \\
&= \|\,(1-\alpha_n)\,(x_n - p) + \alpha_n (Sy_n - p)\|^2 \\
&\leq (1-\alpha_n)^2 \|x_n - p\|^2 + 2\alpha_n \langle Sy_n - p, j(x_{n+1} - p)\rangle \\
&\leq (1-\alpha_n)^2 \|x_n - p\|^2 + 2\alpha_n \langle Tx_{n+1} - p, j(x_{n+1} - p)\rangle \\
&\quad + 2\alpha_n \langle Sy_n - Tx_{n+1}, j(x_{n+1} - p)\rangle \qquad (2.2) \\
&\leq (1-\alpha_n)^2 \|x_n - p\|^2 + 2\alpha_n \|x_{n+1} - p\|^2 \\
&\quad + 2\alpha_n \|Sy_n - Tx_{n+1}\| \, \|x_{n+1} - p\| \\
&\leq (1-\alpha_n)^2 \|x_n - p\|^2 + 2\alpha_n \|x_{n+1} - p\|^2 \\
&\quad + 2M\alpha_n \|Sy_n - Tx_{n+1}\|.
\end{aligned}$$

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Consider

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|(1-\alpha_n)x_n+\alpha_n Sy_n-p\|^2\\ &= \|(1-\alpha_n)(x_n-p)+\alpha_n(Sy_n-p)\|^2\\ &\le (1-\alpha_n)\|x_n-p\|^2+\alpha_n\|Sy_n-p\|^2\\ &\le (1-\alpha_n)\|x_n-p\|^2+M^2\alpha_n,\end{aligned} \tag{2.3}$$

where the first inequality holds by the convexity of $\|\cdot\|^2$.

Substitution of (2.3) in (2.2) yields

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \left((1-\alpha_n)^2+2\alpha_n(1-\alpha_n)\right)\|x_n-p\|^2\\ &\quad+2M\alpha_n\left(M\alpha_n+\|Sy_n-Tx_{n+1}\|\right)\\ &= \left(1-\alpha_n^2\right)\|x_n-p\|^2+2M\alpha_n\left(M\alpha_n+\|Sy_n-Tx_{n+1}\|\right),\end{aligned} \tag{2.4}$$

where

$$\begin{aligned}\|Sy_n-Tx_{n+1}\| &\le \|Sy_n-x_n\|+\|x_n-Tx_{n+1}\|\\ &\le \|Sy_n-Sx_n\|+\|Tx_n-Tx_{n+1}\|\\ &\le \|y_n-x_n\|+L\|x_n-x_{n+1}\|\\ &= \|y_n-x_n\|+L\|x_n-(1-\alpha_n)x_n-\alpha_n Sy_n\|\\ &\le \|y_n-x_n\|+\alpha_n L\|x_n-Sy_n\|\\ &\le \|y_n-x_n\|+\alpha_n L\|Sx_n-Sy_n\|\\ &\le (1+L)\|x_n-y_n\|\\ &= (1+L)\beta_n\|x_n-Tx_n\|\\ &\le (1+L)M\beta_n\\ &\to 0\end{aligned} \tag{2.5}$$

as $n\to\infty$.

For all $n\ge 1$, put

$$\begin{aligned}\rho_n &= \|x_n-p\|,\\ \delta_n &= \alpha_n,\\ b_n &= 2M\alpha_n\left(M\alpha_n+\|Sy_n-Tx_{n+1}\|\right),\end{aligned}$$

then according to Lemma 2.3 and by using (2.5), we obtain from (2.4) that

$$\lim_{n\to\infty} \|x_n - p\| = 0.$$

This completes the proof. □

Corollary 2.5. *Let K be a nonempty closed convex subset of a real Hilbert space E, $S : K \to K$ be nonexpansive and $T : K \to K$ be Lipschitz pseudocontractive mappings such that $F(S) \cap F(T) \neq \emptyset$ and the condition (C).*

Let $\{\alpha_n\}_{n\geq 1}$ and $\{\beta_n\}_{n\geq 1}$ be sequences in $[0, 1]$ *satisfying the conditions* (i)-(iii)*. For arbitrary $x_1 \in K$, let $\{x_n\}_{n\geq 1}$ be a sequence defined iteratively by* (2.1)*. Then the sequence $\{x_n\}_{n\geq 1}$ converges strongly to the common fixed point p of S and T.*

Example 2.6. As a particular case, we may choose for instance $\alpha_n = \frac{1}{\sqrt{n}}$ and $\beta_n = \frac{1}{n^2}$.

Remark 2.7. The condition (C) is not new and is due to Liu et al. [9].

Remark 2.8. (1) We do not need the boundedness assumption on K introduced in [4] and [18].

(2) Our proofs are simple.

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SOME IDENTITIES OF q-BERNOULLI NUMBERS ASSOCIATED p-ADIC CONVOLUTIONS

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Abstract. In this paper, we give some interesting and new identities of q-Bernoulli numbers which are derived from convolutions on the ring of p-adic integers.

1. Introduction

Throughout this paper $\mathbb{Z}_p, \mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic numbers and the completion of algebraic closure of $\mathbb{Q}_p$. The p-adic norm is defined by $|p|_p = p^{-v_p(p)} = p^{-1}$. Now, we set

$$U_p = \{\alpha \in \mathbb{C}_p |\ |\alpha - 1|_p < 1\}$$

and

$$T_p = \left\{ w \in \mathbb{C}_p | w^{p^n} = 1 \text{ for some } n \geq 0 \right\},$$

so that T_p is the union of cyclic (multiplicative) group $\mathbb{C}_{p^n}$ of order p^n $(n \geq 0)$ and $T_p \subset U_p$ (see [12]). Let $UD(\mathbb{Z}_p)$ and $C(\mathbb{Z}_p)$ be the space of uniformly differentiable and continuous function on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the p-adic invariant integral on $\mathbb{Z}_p$ is defined as follows :

$$I_0(f) = \int_{\mathbb{Z}_p} f(x) d\mu_0(x) = \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \text{ (see [11]).} \tag{1.1}$$

For $w \in T_p$, the p-adic Fourier transform of f is given by

$$\hat{f}_w = I_0(f\phi_w) = \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x)\phi_w(x), \text{ (see [12]),} \tag{1.2}$$

where $\phi_w(x) = w^x$.

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Let $f, g \in UD(\mathbb{Z}_p)$. Then *C. F. Woodcock* defined the convolution of f and g as follows :

$$f * g = \sum_w \widehat{f}_w \widehat{g}_w \phi_{w^{-1}}, \quad \text{(see [12])}. \tag{1.3}$$

From (1.3), we note that

$$f * g \in UD(\mathbb{Z}_p), \quad (\widehat{f * g})_w = \widehat{f}_w \widehat{g}_w, \ (\forall w \in T_p), \tag{1.4}$$

and $(UD(\mathbb{Z}_p), +, *, V)$ is Banach algebra, where $V(f) = \min\{\nu(f), R(f)\}, \nu(f) = \inf_{x \in \mathbb{Z}_p} \nu_p(f(x))$ and $R(f) = \inf \nu_p\left(\frac{f(x)-f(y)}{x-y}\right), x \neq y \in \mathbb{Z}_p$, (see [12]).

Let $int\ \mathbb{Z}_p = \{f \in UD(\mathbb{Z}_p) | f' = 0\}$. Then $int\ \mathbb{Z}_p$ is $*$-ideal of $UD(\mathbb{Z}_p)$. Differentiation induced a linear isometry

$$UD(\mathbb{Z}_p)/int\mathbb{Z}_p \longrightarrow C(\mathbb{Z}_p) \ \text{ by } \ (f * g)' = -f' \circledast g', \tag{1.5}$$

where $f, g \in UD(\mathbb{Z}_p)$ (see [12]). By (1.5), we get

$$(f \circledast g)(n) = \sum_{i=0}^{n} f(i)g(n-i), \tag{1.6}$$

where $f, g \in C(\mathbb{Z}_p)$.

For $f, g \in UD(\mathbb{Z}_p)$ and $z \in \mathbb{Z}_p$, it is known that

$$f * g(z) = I_0^{(x)}(f(x)g(z-x)) - f \circledast g'(z), \tag{1.7}$$

where $I_0^{(x)}(f) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x)$.

When one talks of q-extensions, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}_p$, then we normally assume $|1-q|_p < p^{\frac{-1}{p-1}}$ so that $q^x = \exp(\log q)$ for $|x|_p \leq 1$. We use the notation $[x]_q = [x : q] = \frac{1-q^x}{1-q}$. Thus, $\lim_{q\to 1} [x]_q = x$.

As is well known, the usual *Bernoulli numbers* are defined by

$$B_0 = 1, \quad (B+1)^n - B_n = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{if } n > 1, \end{cases} \tag{1.8}$$

with the usual convention about replacing β^i by β_i (see [1-10]). In [1.3], *Carlitz* defined the q-extension of *Bernoulli numbers* as follows:

$$\beta_{0,q} = 1, \quad q(q\beta_q + 1)^n - \beta_{n,q} = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{if } n > 1, \end{cases} \tag{1.9}$$

with the usual convention about replacing β_q^i by $\beta_{i,q}$. By (1.9), we easily see that

$$\beta_{0,q} = 1, \ \beta_{1,q} = \frac{-1}{[2]_q}, \ \beta_{2,q} = \frac{q}{[2]_q[3]_q}, \ \beta_{3,q} = \frac{(1-q)}{[3]_q[4]_q}, \cdots .$$

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In this paper, we consider the modified q-*Bernoulli numbers*, which are slightly different *Carlitz's q-Bernoulli numbers*, as follows :

$$\widetilde{\beta}_{0,q} = 1, \ \left(q\widetilde{\beta}_q + 1\right)^n - \widetilde{\beta}_{n,q} = \begin{cases} \frac{logq}{q-1}, & \text{if } n = 1 \\ 0, & \text{if } n > 1, \end{cases} \tag{1.10}$$

with the usual convention about replacing $\widetilde{\beta}_q^i$ by $\widetilde{\beta}_{i,q}$.

The purpose of our paper is to give some interesting and new identities of the modified $q-$ *Bernoulli numbers* $\widetilde{\beta}_{n,q}$ which are derived from convolutions on the ring of p-adic integers.

2. Some identities of q-Bernoulli numbers

Let us consider the following q-extension of *Bernoulli numbers* :

$$\widetilde{\beta}_{0,q} = 1, \ \left(q\widetilde{\beta}_q + 1\right)^n - \widetilde{\beta}_{n,q} = \begin{cases} \frac{logq}{q-1}, & \text{if } n = 1 \\ 0, & \text{if } n > 1, \end{cases} \tag{2.1}$$

Then, by (1.1), we easily see that

$$\begin{aligned} \widetilde{\beta}_{n,q} &= \int_{\mathbb{Z}_p} [x]_q^n d\mu_0(x) \\ &= \frac{\log q}{(1-q)^{n+1}} \sum_{l=0}^{n} \binom{n}{l} (-1)^{l-1} \frac{l}{[l]_q}, \quad n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}\,. \end{aligned} \tag{2.2}$$

In the equation (1.6) and (1.7), if we take p-adic integral on $\mathbb{Z}_p$ with respect to variable $\mathbb{Z}$, then we have

$$I_0^{(z)}(f * g) = I_0^{(z)} \left(I_0^{(x)}(f(x)g(z-x))\right) - I_0^{(z)} \left(f \circledast g'(z)\right). \tag{2.3}$$

By (1.4) and (2.3), we get

$$I_0^{(z)} \left(f \circledast g'(z)\right) = I_0^{(z)} \left(I_0^{(x)}(f(x)g(z-x))\right) - I_0^{(z)}(f) I_0^{(z)}(g). \tag{2.4}$$

Let us take $f(x) = [x]_{q^{-1}}^m$, $g[x] = [x]_q^n$. Then

$$q'(x) = n\frac{\log q}{q-1}[x]_q^{n-1}, \quad (m, n \in \mathbb{N}).$$

Now, we set

$$A_{m,n}^q = I_0^{(z)} \left([z]_{q^{-1}}^m \circledast [z]_q^{n-1}\right), \quad m, n \in \mathbb{N}. \tag{2.5}$$

From (2.4) and (2.5), we can derive

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$$n\frac{\log q}{q-1}A^q_{m,n-1} = \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}[x]^m_{q^{-1}}[z-x]^n_q d\mu_0(x)d\mu_0(z) - \widetilde{\beta}_{m,q^{-1}}\widetilde{\beta}_{n,q}. \tag{2.6}$$

Note that

$$[z-x]^n_{q^{-1}} = \left([z]_q - q^{-1}q^z[x]_{q^{-1}}\right)^n = \sum_{l=0}^{n}\binom{n}{l}[z]^{n-l}_q(-1)^l q^{-l}q^{lz}[x]^l_{q^{-1}}. \tag{2.7}$$

By (2.6) and (2.7), we get

$$\begin{aligned}
n\frac{\log q}{q-1}A^q_{m,n} =& \sum_{l=1}^{n}\binom{n}{l}(-1)^l q^{-l}\int_{\mathbb{Z}_p}[x]^{m+l}_{q-1}d\mu_0(x)\int_{\mathbb{Z}_p}[z]^{n-l}_q q^{lz}d\mu_0(z)\\
=&\sum_{l=1}^{n}\binom{n}{l}(-1)^k q^{-l}\widetilde{\beta}_{m+l,q^{-1}}\sum_{k=0}^{l}\binom{l}{k}(q-1)^k\int_{\mathbb{Z}_p}[z]^{n+k-l}_q d\mu_0(z)\\
=&\sum_{l=1}^{n}\sum_{k=0}^{l}\binom{n}{l}\binom{l}{k}(-1)^l q^{-l}\widetilde{\beta}_{m+l,q^{-1}}(q-1)^k\widetilde{\beta}_{m+k-l,q}.
\end{aligned} \tag{2.8}$$

Thus, from (2.8), we have

$$A^q_{m,n} = \frac{q-1}{\log q}\frac{1}{n}\sum_{l=1}^{n}\sum_{k=0}^{l}\binom{n}{l}\binom{l}{k}(-1)^l q^{-l}(q-1)^k\widetilde{\beta}_{m+l,q^{-1}}\widetilde{\beta}_{n+k-l,q}. \tag{2.9}$$

By (2.5), we easily see that

$$\begin{aligned}
A^q_{m,n} =& I^{(z)}_0\left([z]^m_{q^{-1}} \circledast [z]^{n-1}_q\right)\\
=& I^{(z)}_0\left([z]^{n-1}_q \circledast [z]^m_{q^{-1}}\right)\\
=& A^q_{n-1,m+1}.
\end{aligned} \tag{2.10}$$

From (2.5), we have

$$\begin{aligned}
\nu_p(A^q_{m,n}) =& \nu_p\left(I_0\left([z]^m_{q^{-1}} \circledast [z]^{n-1}_q\right)\right)\\
\geq& \nu\left([z]^m_{q^{-1}} \circledast [z]^{n-1}_q\right) - 1\\
\geq& \nu([z]^m_{q^{-1}}) + \nu([z]^{n-1}_q) - 1 - 1\\
\geq& -2.
\end{aligned} \tag{2.11}$$

Therefore, by (2.9), (2.10) and (2.11), we obtain the following theorem.

Theorem 2.1. *For $m \in \mathbb{Z}_+$ and $n \in \mathbb{N}$, we have*

$$A_{m,n}^q = \frac{q-1}{\log q}\frac{1}{n}\sum_{l=1}^{n}\sum_{k=0}^{l}\binom{n}{l}\binom{l}{k}(-1)^l q^{-l}(q-1)^k \widetilde{\beta}_{m+l,q^{-1}}\widetilde{\beta}_{n+k-l,q}.$$

Furthermore,

$$A_{m,n}^q = A_{n-1,m+1}^q, \quad \nu_p(A_{m,n}^q) \geq -2.$$

In particular, if we take $m = 0$, then by (2.10), we get

$$A_{0,n}^q = A_{n-1,1}^q \quad (q\text{-analogue of } \textit{Euler identity}). \tag{2.12}$$

From (2.2), we note that

$$\begin{aligned}
\widetilde{\beta}_{n,q} =& \frac{\log q}{(1-q)^{n+1}}\sum_{l=0}^{n}\binom{n}{l}(-1)^{l-1} l \frac{1-q}{1-q^l} \\
=& n\frac{\log q}{(1-q)^n}\sum_{l=1}^{n}\binom{n-1}{l-1}(-1)^{l-1}\sum_{m=0}^{\infty} q^{lm} \\
=& n\frac{\log q}{1-q}\frac{1}{(1-q)^{n-1}}\sum_{l=0}^{n-1}\binom{n-1}{l}(-1)^l\sum_{m=0}^{\infty} q^{(l+1)m} \\
=& n\frac{\log q}{1-q}\sum_{m=0}^{\infty} q^m [m]_q^{n-1},
\end{aligned} \tag{2.13}$$

where $n \in \mathbb{N}$. Thus, by (2.13), we obtain the following theorem.

Theorem 2.2. *For $n \in \mathbb{N}$, we have*

$$-\frac{\widetilde{\beta}_{n,q}}{n} = \frac{\log q}{q-1}\sum_{m=1}^{\infty} q^m [m]_q^{n-1}.$$

Let $F_q(t)$ be the generating function for $\widetilde{\beta}_{n,q}$ with $F_q(t) = \sum_{k=0}^{\infty}\frac{1}{k!}\widetilde{\beta}_{k,q}t^k$. Then, by (2.13), we get

$$\begin{aligned}
F_q(t) =& \sum_{k=0}^{\infty}\frac{\widetilde{\beta}_{k,q}}{k!}t^k \\
=& \frac{\log q}{1-q}\sum_{m=0}^{\infty} q^m \left\{1 + \sum_{k=1}^{\infty}\frac{k}{k!}[m]_q^{k-1}t^k\right\} \\
=& \frac{\log q}{(1-q)^2} + t\frac{\log q}{1-q}\sum_{m=0}^{\infty} q^m e^{[m]_q t}.
\end{aligned} \tag{2.14}$$

Therefore, by (2.14), we obtain the following theorem.

Theorem 2.3. *Let $F_q(t) = \sum_{k=0}^{\infty} \widetilde{\beta}_{k,q} \frac{t^k}{k!}$. Then we have*

$$F_q(t) = \frac{\log q}{(1-q)^2} + t\frac{\log q}{1-q} \sum_{m=0}^{\infty} q^m e^{[m]_q^t}.$$

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HYERS-ULAM STABILITY OF GENERAL JENSEN TYPE MAPPINGS

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ABSTRACT. In this paper, we introduce general Jensen mappings of type I and II, and prove the Hyers-Ulam stability of these mappings.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [16] concerning the stability of group homomorphisms. Hyers [9] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [13] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [8] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. The stability problems for several functional equations or inequations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [2]–[7], [10]–[12], [14, 15]).

Throughout this paper, assume that X is a real normed space, and Y is a real Banach space. Let α and β be positive real numbers.

Definition 1.1. A mapping $f : X \to Y$ is called a *general Jensen mapping of type I* if f satisfies the functional equation

$$f(\alpha x + \beta y) + f(\alpha x - \beta y) = 2\alpha f(x) \tag{1.1}$$

for all $x, y \in X$. We note that (1.1) is equivalent to the equation

$$f(x) + f(y) = 2\alpha f\left(\frac{x+y}{2\alpha}\right)$$

for $x, y \in X$.

Definition 1.2. A mapping $f : X \to Y$ is called a *general Jensen mapping of type II* if f satisfies the functional equation

$$f(\alpha x + \beta y) - f(\alpha x - \beta y) = 2\beta f(y) \tag{1.2}$$

for all $x, y \in X$. We note that (1.2) is equivalent to the equation

$$f(x) - f(y) = 2\beta f\left(\frac{x-y}{2\beta}\right)$$

for $x, y \in X$.

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2. Hyers-Ulam stability of the general Jensen mapping (1.1) of type I

In this section, we prove the Hyers-Ulam stability of the functional equation (1.1) with $\alpha \neq 1$.

Theorem 2.1. *Let $f : X \to Y$ be a mapping for which there exists a function $\varphi : X \times X \to [0, \infty)$ with $\varphi(0,0) = 0$ such that*

$$\widetilde{\varphi}(x,y) := \frac{1}{2}\sum_{j=0}^{\infty} \alpha^j \varphi\left(\frac{x}{\alpha^{j+1}}, \frac{y}{\alpha^{j+1}}\right) < \infty,$$

$$\|f(\alpha x + \beta y) + f(\alpha x - \beta y) - 2\alpha f(x)\| \leq \varphi(x,y) \tag{2.1}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \widetilde{\varphi}(x,0) \tag{2.2}$$

for all $x \in X$.

Proof. Letting $x = y = 0$ in (2.1), we get

$$\|(2-2\alpha)f(0)\| \leq \varphi(0,0).$$

So $f(0) = 0$.

Letting $y = 0$ in (2.1), we get

$$\left\|f(x) - \alpha f\left(\frac{x}{a}\right)\right\| \leq \frac{1}{2}\varphi\left(\frac{x}{\alpha}, 0\right)$$

for all $x \in X$.

Hence one may have the following formula for positive integers m, l with $m > l$,

$$\left\|\alpha^l f\left(\frac{x}{\alpha^l}\right) - \alpha^m f\left(\frac{x}{\alpha^m}\right)\right\| \leq \frac{1}{2}\sum_{i=l}^{m-1} \alpha^i \varphi\left(\frac{x}{\alpha^{i+1}}, 0\right). \tag{2.3}$$

It follows from (2.3) that the sequence $\left\{\alpha^k f\left(\frac{x}{\alpha^k}\right)\right\}$ is a Cauchy sequence for all $x \in X$. Since Y is a real Banach space, the sequence $\left\{\alpha^k f\left(\frac{x}{\alpha^k}\right)\right\}$ converges. So one may define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty} \alpha^k f\left(\frac{x}{\alpha^k}\right), \quad \forall x \in X.$$

Next, we show that $A(x)$ is an additive mapping. Letting $x = \alpha x + \beta y$ and $y = \alpha x - \beta y$ in (2.1) , we get

$$\left\|f(x) + f(y) - 2\alpha f\left(\frac{x+y}{2\alpha}\right)\right\| \leq \varphi(\alpha x + \beta y, \alpha x - \beta y)$$

for all $x, y \in X$.

Replacing y by $\frac{\alpha}{\beta}x$ in (2.1), we get

$$\|f(2\alpha x) - 2\alpha f(x)\| \leq \varphi\left(x, \frac{\alpha}{\beta}x\right)$$

and so

$$\left\|f(x) - 2\alpha f\left(\frac{x}{2\alpha}\right)\right\| \leq \varphi\left(\frac{x}{2\alpha}, \frac{x}{2\beta}\right)$$

for all $x \in X$.

$$\begin{aligned}
\|A(x)+A(y)-A(x+y)\| &= \lim_{k\to\infty}\left\|\alpha^k f\left(\frac{x}{\alpha^k}\right)+\alpha^k f\left(\frac{y}{\alpha^k}\right)-\alpha^k f\left(\frac{x+y}{\alpha^k}\right)\right\| \\
&\le \lim_{k\to\infty}\alpha^k\left\|f\left(\frac{x}{\alpha^k}\right)+f\left(\frac{y}{\alpha^k}\right)-2\alpha f\left(\frac{x+y}{2\alpha^{k+1}}\right)\right\| \\
&+\lim_{k\to\infty}\alpha^k\left\|f\left(\frac{x+y}{\alpha^k}\right)-2\alpha f\left(\frac{x+y}{2\alpha^{k+1}}\right)\right\| \\
&\le \lim_{k\to\infty}\alpha^k\varphi\left(\alpha\frac{x}{\alpha^k}+\beta\frac{y}{\alpha^k},\alpha\frac{x}{\alpha^k}-\beta\frac{y}{\alpha^k}\right) \\
&+\lim_{k\to\infty}\alpha^k\varphi\left(\frac{1}{2\alpha}\frac{x+y}{\alpha^k},\frac{1}{2\beta}\frac{x+y}{\alpha^k}\right)=0
\end{aligned}$$

for all $x, y \in X$. Therefore, the mapping $A : X \to Y$ is additive.

Now, we show that the uniqueness of A. Let $T : X \to Y$ be another additive mapping satisfying (2.2). Then one have

$$\begin{aligned}
\|A(x)-T(x)\| &= \left\|\alpha^k A\left(\frac{x}{\alpha^k}\right)-\alpha^k T\left(\frac{x}{\alpha^k}\right)\right\| \\
&\le \alpha^k\left(\left\|A\left(\frac{x}{\alpha^k}\right)-f\left(\frac{x}{\alpha^k}\right)\right\|+\left\|T\left(\frac{x}{\alpha^k}\right)-f\left(\frac{x}{\alpha^k}\right)\right\|\right) \\
&\le 2\alpha^k\widetilde{\varphi}\left(\frac{x}{\alpha^k},0\right),
\end{aligned}$$

which tends to zero as $k \to \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. This completes the proof. $\square$

Corollary 2.2. *Let p, θ and α be positive real numbers with $p > 1$ and $\alpha > 1$ or $p < 1$ and $\alpha < 1$. Let $f : X \to Y$ be a mapping satisfying*

$$\|f(\alpha x+\beta y)+f(\alpha x-\beta y)-2\alpha f(x)\| \le \theta(\|x\|^p+\|y\|^p)$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x)-A(x)\| \le \frac{\|x\|^p}{2(\alpha^p-\alpha)}$$

for all $x \in X$.

3. Hyers-Ulam stability of the general Jensen mapping (1.2) of type II

In this section, we prove the Hyers-Ulam stability of the functional equation (1.2) with $\beta \neq \frac{1}{2}$.

Theorem 3.1. *Let $f : X \to Y$ be a mapping for which there exists a function $\varphi : X\times X \to [0,\infty)$ with $\varphi(0,0) = 0$ such that*

$$\widetilde{\varphi}(x,y) := \sum_{j=0}^{\infty}(2\beta)^j\varphi\left(\frac{x}{(2\beta)^{j+1}},\frac{y}{(2\beta)^{j+1}}\right)<\infty,$$

$$\|f(\alpha x+\beta y)-f(\alpha x-\beta y)-2\beta f(y)\| \le \varphi(x,y) \tag{3.1}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \widetilde{\varphi}\left(\frac{2\alpha x}{2\beta}, x\right) \tag{3.2}$$

for all $x \in X$.

Proof. Letting $x = y = 0$ in (3.1), we get

$$\|2\beta f(0)\| \le \varphi(0,0).$$

So $f(0) = 0$.

Replacing x and y by $\frac{\beta}{\alpha}x$ and x in (3.1), respectively, we get

$$\|f(x) - 2\beta f(\frac{x}{2\beta})\| \le \varphi(\frac{x}{2\alpha}, \frac{x}{2\beta})$$

for all $x \in X$.

Hence one may have the following formula for positive integers m, l with $m > l$,

$$\left\|(2\beta)^l f\left(\frac{x}{(2\beta)^l}\right) - (2\beta)^m f\left(\frac{x}{(2\beta)^m}\right)\right\| \le \sum_{i=l}^{m-1} (2\beta)^i \varphi\left(\frac{x}{2\alpha(2\beta)^i}, \frac{x}{(2\beta)^{i+1}}\right). \tag{3.3}$$

It follows from (3.3) that the sequence $\left\{(2\beta)^k f\left(\frac{x}{(2\beta)^k}\right)\right\}$ is a Cauchy sequence for all $x \in X$. Since Y is a real Banach space, the sequence $\left\{(2\beta)^k f\left(\frac{x}{(2\beta)^k}\right)\right\}$ converges. So one may define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty} (2\beta)^k f\left(\frac{x}{(2\beta)^k}\right), \quad \forall x \in X.$$

Moreover, by letting $l = 0$ and passing the limit $m \to \infty$, we get (3.2).

Next, we show that $A(x)$ is an additive mapping.

Replacing x and y by $\alpha x + \beta y$ and $\alpha x - \beta y$ in (3.1), respectively, we get

$$\left\|f(x) - f(y) - 2\beta f\left(\frac{x-y}{2\beta}\right)\right\| \le \varphi(\alpha x + \beta y, \alpha x - \beta y)$$

for all $x, y \in X$.

Replacing x and y by $\frac{x}{\alpha}$ and $\frac{y}{\beta}$ in (3.1), respectively, we get

$$\left\|f(x+y) - f(x-y) - 2\beta f\left(\frac{y}{\beta}\right)\right\| \le \varphi\left(\frac{x}{\alpha}, \frac{y}{\beta}\right)$$

for all $x, y \in X$.

Replacing x by $\frac{\beta y}{\alpha}$ in (3.1), we get

$$\left\|f(y) - 2\beta f\left(\frac{y}{2\beta}\right)\right\| \le \varphi\left(\frac{y}{2\alpha}, \frac{y}{2\beta}\right) \tag{3.4}$$

for all $y \in X$.

Letting $x = 0$ in (3.1), we get

$$\left\|f(y) - f(-y) - 2\beta f\left(\frac{y}{\beta}\right)\right\| \le \varphi\left(0, \frac{y}{\beta}\right)$$

for all $y \in X$.

Replacing x by $-\frac{\beta y}{\alpha}$ in (3.1), we get

$$\left\| f(-y) + 2\beta f\left(\frac{y}{2\beta}\right) \right\| \leq \varphi\left(-\frac{y}{2\alpha}, \frac{y}{2\beta}\right) \tag{3.5}$$

for all $y \in X$. By (3.4) and (3.5), we have

$$\|f(-y) + f(y)\| \leq \varphi\left(\frac{y}{2\alpha}, \frac{y}{2\beta}\right) + \varphi\left(-\frac{y}{2\alpha}, \frac{y}{2\beta}\right)$$

for all $y \in X$. So

$$\begin{aligned}
&\|A(x+y) - A(x) - A(y)\| \\
&= \lim_{k\to\infty} \left\| (2\beta)^k f\left(\frac{x+y}{(2\beta)^k}\right) - (2\beta)^k f\left(\frac{x}{(2\beta)^k}\right) - (2\beta)^k f\left(\frac{y}{(2\beta)^k}\right) \right\| \\
&\leq \lim_{k\to\infty} (2\beta)^k \left(\left\| f\left(\frac{x+y}{(2\beta)^k}\right) - f\left(\frac{x-y}{(2\beta)^k}\right) - 2f\left(\frac{y}{(2\beta)^k}\right) \right\| \right. \\
&\left. + \left\| f\left(\frac{x-y}{(2\beta)^k}\right) - f\left(\frac{x}{(2\beta)^k}\right) + f\left(\frac{y}{(2\beta)^k}\right) \right\| \right) \\
&\leq \lim_{k\to\infty} (2\beta)^k \left(\left\| f\left(\frac{x+y}{(2\beta)^k}\right) - f\left(\frac{x-y}{(2\beta)^k}\right) - 2\beta f\left(\frac{y}{\beta(2\beta)^k}\right) \right\| \right. \\
&+ \left\| 2\beta f\left(\frac{y}{\beta(2\beta)^k}\right) - 2f\left(\frac{y}{(2\beta)^k}\right) \right\| + \left\| 2\beta f\left(\frac{x-y}{(2\beta)^{k+1}}\right) - f\left(\frac{x}{(2\beta)^k}\right) + f\left(\frac{y}{(2\beta)^k}\right) \right\| \\
&\left. + \left\| 2\beta f\left(\frac{x-y}{(2\beta)^{k+1}}\right) - f\left(\frac{x-y}{(2\beta)^k}\right) \right\| \right) \\
&\leq \lim_{k\to\infty} (2\beta)^k \left(\left\| f\left(\frac{x+y}{(2\beta)^k}\right) - f\left(\frac{x-y}{(2\beta)^k}\right) - 2\beta f\left(\frac{y}{\beta(2\beta)^k}\right) \right\| \right. \\
&+ \left\| 2\beta f\left(\frac{y}{\beta(2\beta)^k}\right) - f\left(\frac{y}{(2\beta)^k}\right) + f\left(-\frac{y}{(2\beta)^k}\right) \right\| + \left\| -f\left(\frac{-y}{(2\beta)^k}\right) - f\left(\frac{y}{(2\beta)^k}\right) \right\| \\
&\left. + \left\| 2\beta f\left(\frac{x-y}{(2\beta)^{k+1}}\right) - f\left(\frac{x}{(2\beta)^k}\right) + f\left(\frac{y}{(2\beta)^k}\right) \right\| + \left\| 2\beta f\left(\frac{x-y}{(2\beta)^{k+1}}\right) - f\left(\frac{x-y}{(2\beta)^k}\right) \right\| \right) \\
&\leq \lim_{k\to\infty} (2\beta)^k \left(\varphi\left(\frac{x}{\alpha(2\beta)^k}, \frac{y}{\beta(2\beta)^k}\right) + \varphi\left(0, \frac{y}{\beta(2\beta)^k}\right) + \varphi\left(\frac{y}{2\alpha(2\beta)^k}, \frac{y}{(2\beta)^{k+1}}\right) \right. \\
&+ \varphi\left(-\frac{y}{2\alpha(2\beta)^k}, \frac{y}{(2\beta)^{k+1}}\right) + \varphi\left(\alpha\frac{x}{(2\beta)^k} + \beta\frac{y}{(2\beta)^k}, \alpha\frac{x}{(2\beta)^k} - \beta\frac{y}{(2\beta)^k}\right) \\
&\left. + \varphi\left(\frac{x-y}{2\alpha(2\beta)^k}, \frac{x-y}{(2\beta)^{k+1}}\right) \right) = 0
\end{aligned}$$

for all $x, y \in X$. Therefore, the mapping $A : X \to Y$ is additive.

Now, we show that the uniqueness of A. Let $T : X \to Y$ be another additive mapping satisfying (3.2). Then one have

$$\begin{aligned}\|A(x)-T(x)\| &= \left\|(2\beta)^k A\left(\frac{x}{(2\beta)^k}\right)-(2\beta)^k T\left(\frac{x}{(2\beta)^k}\right)\right\| \\ &\le (2\beta)^k\left(\left\|A\left(\frac{x}{(2\beta)^k}\right)-f\left(\frac{x}{(2\beta)^k}\right)\right\|+\left\|T\left(\frac{x}{(2\beta)^k}\right)-f\left(\frac{x}{(2\beta)^k}\right)\right\|\right) \\ &\le 2(2\beta)^k\widetilde{\varphi}\left(\frac{\frac{2\alpha}{2\beta}x}{(2\beta)^k},\frac{x}{(2\beta)^k}\right),\end{aligned}$$

which tends to zero as $k \to \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. This completes the proof. □

Corollary 3.2. *Let p, θ and β be positive real numbers with $p > 1$ and $\beta > \frac{1}{2}$ or $p < 1$ and $\beta < \frac{1}{2}$. Let $f : X \to Y$ be a mapping satisfying*

$$\|f(\alpha x+\beta y)-f(\alpha x-\beta y)-2\beta f(y)\| \le \theta(\|x\|^p+\|y\|^p)$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x)-A(x)\| \le \frac{(2\beta)^p+(2\alpha)^p}{2\beta((2\beta)^p-2\beta)}\theta\|x\|^p$$

for all $x \in X$.

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Identities involving r-stirling numbers

by
Dae San Kim and Taekyun Kim

Abstract

In this paper, we investigate some interesting identities involving r-stirling numbers which are derived from the transfer formula for the associated sequences.

1 Introduction

Let $\mathcal{F}$ be the set of all formal power series in the variable t over $\mathbf{C}$ with

$$\mathcal{F} = \left\{ f(t) := \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \middle| a_k \in \mathbf{C} \right\}. \tag{1}$$

and let $\mathbb{P}$ be the algebra of polynomials in the variable x over $\mathbf{C}$ and $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. $\langle L|p(x)\rangle$ denotes the action of the linear functional L on a polynomial $p(x)$. For $f(t) \in \mathcal{F}$, let us define the continuous linear functional $f(t)$ on $\mathbb{P}$ by

$$\langle f(t)|x^n\rangle = a_n,\ (n \geq 0),\ \ (\text{see } [2,3,4,5]). \tag{2}$$

Thus, by (1) and (2), we get

$$\langle t^k|x^n\rangle = n!\delta_{n,k},\ (n,k \geq 0),\ \ (\text{see } [4,5]), \tag{3}$$

where $\delta_{n,k}$ is the Kronecker's symbol.
Let $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L|x^k\rangle}{k!} t^k$. Then, by (1), (2) and (3), we easily see that $\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle$. The map $L \longmapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ is thought of as both a formal power series and a linear functional (see [3,5]). We call $\mathcal{F}$ the umbral algebra. The umbral calculus is the study of umbral algebra. The order $O(f(t))$ of the non-zero

power series $f(t)$ is the smallest integer k for which the coefficient of t^k does not vanish (see [2, 3, 4]). If $O(f(t)) = 1$, then $f(t)$ is called a delta series and if $O(f(t)) = 0$, then $f(t)$ is called an invertible series. Let $O(f(t)) = 1$ and $O(g(t)) = 0$. Then there exists a unique sequence $s_n(x)$ of polynomials such that $\langle g(t)f(t)^k | s_n(x)\rangle = n!\delta_{n,k}$, where $n, k \geq 0$ (see [3, 4, 5]). The sequence $s_n(x)$ is called the Sheffer sequence for $(g(t), f(t))$, which is denoted by $s_n(x) \sim (g(t), f(t))$. If $s_n(x) \sim (1, f(t))$, then $s_n(x)$ is called the associated sequence for $f(t)$. Let $f(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$. Then we have

$$f(t) = \sum_{k=0}^{\infty} \frac{\langle f(t)|x^k\rangle}{k!} t^k, \quad p(x) = \sum_{k=0}^{\infty} \frac{\langle t^k|p(x)\rangle}{k!} x^k, \quad \text{(see [4, 5])}. \tag{4}$$

From (4), we have

$$p^{(k)}(0) = \langle t^k|p(x)\rangle, \quad \langle 1|p^{(k)}(x)\rangle = p^{(k)}(0). \tag{5}$$

By (5), we easily get

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k}, \ (k \geq 0), \quad \text{(see [3, 4, 5])}. \tag{6}$$

Let $p_n(x) \sim (1, f(t))$, $q_n(x) \sim (1, g(t))$. Then the transfer formula for the associated sequences is given by

$$q_n(x) = x \left(\frac{f(t)}{g(t)}\right)^n x^{-1} p_n(x). \tag{7}$$

Broder has worked out the generalization of Stirling numbers to the so-called r-stirling numbers (see [1]).
In this paper, we investigate some interesting identities involving r-stirling numbers which are derived from the transfer formula for the associated sequences.

2 Identities involving r-stirling numbers

In this section, we assume that r is natural number. In [1], the identity involving r-stirling numbers of the first kind is given by

$$m! \sum_{k=0}^{\infty} \left[\begin{array}{c} k+m+r \\ m+r \end{array} \right]_r \frac{t^k}{(k+m)!} = \frac{t\left(-\frac{\log(1-t)}{t}\right)^m}{t(1-t)^r}, \tag{8}$$

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where $\begin{bmatrix} n \\ m \end{bmatrix}_r$ is the r-stirling number of the first kind.
Let $q_n(x) \sim (1, t(1-t)^r)$. For $n \geq 1$, from (7) and $x^n \sim (1,t)$, we have

$$q_n(x) = \sum_{k=1}^{n} \binom{rn+n-k-1}{n-k} (n-1)_{n-k} x^k, \tag{9}$$

where $(x)_n = x(x-1)\cdots(x-n+1)$.
Let us assume that

$$p_n(x) \sim \left(1, t\left(-\frac{\log(1-t)}{t}\right)^m\right). \tag{10}$$

For $n \geq 1$, by (8), (9) and $x^n \sim (1,t)$, we get

$$p_n(x) = x \left(\frac{-t}{\log(1-t)}\right)^{mn} x^{n-1}. \tag{11}$$

As is well known, we note that

$$\left(\frac{t}{\log(1+t)}\right)^n = \sum_{l=0}^{n} B_l^{(l-n+1)}(1) \frac{t^l}{l!}, \tag{12}$$

where $B_l^{(n)}(x)$ is the l-th Bernoulli polynomial of order n.
From (11) and (12), we can derive the following identity:

$$p_n(x) = \sum_{k=1}^{n} (-1)^{n-k} \binom{n-1}{k-1} B_{n-k}^{(n(1-m)-k+1)}(1) x^k. \tag{13}$$

For $n \geq 1$, by (7) and (10), we get

$$\begin{aligned} q_n(x) = (m!)^n \sum_{k=1}^{n} \Bigg\{ \sum_{l=k}^{n} \sum_{j_1+\cdots+j_n=l-k} (-1)^{n-l} \binom{n-1}{l-1} (l-1)_{l-k} B_{n-l}^{((1-m)n-l+1)}(1) \\ \times \left(\prod_{i=1}^{n} \begin{bmatrix} j_i+m+r \\ m+r \end{bmatrix}_r \frac{1}{(j_i+m)!} \right) \Bigg\} x^k. \end{aligned} \tag{14}$$

Therefore, by (9) and (14), we obtain the following theorem.

3

Theorem 1. *For $n \geq 1$, $1 \leq k \leq n$, we have*

$$\binom{(r+1)n-k-1}{n-k}(n-1)_{n-k} = (m!)^n \sum_{l=k}^{n} \sum_{j_1+\cdots+j_n=l-k} (-1)^{n-l}\binom{n-1}{l-1}$$
$$\times (l-1)_{l-k} B_{n-l}^{((1-m)n-l+1)}(1) \left(\prod_{i=1}^{n} \begin{bmatrix} j_i+m+r \\ m+r \end{bmatrix}_r \frac{1}{(j_i+m)!} \right).$$

Remark. By the same method of Theorem 1, we get

$$\begin{aligned}
&\binom{(r+1)n-k-1}{n-k}(n-1)_{n-k} \\
&= (m!)^n \sum_{l=k}^{n} \sum_{j_1+\cdots+j_n=l-k} (-1)^{n-l}\binom{n-1}{l-1}(l-1)_{n-k} N_{n-l}^{(-mn)} \\
&\quad \times \left(\prod_{i=1}^{n} \begin{bmatrix} j_i+m+r \\ m+r \end{bmatrix}_r \frac{1}{(j_i+m)!} \right),
\end{aligned} \tag{15}$$

where $N_l^{(-n)}$ is Narumi number of order $-n$.
It is known that

$$m! \sum_{k=0}^{\infty} \left\{ \begin{matrix} k+m+r \\ m+r \end{matrix} \right\}_r \frac{t^k}{(k+m)!} = \frac{t\left(\frac{e^t-1}{t}\right)^m}{te^{-rt}}, \quad \text{(see [1])}, \tag{16}$$

where $\left\{ \begin{matrix} n \\ m \end{matrix} \right\}_r$ is the r-stirling number of the second kind.
Let

$$q_n(x) \sim \left(1, te^{-rt}\right), \quad p_n(x) \sim \left(1, t\left(\frac{e^t-1}{t}\right)^m\right). \tag{17}$$

From (7), (17) and $x^n \sim (1,t)$, we can derive the following equations:

$$q_n(x) = x(x+rn)^{n-1}, \quad p_n(x) = xB_{n-1}^{(mn)}(x). \tag{18}$$

By (7) and (17), we can also see that

$$\begin{aligned}
q_n(x) = &(m!)^n \sum_{k=1}^{n-1} \sum_{j_1+\cdots+j_n=n-1-k} (n-1)_{n-1-k} \\
&\times \left(\prod_{i=1}^{n} \left\{ \begin{matrix} j_i+m+r \\ m+r \end{matrix} \right\}_r \frac{1}{(j_i+m)!} \right) xB_k^{(mn)}(x).
\end{aligned} \tag{19}$$

Therefore, by (18) and (19), we obtain the following theorem.

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Theorem 2. *For $n \geq 1$, we have*

$$x(x+rn)^{n-1} = (m!)^n \sum_{k=0}^{n-1} \sum_{j_1+\cdots+j_n=n-1-k} (n-1)_{n-1-k} \times \left(\prod_{i=1}^{n} \left\{ \begin{matrix} j_i+m+r \\ m+r \end{matrix} \right\}_r \frac{1}{(j_i+m)!} \right) x B_k^{(mn)}(x).$$

In [1], it is known that

$$\sum_{k,m=0}^{\infty} \left\{ \begin{matrix} k+r \\ m+r \end{matrix} \right\}_r \frac{t^k}{k!} a^m = \exp\left(a\left(e^t-1\right)+rt\right), \tag{20}$$

where a is constatant.
Thus, by (20), we get

$$\sum_{k,m=0}^{\infty} \left\{ \begin{matrix} k+r \\ m+r \end{matrix} \right\}_r \frac{t^k}{k!} a^m = \frac{t \exp\left(a\left(e^t-1\right)\right)}{te^{-rt}}. \tag{21}$$

Let

$$q_n(x) \sim \left(1, te^{-rt}\right), \quad p_n(x) \sim \left(1, t\exp\left(a\left(e^t-1\right)\right)\right). \tag{22}$$

For $n \geq 1$, from (7), (22) and $x^n \sim (1,t)$, we have

$$q_n(x) = x(x+rn)^{n-1}, \quad p_n(x) = \sum_{k=1}^{n} \binom{n-1}{k-1} \phi_{n-k}(-an)x^k, \tag{23}$$

where $\phi_n(x)$ is the exponential polynomial.
By (7), (22) and (23), we get

$$q_n(x) = \sum_{k=1}^{n} \left\{ \sum_{l=k}^{n} \sum_{m=0}^{\infty} \sum_{\substack{j_1+\cdots+j_n=l-k \\ m_1+\cdots+m_n=m}} \binom{n-1}{l-1} (l-1)_{l-k} a^m \phi_{n-l}(-an) \times \left(\prod_{i=1}^{n} \left\{ \begin{matrix} j_i+r \\ m_i+r \end{matrix} \right\}_r \frac{1}{j_i!} \right) \right\} x^k. \tag{24}$$

Therefore, by (23) and (24), we obtain the following theorem.

Theorem 3. *For $n \geq 1$, $1 \leq k \leq n$, we have*

$$\binom{n-1}{k-1}(rn)^{n-k} = \sum_{l=k}^{n} \sum_{m=0}^{\infty} \sum_{\substack{j_1+\cdots+j_n=l-k \\ m_1+\cdots+m_n=m}} \binom{n-1}{l-1}(l-1)_{l-k} a^m \phi_{n-l}(-an) \times \left(\prod_{i=1}^{n} \left\{ \begin{matrix} j_i+r \\ m_i+r \end{matrix} \right\}_r \frac{1}{j_i!} \right)$$

It is known that

$$\sum_{k,m=0}^{\infty} \left[\begin{matrix} k+r \\ m+r \end{matrix} \right]_r \frac{t^k}{k!} a^m = \frac{t}{t(1-t)^{r+a}}, \quad \text{where } a \in \mathbf{Z}_{\geq 0}, \quad (\text{see [1]}). \tag{25}$$

Let

$$q_n(x) \sim \left(1, t(1-t)^{r+a}\right), \quad x^n \sim (1,t). \tag{26}$$

For $n \geq 1$, by (7) and (26), we get

$$\begin{aligned} q_n(x) &= x(1-t)^{-(r+a)n} x^{n-1} = x \sum_{l=0}^{\infty} \binom{(r+a)n+l-1}{l} t^l x^{n-1} \\ &= \sum_{k=1}^{n} \binom{(r+a+1)n-k-1}{n-k} (n-1)_{n-k} x^k. \end{aligned} \tag{27}$$

From (7), (25) and (26), we can also derive

$$\begin{aligned} q_n(x) &= x \left(\frac{t}{t(1-t)^{r+a}} \right)^n x^{n-1} = x \left(\sum_{j,m=0}^{\infty} \left[\begin{matrix} j+r \\ m+r \end{matrix} \right]_r \frac{t^j}{j!} a^m \right)^n x^{n-1} \\ &= \sum_{k=1}^{n} \sum_{m=0}^{\infty} \sum_{\substack{j_1+\cdots+j_n=n-k \\ m_1+\cdots+m_n=m}} (n-1)_{n-k} a^m \left(\prod_{i=1}^{n} \left[\begin{matrix} j_i+r \\ m_i+r \end{matrix} \right]_r \frac{1}{j_i!} \right) x^k. \end{aligned} \tag{28}$$

Therefore, by (27) and (28), we obtain the following theorem.

Theorem 4. *For $n \geq 1$, $1 \leq k \leq n$, we have*

$$\begin{aligned} &\binom{(r+a+1)n-k-1}{n-k} (n-1)_{n-k} \\ &= \sum_{m=0}^{n(n-k)} \sum_{\substack{j_1+\cdots+j_n=n-k \\ m_1+\cdots+m_n=m}} (n-1)_{n-k} a^m \left(\prod_{i=1}^{n} \left[\begin{matrix} j_i+r \\ m_i+r \end{matrix} \right]_r \frac{1}{j_i!} \right). \end{aligned}$$

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Ternary Jordan C^*-homomorphisms and ternary Jordan C^*-derivations for a generalized Cauchy–Jensen functional equation

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Abstract. In this paper, we prove the Hyers-Ulam stability of ternary Jordan C^*-homomorphisms and ternary Jordan C^*-derivations associated with the following generalized Cauchy-Jensen functional equation:

$$\sum_{i=1}^{p} f\left(\frac{1}{k}\sum_{\substack{j=1\\ j\neq i}}^{p} x_j + x_i\right) = \frac{p+k-1}{k}\sum_{i=1}^{p} f(x_i) \qquad (\,*\,)$$

by proving the generalization of Ǧavruţa's theorem.

Keywords: Hyers-Ulam stability; C^*-ternary algebra; Cauchy-Jensen functional equation; Ternary Jordan C^*-homomorphism; Ternary Jordan C^*-derivation.

2010 MSC: 39B52, 17A40, 47B47, 17B40.

1. Introduction and preliminaries

Consider the following Cauchy-Jensen functional equation:

$$f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x+z}{2}+y\right)+f\left(\frac{y+z}{2}+x\right)=2\left[f(x)+f(y)+f(z)\right].$$

A generalized form of the Cauchy-Jensen functional equation is

$$f\left(\frac{x+y}{k}+z\right)+f\left(\frac{x+z}{k}+y\right)+f\left(\frac{y+z}{k}+x\right)=\frac{k+2}{k}\left[f(x)+f(y)+f(z)\right] \qquad (1.1)$$

with $k \in \mathbb{N}$. We can generalize the functional equation (1.1) again. It is clear that if we put $p = 3$, $x_1 = x$, $x_2 = y$ and $x_3 = z$ in the functional equation ($*$), then we lead to (1.1). In order to investigate of the functional equation ($*$), we will suppose that $p \geq 2$ and $k \in \mathbb{N}$, and so the simplest case of ($*$) is Cauchy's additive equation; ($p = 2$, $k = 1$).

We say a functional equation (ξ) is *stable* if any function g satisfying the equation (ξ) "approximately" is near to a true solution of that. The stability of functional equations was first originated by Ulam [28] in 1940. A classical question in this theory, was that "when does an exact solution of functional equation (ξ), near an approximately solution of that exist?" More precisely, Ulam proposed the following problem: Let G_1 be a group and (G_2, d) be a metric group. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $f : G_1 \to G_2$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $T : G_1 \to G_2$ with $d(f(x), T(x)) < \varepsilon$ for any $x \in G_1$?

A year later, Hyers [11] affirmatively answered to this question of Ulam for the case where G_1 and G_2 are Banach spaces. In 1950, Aoki [1] generalized the Hyers' theorem for approximately additive mappings. In 1978, Rassias [23] generalized the theorem of Hyers by considering the stability problem with unbounded Cauchy differences.

Theorem 1.1. [23] *Let E, E' be two Banach spaces, and let $f : E \to E'$ be a mapping from E into E', subject to the inequality*

$$\| f(x+y) - f(x) - f(y) \| \leq \theta\,(\| x \|^p + \| y \|^p)$$

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for all $x, y \in E$, where θ and p are constants with $\theta > 0$ and $0 \leq p < 1$. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$$

exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping that satisfies

$$\| f(x) - L(x) \| \leq \frac{2\theta}{2-2^p} \| x \|^p$$

for all $x \in$. If $p < 0$, then the top inequalities are true for $x, y \neq 0$. In addition, if for each $x \in E$ the mapping $f(tx)$ is continuous in $t \in \mathbb{R}$, then L is $\mathbb{R}$-linear.

In 1991, Gajda [9] showed that Rassias' theorem is also true for $p > 1$. Rassias and Šemrl [25] proved that the similar result does not hold for the case $p = 1$. In 1994, a generalization of Rassias' theorem was obtained by Ĝavruţa [10], who replaced the factors $\| x \|^p + \| y \|^p$ by a general control function $\varphi(x, y)$.

Theorem 1.2. [10] *Let G be an abelian group and E a Banach space. Denote by $\varphi : G \times G \to [0, \infty)$ a mapping such that*

$$\phi(x, y) := \sum_{n=0}^{\infty} 2^{-(n+1)} \varphi(2^n x, 2^n y) < \infty$$

for all $x, y \in G$. Suppose $f : G \to E$ is a mapping satisfying

$$\| f(x + y) - f(x) - f(y) \| \leq \varphi(x, y)$$

for all $x, y \in G$. Then there exists a unique additive mapping $A : G \to E$ such that

$$\| f(x) - A(x) \| \leq \phi(x, x)$$

for all $x \in G$.

The stability problems of several functional equations have been extensively investigated by a number of authors, and there are many interesting results concerning this problem. A list of references concerning this results can be found in [3, 4, 5, 6, 7, 8, 12, 16, 17, 18, 19, 20, 22, 24, 26, 27].

A C^*-ternary algebra is a complex Banach space A, equipped with a ternary product, $(x, y, z) \longmapsto [x, y, z]$ of A^3 into A, which is $\mathbb{C}$-linear in the outer variables, conjugate $\mathbb{C}$-linear in the middle variable, and associative in the sense that $[x, y, [z, u, v]] = [x, [u, z, y], v] = [[x, y, z]u, v]$, and satisfies $\| [x, y, z] \| \leq \| x \| . \| y \| . \| z \|$ and $\| [x, x, x] \| = \| x \|^3$. If a C^*-ternary algebra $(A, [., ., .])$ has an identity, i.e., an element $e \in A$ such that $x = [x, e, e] = [e, e, x]$ for all $x \in A$, then it is routine to verify that A, endowed with $x \circ y := [x, e, y]$ and $x^* = [e, x, e]$, is a unital C^*-algebra. Conversely, if $(A, \circ)$ is a unital C^*-algebra, then $[x, y, z] := x \circ y^* \circ z$ makes A into a C^*-ternary algebra [13, 14, 15, 20, 29].

A $\mathbb{C}$-linear mapping $H : A \to B$ is called a C^*-ternary homomorphism if

$$H([x, y, z]) = [H(x), H(y), H(z)]$$

for all $x, y, z \in A$. If, in addition,

$$H(x^*) = H(x)^* \tag{1.2}$$

for all $x \in A$, then H is called a ternary C^*-homomorphism. A $\mathbb{C}$-linear mapping $H : A \to B$ is called a C^*-ternary Jordan homomorphism if

$$H([x, x, x]) = [H(x), H(x), H(x)]$$

for all $x \in A$. If, in addition, H satisfies (1.2), then H is called a ternary Jordan C^*-homomorphism.

A $\mathbb{C}$-linear mapping $\delta : A \to A$ is called a C^*-ternary derivation if

$$\delta([x, y, z]) = [\delta(x), y, z] + [x, \delta(y), z] + [x, y, \delta(z)]$$

for all $x, y, z \in A$. If, in addition,

$$\delta(x^*) = \delta(x)^* \tag{1.3}$$

for all $x \in A$, then δ is called a ternary C^*-derivation. A $\mathbb{C}$-linear mapping $\delta : A \to A$ is called a C^*-ternary Jordan derivation if

$$\delta([x,x,x]) = [\delta(x),x,x] + [x,\delta(x),x] + [x,x,\delta(x)]$$

for all $x \in A$. If, in addition, δ satisfies (1.3), then δ is called a ternary Jordan C^*-derivation [13, 14, 15, 16, 20].

2. Hyers-Ulam stability of ternary Jordan C^*-homomorphisms

Throughout this section, assume that A and B are C^*-ternary algebras with norms $\|\cdot\|_A$ and $\|\cdot\|_B$, respectively.

We prove the Hyers-Ulam stability of ternary Jordan C^*-homomorphisms associated with functional equation ($*$), by proving the generalization of Găvruţa's theorem.

For a given mapping $f : A \to B$, we define

$$\begin{aligned} \Gamma_\mu f(x_1,\cdots,x_p) &:= \sum_{i=1}^{p} f\left(\frac{1}{k}\sum_{\substack{j=1\\ j\neq i}}^{p} \mu x_j + \mu x_i\right) - \frac{p+k-1}{k}\sum_{i=1}^{p}\mu f(x_i) , \\ \Gamma^h f(x,y) &:= f([x,x,x]) - [f(x),f(x),f(x)] + f(y^*) - (f(y))^* \end{aligned}$$

for all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and all $x, y, x_1, \cdots, x_p \in A$, and also for any $f : A \to A$

$$\Gamma^d f(x,y) := f([x,x,x]) - [f(x),x,x] - [x,f(x),x] - [x,x,f(x)] + f(y^*) - (f(y))^*$$

for all $x, y \in A$.

One can easily show that a mapping $f : A \to B$ satisfies

$$\Gamma_\mu f(x_1,\cdots,x_p) = 0$$

for all $\mu \in \mathbb{T}^1$ and all $x_1,\cdots,x_p \in A$ if and only if

$$f(\mu x + \lambda y) = \mu\ f(x) + \lambda\ f(y)$$

for all $\mu, \lambda \in \mathbb{T}^1$ and all $x, y \in A$.

We will use the following lemmas in the proof of our theorems.

Lemma 2.1. [2] *Let X and Y be linear spaces. If $f : X \to Y$ is an additive mapping, then*

$$f(rx) = r\ f(x)$$

for all $x \in X$ and $r \in \mathbb{Q}$.

Lemma 2.2. *Let $f : A \to B$ be a mapping. Then f is $\mathbb{C}$-linear if and only if*

$$f(\mu x + \lambda y) = \mu\ f(x) + \lambda\ f(y) \tag{2.1}$$

for all $\mu, \lambda \in \mathbb{T}^1$ and all $x, y \in A$.

Proof. By letting $\mu = \lambda = 1$ in (2.1), we will see that f is an additive mapping. So it's just necessary to show

$$f(\lambda x) = \lambda\ f(x)$$

for all $\lambda \in \mathbb{C}$ and $x \in A$. Now assume that $\lambda \in \mathbb{C}$ and k be an integer number such that $k > 2|\lambda| \geq 0$. By Lemma 2.1, we have

$$f\left(\frac{k}{2}x\right) = \frac{k}{2}\ f(x) \tag{2.2}$$

for all $x \in A$. Since $0 \leq \left|\frac{\lambda}{k}\right| < \frac{1}{2}$, there is $t \in (\frac{\pi}{3}, \frac{\pi}{2}]$ such that

$$0 \leq \left|\frac{\lambda}{k}\right| = \mathbf{cos}\ t = \frac{e^{it} + e^{-it}}{2} < \frac{1}{2} \tag{2.3}$$

We know there exists $\mu \in \mathbb{T}^1$ such that $\frac{\lambda}{k} = \left| \frac{\lambda}{k} \right| \mu$. By (2.1), (2.2) and (2.3), one can obtain

$$\begin{aligned} f(\lambda x) &= f\left(k\frac{\lambda}{k}\, x\right) = f\left(k\left|\frac{\lambda}{k}\right|\mu\, x\right) = f\left(k\frac{e^{it}+e^{-it}}{2}\mu\, x\right) \\ &= \frac{ke^{it}\mu}{2}\, f(x) + \frac{ke^{-it}\mu}{2}\, f(x) = k\frac{e^{it}+e^{-it}}{2}\mu\, f(x) = \lambda\, f(x) \end{aligned}$$

for all $\lambda \in \mathbb{C}$ and $x \in A$. □

Lemma 2.3. *A mapping $f : A \to B$ is a $\mathbb{C}$-linear mapping if and only if*

$$\Gamma_\mu f(x_1, \cdots, x_p) = 0$$

for all $\mu \in \mathbb{T}^1$ and all $x_1, \cdots, x_p \in A$.

Proof. It's clear from Lemma 2.2 and the initial descriptions of this section. □

Lemma 2.4. [14] *Let $\{x_n\}_n$, $\{y_n\}_n$ and $\{z_n\}_n$ be convergent sequences in A. Then the sequence $\{[x_n, y_n, z_n]\}_n$ is a convergent sequence in A.*

Theorem 2.5. *Let $\varphi : A^p \to [0, \infty)$ be a function with $\varphi(0, \cdots, 0) = 0$. Denote by ϕ a function such that*

$$\phi(x_1, \cdots, x_p) := \sum_{n=0}^{\infty} K^n \varphi\left(K^{-(n+1)}x_1, \cdots, K^{-(n+1)}x_p\right) < \infty \tag{2.4}$$

for all $x_1, \cdots, x_p \in A$, where $K = \frac{p+k-1}{k}$. Suppose that $f : A \to B$ is a mapping satisfying

$$\begin{aligned} \| \Gamma_\mu f(x_1, \cdots, x_p) \|_B &\leq \varphi(x_1, \cdots, x_p), &(2.5)\\ \| \Gamma^h f(x, y) \|_B &\leq \varphi(x, y, 0, \cdots, 0) &(2.6)\end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, x_1, \cdots, x_p \in A$. If $\lim_{n\to\infty} K^{3n}\varphi\left(K^{-n}x, 0, \cdots, 0\right) = 0$ for all $x \in A$, then there exists a unique ternary Jordan C^*-homomorphism $H : A \to B$ such that

$$\| f(x) - H(x) \|_B \leq \frac{1}{p}\ \phi(x, \cdots, x) \tag{2.7}$$

for all $x \in A$.

Proof. Putting $\mu = 1$ and $x_1 = \cdots = x_p = x$ in (2.5), we get

$$\begin{aligned} \left\| pf\left(\frac{p+k-1}{k}\ x\right) - \frac{p+k-1}{k}p\ f(x) \right\|_B &\leq \varphi(x, \cdots, x), \\ \| f(K\ x) - K\ f(x) \|_B &\leq \frac{1}{p}\ \varphi(x, \cdots, x) \end{aligned}$$

for all $x \in A$. Replacing x by $\frac{x}{K}$, we see that

$$\left\| f(x) - K\ f\left(\frac{x}{K}\right) \right\|_B \leq \frac{1}{p}\ \varphi\left(\frac{x}{K}, \cdots, \frac{x}{K}\right) \tag{2.8}$$

for all $x \in A$. Now we claim

$$\left\| f(x) - K^n f\left(\frac{x}{K^n}\right) \right\|_B \leq \frac{1}{p} \sum_{s=0}^{n-1} K^s \varphi\left(K^{-(s+1)}x, \cdots, K^{-(s+1)}x\right) \tag{2.9}$$

for each $n \geq 1$ and all $x \in A$. We verify it by induction on n. The relation (2.8) shows that (2.9) is true for the case $n = 1$. For the case $n + 1$, replacing x by $\frac{x}{K^n}$ and putting $n = 1$ in (2.9), we get

$$\left\| f\left(\frac{x}{K^n}\right) - Kf\left(\frac{x}{K^{n+1}}\right) \right\|_B \leq \frac{1}{p}\ \varphi\left(K^{-(n+1)}x, \cdots, K^{-(n+1)}x\right).$$

From this and (2.9), we obtain

$$\begin{aligned} \left\| f(x) - K^{n+1} f\left(\frac{x}{K^{n+1}}\right) \right\|_B &\le \left\| f(x) - K^n f\left(\frac{x}{K^n}\right) \right\|_B + K^n \left\| f\left(\frac{x}{K^n}\right) - K f\left(\frac{x}{K^{n+1}}\right) \right\|_B \\ &\le \frac{1}{p} \sum_{s=0}^{n-1} K^s \varphi\left(K^{-(s+1)}x, \cdots, K^{-(s+1)}x\right) \\ &\quad + K^n \left(\frac{1}{p}\, \varphi\left(K^{-(n+1)}x, \cdots, K^{-(n+1)}x\right)\right) \\ &= \frac{1}{p} \sum_{s=0}^{n} K^s \varphi\left(K^{-(s+1)}x, \cdots, K^{-(s+1)}x\right). \end{aligned}$$

Accordingly, the assertion (2.9) holds for all $n \ge 1$ and all $x \in A$.

Now assume that m, l are positive integers, with $m > l$. By (2.9) for $m - l > 0$ and $\frac{x}{K^l}$, we have

$$\begin{aligned} \left\| K^m f\left(\frac{x}{K^m}\right) - K^l f\left(\frac{x}{K^l}\right) \right\|_B &= K^l \left\| K^{m-l} f\left(\frac{1}{K^{m-l}}\, \frac{x}{K^l}\right) - f\left(\frac{x}{K^l}\right) \right\|_B \\ &\le \frac{1}{p} \sum_{s=l}^{m-1} K^s \varphi\left(K^{-(s+1)}x, \cdots, K^{-(s+1)}x\right) \\ &\le \frac{1}{p} \sum_{s=l}^{\infty} K^s \varphi\left(K^{-(s+1)}x, \cdots, K^{-(s+1)}x\right). \end{aligned}$$

Now, the relation (2.4) shows when in the top line $l \to \infty$, the right side of it converges to 0, and this clarifies that the sequence $\{K^n f(\frac{x}{K^n})\}$ is a Cauchy sequence. Since A is a complete space, the sequence $\{K^n f(\frac{x}{K^n})\}$ is a convergent sequence. Therefore, we can define, for all $x \in A$, the mapping $H : A \to B$ by

$$H(x) := \lim_{n\to\infty} K^n f\left(\frac{x}{K^n}\right).$$

Passing the limit $n \to \infty$ in (2.9) and looking to (2.4), we obtain (2.7).

It follows from (2.5) and (2.4) that

$$\begin{aligned} \| \Gamma_\mu H(x_1, \cdots, x_p) \|_B &= \lim_{n\to\infty} K^n \left\| \Gamma_\mu f\left(\frac{x_1}{K^n}, \cdots, \frac{x_p}{K^n}\right) \right\|_B \\ &\le \lim_{n\to\infty} K^n\, \varphi\left(\frac{x_1}{K^n}, \cdots, \frac{x_p}{K^n}\right) = 0 \end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x_1, \cdots, x_p \in A$. It means by Lemma 2.3 that H is $\mathbb{C}$-linear.

Putting $\mu = 1$ and $x_1 = \cdots = x_p = 0$ in (2.5), we get $f(0) = 0$. Letting $y = 0$ and replacing x by $\frac{x}{K^n}$ in (2.6) and by Lemma 2.4 and the assumption on φ, we obtain

$$\begin{aligned} \| H([x,x,x]) - [H(x), H(x), H(x)] \|_B &= \lim_{n\to\infty} K^{3n} \left\| f\left(\left[\frac{x}{K^n}, \frac{x}{K^n}, \frac{x}{K^n}\right]\right) - \left[f\left(\frac{x}{K^n}\right), f\left(\frac{x}{K^n}\right), f\left(\frac{x}{K^n}\right)\right] \right\|_B \\ &\le \lim_{n\to\infty} K^{3n} \varphi\left(\frac{x}{K^n}, 0, \cdots, 0\right) = 0 \end{aligned}$$

for all $x \in A$. Thus $H([x,x,x]) = [H(x), H(x), H(x)]$ for all $x \in A$.

Letting $x = 0$ and replacing y by $\frac{y}{K^n}$ in (2.6) and by (2.4), we obtain

$$\begin{aligned} \| H(y^*) - (H(y))^* \|_B &= \lim_{n\to\infty} K^n \left\| f\left(\left(\frac{y}{K^n}\right)^*\right) - \left(f\left(\frac{y}{K^n}\right)\right)^* \right\|_B \\ &\le \lim_{n\to\infty} K^n \varphi\left(0, \frac{y}{K^n}, 0, \cdots, 0\right) = 0 \end{aligned}$$

for all $y \in A$. Thus $H(y^*) = H(y)^*$ for all $y \in A$. Therefore, $H : A \to B$ is a ternary Jordan C^*-homomorphism.

Let $G : A \to B$ be another ternary Jordan C^*-homomorphism that satisfies (2.7), we will have

$$\begin{aligned} \| H(x) - G(x) \|_B &\leq K^n \left(\left\| f\left(\frac{x}{K^n}\right) - H\left(\frac{x}{K^n}\right) \right\|_B + \left\| f\left(\frac{x}{K^n}\right) - G\left(\frac{x}{K^n}\right) \right\|_B \right) \\ &\leq K^n \left(\frac{2}{p}\, \phi\left(\frac{x}{K^n}, \cdots, \frac{x}{K^n}\right)\right) \\ &\leq \frac{2K^n}{p} \sum_{s=0}^{\infty} K^s \varphi\left(K^{-(s+1)}\frac{x}{K^n}, \cdots, K^{-(s+1)}\frac{x}{K^n}\right) \\ &= \frac{2}{p} \sum_{s=n}^{\infty} K^s \varphi\left(K^{-(s+1)}x, \cdots, K^{-(s+1)}x\right) \end{aligned}$$

for all $x \in A$. Now if in the top line $n \to \infty$, then (2.4) shows that the right side of it converges to 0. This clearly means that H is unique. □

Theorem 2.6. *Let $\varphi : A^p \to [0, \infty)$ be a function with $\varphi(0, \cdots, 0) = 0$. Denote by ϕ a function such that*

$$\phi(x_1, \cdots, x_p) := \sum_{n=0}^{\infty} K^{-(n+1)} \varphi(K^n x_1, \cdots, K^n x_p) < \infty \tag{2.10}$$

for all $x_1, \cdots, x_p \in A$, where $K = \frac{p+k-1}{k}$. Suppose that $f : A \to B$ is a mapping satisfying (2.5) and (2.6). Then there exists a unique ternary Jordan C^*-homomorphism $H : A \to B$ satisfying (2.7).

Proof. It follows from (2.5) that

$$\begin{aligned} \| f(K\, x) - K\, f(x) \|_B &\leq \frac{1}{p}\, \varphi(x, \cdots, x), \\ \left\| \frac{1}{K}\, f(Kx) - f(x) \right\|_B &\leq \frac{1}{pK}\, \varphi(x, \cdots, x) \end{aligned} \tag{2.11}$$

for all $x \in A$. Now we claim

$$\left\| \frac{1}{K^n} f(K^n x) - f(x) \right\|_B \leq \frac{1}{p} \sum_{s=0}^{n-1} K^{-(s+1)} \varphi(K^s x, \cdots, K^s x) \tag{2.12}$$

for each $n \geq 1$ and all $x \in A$. We verify it by induction on n. The relation (2.11) shows that (2.12) is true for the case $n = 1$. For the case $n + 1$, replacing x by $K^n x$ and putting $n = 1$ in (2.12), we get

$$\left\| \frac{1}{K} f\left(K^{n+1}x\right) - f(K^n x) \right\|_B \leq \frac{1}{pK}\, \varphi(K^n x, \cdots, K^n x).$$

From this and (2.12), we obtain

$$\begin{aligned} \left\| \frac{1}{K^{n+1}} f\left(K^{n+1}x\right) - f(x) \right\|_B &\leq \frac{1}{K^n} \left\| \frac{1}{K} f\left(K^{n+1}x\right) - f(K^n x) \right\|_B + \left\| \frac{1}{K^n} f(K^n x) - f(x) \right\|_B \\ &\leq \frac{1}{K^n} \left(\frac{1}{pK}\, \varphi(K^n x, \cdots, K^n x)\right) \\ &\quad + \frac{1}{p} \sum_{s=0}^{n-1} K^{-(s+1)} \varphi(K^s x, \cdots, K^s x) \\ &= \frac{1}{p} \sum_{s=0}^{n} K^{-(s+1)} \varphi(K^s x, \cdots, K^s x). \end{aligned}$$

Accordingly, the assertion (2.12) holds for all $n \geq 1$ and all $x \in A$.

Now assume that m, l are positive integers, with $m > l$. By (2.12) for $m - l > 0$ and $K^l x$, we have

$$\begin{aligned}\left\| \frac{1}{K^m} f(K^m x) - \frac{1}{K^l} f\left(K^l x\right) \right\|_B &= \frac{1}{K^l} \left\| \frac{1}{K^{m-l}} f\left(K^{m-l} K^l x\right) - f\left(K^l x\right) \right\|_B \\ &\leq \frac{1}{p} \sum_{s=l}^{m-1} K^{-(s+1)} \varphi(K^s x, \cdots, K^s x) \\ &\leq \frac{1}{p} \sum_{s=l}^{\infty} K^{-(s+1)} \varphi(K^s x, \cdots, K^s x).\end{aligned}$$

Now, the relation (2.10) shows when in the top line $l \to \infty$, the right side of it converges to 0, and this clarifies that the sequence $\left\{\frac{1}{K^n} f(K^n x)\right\}$ is a Cauchy sequence. Since A is a complete space, the sequence $\left\{\frac{1}{K^n} f(K^n x)\right\}$ is a convergent sequence and we can define for all $x \in A$, the mapping $H : A \to B$ by

$$H(x) := \lim_{n\to\infty} \frac{1}{K^n} f(K^n x).$$

Passing the limit $n \to \infty$ in (2.12) and by (2.10), we obtain (2.7).

Letting $y = 0$ and replacing x by $K^n x$ in (2.6) and by Lemma 2.4 and (2.10), we obtain

$$\begin{aligned}\| H([x, x, x]) - [H(x), H(x), H(x)] \|_B &= \lim_{n\to\infty} \frac{1}{K^{3n}} \| f([K^n x, K^n x, K^n x]) - [f(K^n x), f(K^n x), f(K^n x)] \|_B \\ &\leq \lim_{n\to\infty} \frac{1}{K^{3n}} \varphi(K^n x, 0, \cdots, 0) \leq \lim_{n\to\infty} \frac{1}{K^n} \varphi(K^n x, 0, \cdots, 0) = 0\end{aligned}$$

for all $x \in A$. Thus $H([x, x, x]) = [H(x), H(x), H(x)]$ for all $x \in A$.

The rest of the proof is similar to the proof of the previous theorem. □

Corollary 2.7. *Let θ be a nonnegative real number and, for every $1 \leq j \leq p$, q_j be positive real numbers such that all $q_j > 1$ with $q_1 > 3$ or all $q_j < 1$ and let $f : A \to B$ be a mapping satisfying*

$$\begin{aligned}\| \Gamma_\mu f(x_1, \cdots, x_p) \|_B &\leq \theta \left(\sum_{j=1}^{p} \| x_j \|_A^{q_j} \right), \\ \| \Gamma^h f(x, y) \|_B &\leq \theta \left(\| x \|_A^{q_1} + \| y \|_A^{q_2}\right)\end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, x_1, \cdots, x_p \in A$. Then there exists a unique ternary Jordan C^*-homomorphism $H : A \to B$ such that

$$\| f(x) - H(x) \|_B \leq \sum_{j=1}^{p} \frac{k^{q_j+1}}{p\,|k\ (p+k-1)^{q_j} - (p+k-1)\ k^{q_j}|} \theta \| x \|_A^{q_j}$$

for all $x \in A$.

Proof. Defining

$$\varphi(x_1, \cdots, x_p) = \theta \left(\sum_{j=1}^{p} \| x_j \|_A^{q_j} \right)$$

and applying Theorem 2.5 for the case that all $q_j > 1$ with $q_1 > 3$, and Theorem 2.6 for the case that all $q_j < 1$, we get the result. □

Corollary 2.8. *Let θ be a nonnegative real number and let q be a positive real number with $q > 3$ or $q < 1$. Let $f : A \to B$ be a mapping such that*

$$\begin{aligned}\| \Gamma_\mu f(x_1, \cdots, x_p) \|_B &\leq \theta \left(\sum_{j=1}^{p} \| x_j \|_A^{q} \right), \\ \| \Gamma^h f(x, y) \|_B &\leq \theta \left(\| x \|_A^{q} + \| y \|_A^{q}\right)\end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, x_1, \cdots, x_p \in A$. Then there exists a unique ternary Jordan C^*-homomorphism $H : A \to B$ such that

$$\| f(x) - H(x) \|_B \leq \frac{k^{q+1}}{|k\ (p+k-1)^q - (p+k-1)\ k^q|} \theta \| x \|_A^q$$

for all $x \in A$.

Proof. Putting $q_1 = \cdots = q_p = q$ and applying Corollary 2.7, we get the result. □

3. Hyers-Ulam stability of ternary Jordan C^*-derivations

Throughout this section, assume that A is a C^*-ternary algebra with norm $\| \cdot \|$. In this section, we will prove the results for ternary Jordan C^*-derivations.

Theorem 3.1. *Let $\varphi : A^p \to [0, \infty)$ be a function with $\varphi(0, \cdots, 0) = 0$. Let $\phi : A^p \to [0, \infty)$ be a function satisfying (2.4) and let $f : A \to A$ be a mapping such that*

$$\begin{aligned} \| \Gamma_\mu f(x_1, \cdots, x_p) \| &\leq \varphi(x_1, \cdots, x_p), && (3.1)\\ \| \Gamma^d f(x, y) \| &\leq \varphi(x, y, 0, \cdots, 0) && (3.2) \end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, x_1, \cdots, x_p \in A$. If $\lim_{n\to\infty} K^{3n} \varphi\left(K^{-n} x, 0, \cdots, 0\right) = 0$ for all $x \in A$, then there exists a unique ternary Jordan C^*-derivation $\delta : A \to A$ such that

$$\| f(x) - \delta(x) \| \leq \frac{1}{p}\ \phi(x, \cdots, x) \tag{3.3}$$

for all $x \in A$.

Proof. By the same argument as in the proof of Theorem 2.5, one can show that $\delta : A \to A$, given by

$$\delta(x) := \lim_{n\to\infty}\ K^n f\left(\frac{x}{K^n}\right)$$

for all $x \in A$, is a $\mathbb{C}$-linear mapping and satisfying (3.3).

Putting $\mu = 1$ and $x_1 = \cdots = x_p = 0$ in (3.1), we get $f(0) = 0$. Letting $y = 0$ and replacing x by $\frac{x}{K^n}$ in (3.2) and by Lemma 2.4 and the assumption on φ, we obtain

$$\begin{aligned} &\| \delta([x, x, x]) - [\delta(x), x, x] - [x, \delta(x), x] - [x, x, \delta(x)] \| \\ &= \lim_{n\to\infty}\ K^{3n} \left\| f\left(\left[\frac{x}{K^n}, \frac{x}{K^n}, \frac{x}{K^n}\right]\right) - \left[f\left(\frac{x}{K^n}\right), \frac{x}{K^n}, \frac{x}{K^n}\right] - \left[\frac{x}{K^n}, f\left(\frac{x}{K^n}\right), \frac{x}{K^n}\right] - \left[\frac{x}{K^n}, \frac{x}{K^n}, f\left(\frac{x}{K^n}\right)\right] \right\| \\ &\leq \lim_{n\to\infty} K^{3n} \varphi\left(\frac{x}{K^n}, 0, \cdots, 0\right) = 0 \end{aligned}$$

for all $x \in A$. Thus $\delta([x, x, x]) = [\delta(x), x, x] + [x, \delta(x), x] + [x, x, \delta(x)]$ for all $x \in A$.

Letting $x = 0$ and replacing y by $\frac{y}{K^n}$ in (3.2) and by (2.4), we obtain

$$\begin{aligned} \| \delta(y^*) - (\delta(y))^* \| &= \lim_{n\to\infty}\ K^n \left\| f\left(\frac{y^*}{K^n}\right) - \left(f\left(\frac{y}{K^n}\right)\right)^* \right\| \\ &\leq \lim_{n\to\infty} K^n \varphi\left(0, \frac{y}{K^n}, 0, \cdots, 0\right) = 0 \end{aligned}$$

for all $y \in A$. Thus $\delta(y^*) = \delta(y)^*$ for all $y \in A$. Therefore, $\delta : A \to A$ is a ternary Jordan C^*-derivation.

The rest of the proof is similar to the proof of Theorem 2.5. □

Theorem 3.2. *Let $\varphi : A^p \to [0, \infty)$ be a function with $\varphi(0, \cdots, 0) = 0$. Let $\phi : A^p \to [0, \infty)$ be a function satisfying (2.10) and let $f : A \to A$ be a mapping satisfying (3.1) and (3.2). Then there exists a unique ternary Jordan C^*-derivation $\delta : A \to A$ satisfying (3.3).*

Proof. By the same argument as in the proof of Theorem 2.6, one can obtain a $\mathbb{C}$-linear mapping $\delta : A \to A$, given by $\delta(x) = \lim_{n\to\infty} \frac{1}{K^n} f\left(K^n x\right)$ for all $x \in A$, and satisfying (3.3).

The rest of the proof is similar to the proofs of Theorems 2.6 and 3.1. □

Corollary 3.3. *Let θ be a nonnegative real number and, for every $1 \leq j \leq p$, q_j be positive real numbers such that all $q_j > 1$ with $q_1 > 3$ or all $q_j < 1$ and let $f : A \to A$ be a mapping satisfying*

$$\begin{aligned} \| \Gamma_\mu f(x_1, \cdots, x_p) \| &\leq \theta\left(\sum_{j=1}^{p} \| x_j \|^{q_j}\right), \\ \| \Gamma^d f(x, y) \| &\leq \theta\left(\| x \|^{q_1} + \| y \|^{q_2}\right) \end{aligned}$$

for all $\mu \in \mathbb{T}^1$ *and all* $x, y, x_1, \cdots, x_p \in A$. *Then there exists a unique ternary Jordan* C^**-derivation* $\delta : A \to A$ *such that*

$$\| f(x) - \delta(x) \| \leq \sum_{j=1}^{p} \frac{k^{q_j+1}}{p\,|k\ (p+k-1)^{q_j} - (p+k-1)\ k^{q_j}|}\ \theta \| x \|^{q_j}$$

for all $x \in A$.

Proof. Defining

$$\varphi(x_1, \cdots, x_p) = \theta \left(\sum_{j=1}^{p} \| x_j \|^{q_j} \right)$$

and applying Theorem 3.1 for the case that all $q_j > 1$ with $q_1 > 3$, and Theorem 3.2 for the case that all $q_j < 1$, we get the result. ☐

Corollary 3.4. *Let* θ *be a nonnegative real number and let* q *be a positive real number with* $q > 3$ *or* $q < 1$. *Let* $f : A \to A$ *be a mapping such that*

$$\begin{aligned} \| \Gamma_\mu f(x_1, \cdots, x_p) \| &\leq \theta \left(\sum_{j=1}^{p} \| x_j \|^q \right), \\ \| \Gamma^d f(x, y) \| &\leq \theta (\| x \|^q + \| y \|^q) \end{aligned}$$

for all $\mu \in \mathbb{T}^1$ *and all* $x, y, x_1, \cdots, x_p \in A$. *Then there exists a unique ternary Jordan* C^**-derivation* $\delta : A \to A$ *such that*

$$\| f(x) - \delta(x) \| \leq \frac{k^{q+1}}{|k\ (p+k-1)^q - (p+k-1)\ k^q|}\ \theta \| x \|^q$$

for all $x \in A$.

Proof. Putting $q_1 = \cdots = q_p = q$ and applying Corollary 3.3, we get the result. ☐

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OSTROWSKI TYPE INEQUALITIES FOR FUNCTIONS WHOSE DERIVATIVES ARE $MT-$CONVEX

MEVLÜT TUNÇ

Abstract. Some inequalities of Ostrowski's type for $MT-$convex functions are introduced. An improvements for some Midpoint type inequalities are given. Some applications to special means are also obtained.

1. Introductions

Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a differentiable mapping on I° , the interior of the interval I, such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'(x)| \leq M$, then the following inequality:

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u)\, du \right| \leq \frac{M}{b-a} \left[\frac{(x-a)^2 + (b-x)^2}{2} \right] \tag{1.1}$$

holds. This result is known in the literature as the Ostrowski inequality. For recent results and generalizations concerning Ostrowski's inequality, see [1]-[7] and [9] the references therein.

In [10], $MT-$convex function defined by Tunç and Yıldırım as following.

Definition 1. *A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to belong to the class of $MT(I)$ if it is nonnegative and for all $x, y \in I$ and $t \in (0, 1)$ satisfies the inequality;*

$$f(tx + (1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}} f(y). \tag{1.2}$$

Theorem 1. *Let $f \in MT(I)$, $a, b \in I$ with $a < b$ and $f \in L_1[a, b]$. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)\, dx \tag{1.3}$$

and

$$\frac{2}{b-a} \int_a^b \tau(x) f(x)\, dx \leq \frac{f(a) + f(b)}{2} \tag{1.4}$$

where $\tau(x) = \frac{\sqrt{(b-x)(x-a)}}{b-a}$, $x \in [a, b]$.

The aim of this manuscript is to establish some Ostrowski type inequalities for the class of functions whose derivatives in absolute value are MT-convex functions.

2000 *Mathematics Subject Classification.* 26D10, 26D15.

Key words and phrases. Ostrowski's inequality, $MT-$convex function, Midpoint type inequalities.

2. New Results

In order to prove our main theorems, we need the following lemma that has been obtained in [5]:

Lemma 1. [5]*Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° where $a, b \in I$ with $a < b$. If $f' \in L[a,b]$, then the following equality holds;*

$$\begin{aligned} & f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \\ = & \frac{(x-a)^2}{b-a}\int_0^1 tf'(tx+(1-t)a)\,dt - \frac{(b-x)^2}{b-a}\int_0^1 tf'(tx+(1-t)b)\,dt \end{aligned}$$

for each $x \in [a,b]$.

We shall start with the following refinement of the Ostrowski inequality for MT-convex functions.

Theorem 2. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is MT-convex function on I and $|f'(x)| \leq M$, $x \in [a,b]$, then we have;*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \leq \frac{M\pi\left[(x-a)^2+(b-x)^2\right]}{4(b-a)} \tag{2.1}$$

for each $x \in [a,b]$.

Proof. Using Lemma 1 and MT-convexity of $|f'|$, it follows that

$$\begin{aligned} & \left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \\ \leq & \frac{(x-a)^2}{b-a}\int_0^1 t\left|f'(tx+(1-t)a)\right|dt + \frac{(b-x)^2}{b-a}\int_0^1 t\left|f'(tx+(1-t)b)\right|dt \\ \leq & \frac{(x-a)^2}{b-a}\int_0^1 t\left[\frac{\sqrt{t}}{2\sqrt{1-t}}\left|f'(x)\right| + \frac{\sqrt{1-t}}{2\sqrt{t}}\left|f'(a)\right|\right]dt \\ & + \frac{(b-x)^2}{b-a}\int_0^1 t\left[\frac{\sqrt{t}}{2\sqrt{1-t}}\left|f'(x)\right| + \frac{\sqrt{1-t}}{2\sqrt{t}}\left|f'(b)\right|\right]dt \\ \leq & \frac{M\left[(x-a)^2+(b-x)^2\right]}{2(b-a)}\int_0^1 \left(t^{3/2}(1-t)^{-1/2} + t^{1/2}(1-t)^{1/2}\right)dt. \end{aligned}$$

The proof is completed. □

Remark 1. *In Theorem 2, if we choose $x = (a+b)/2$, then we get*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \leq \frac{M\pi(b-a)}{8}. \tag{2.2}$$

The corresponding version for power of the absolute value of the first derivative is incorporated in the following results.

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Theorem 3. *Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is MT-convex function on $[a, b]$, $q > 1$, $p^{-1} + q^{-1} = 1$ and $|f'(x)| \leq M$, $x \in [a, b]$, then we have;*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u)\,du \right| \leq \frac{M}{(1+p)^{1/p}} \left(\frac{\pi}{8}\right)^{\frac{1}{q}} \left[\frac{(x-a)^2 + (b-x)^2}{b-a} \right] \tag{2.3}$$

for each $x \in [a, b]$.

Proof. Suppose that $p > 1$. From Lemma 1 and using the Hölder inequality, we have

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a} \int_a^b f(u)\,du \right| \\
\leq\ & \frac{(x-a)^2}{b-a} \int_0^1 t\,|f'(tx + (1-t)a)|\,dt + \frac{(b-x)^2}{b-a} \int_0^1 t\,|f'(tx + (1-t)b)|\,dt \\
\leq\ & \frac{(x-a)^2}{b-a} \left(\int_0^1 t^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |f'(tx + (1-t)a)|^q\,dt \right)^{\frac{1}{q}} \\
&+ \frac{(b-x)^2}{b-a} \left(\int_0^1 t^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |f'(tx + (1-t)b)|^q\,dt \right)^{\frac{1}{q}}.
\end{aligned}$$

Since $|f'|^q$ is MT-convex function and $|f'(x)| \leq M$, then we have

$$\begin{aligned}
\int_0^1 |f'(tx + (1-t)a)|^q\,dt &\leq \int_0^1 \left[\frac{\sqrt{t}}{2\sqrt{1-t}} |f'(x)|^q + \frac{\sqrt{1-t}}{2\sqrt{t}} |f'(a)|^q \right] dt \\
&= \frac{\pi}{16} \left[|f'(x)|^q + |f'(a)|^q \right] \leq \frac{\pi}{8} M^q
\end{aligned}$$

and similarly

$$\begin{aligned}
\int_0^1 |f'(tx + (1-t)b)|^q\,dt &\leq \int_0^1 \left[\frac{\sqrt{t}}{2\sqrt{1-t}} |f'(x)|^q + \frac{\sqrt{1-t}}{2\sqrt{t}} |f'(b)|^q \right] dt \\
&= \frac{\pi}{16} \left[|f'(x)|^q + |f'(b)|^q \right] \leq \frac{\pi}{8} M^q
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\left| f(x) - \frac{1}{b-a} \int_a^b f(u)\,du \right| &\leq \frac{(x-a)^2}{b-a} \frac{1}{(1+p)^{1/p}} \left(\frac{\pi}{8} M^q \right)^{\frac{1}{q}} \\
&\quad + \frac{(x-a)^2}{b-a} \frac{1}{(1+p)^{1/p}} \left(\frac{\pi}{8} M^q \right)^{\frac{1}{q}} \\
&= \frac{M}{(1+p)^{1/p}} \left(\frac{\pi}{8}\right)^{\frac{1}{q}} \left[\frac{(x-a)^2 + (b-x)^2}{b-a} \right],
\end{aligned}$$

where $p^{-1} + q^{-1} = 1$, which is required. $\square$

Remark 2. *In Theorem 3, if we choose $x = (a+b)/2$, then we get*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(u)\,du \right| \leq \frac{M.\pi^{\frac{1}{q}}}{(1+p)^{1/p}} \left(\frac{1}{2}\right)^{1+\frac{3}{q}} (b-a). \tag{2.4}$$

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Remark 3. *Since* $\frac{1}{(1+p)^{1/p}} > \frac{1}{2}$ *for any* $p > 1$, *if we write the inequality (2.3) again, then we obtain*

$$\begin{aligned}\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| &\leq \frac{M}{(1+p)^{1/p}}\left(\frac{\pi}{8}\right)^{\frac{1}{q}}\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right] \\ &\leq M\left(\frac{1}{1+p}\right)^{\frac{1}{p}+\frac{3}{pq}}\pi^{\frac{1}{q}}\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right]. \qquad (2.5)\end{aligned}$$

Theorem 4. *Let* $f : I \subset [0,\infty) \to \mathbb{R}$ *be a differentiable mapping on* I° *such that* $f' \in L[a,b]$, *where* $a, b \in I$ *with* $a < b$. *If* $|f'|^q$ *is* MT*-convex function on* $[a,b]$, $q \geq 1$ *and* $|f'(x)| \leq M$, $x \in [a,b]$, *then the following inequality holds:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \leq M.\left(\frac{1}{2}\right)^{1+\frac{1}{q}}.\pi^{\frac{1}{q}}.\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right] \qquad (2.6)$$

for each $x \in [a,b]$.

Proof. From Lemma 1 and using the well known power mean inequality, we have

$$\begin{aligned}&\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \\ \leq\; &\frac{(x-a)^2}{b-a}\int_0^1 t\left|f'(tx+(1-t)a)\right|dt+\frac{(b-x)^2}{b-a}\int_0^1 t\left|f'(tx+(1-t)b)\right|dt \\ \leq\; &\frac{(x-a)^2}{b-a}\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|f'(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}} \\ &+\frac{(b-x)^2}{b-a}\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|f'(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}.\end{aligned}$$

Since $|f'|^q$ is MT-convex function and $|f'(x)| \leq M$, then we have

$$\begin{aligned}\int_0^1 t\left|f'(tx+(1-t)a)\right|^q dt &\leq \int_0^1\left[\frac{t\sqrt{t}}{2\sqrt{1-t}}\left|f'(x)\right|^q+\frac{t\sqrt{1-t}}{2\sqrt{t}}\left|f'(a)\right|^q\right]dt \\ &= \frac{1}{2}\left|f'(x)\right|^q\int_0^1 t^{\frac{3}{2}}(1-t)^{\frac{-1}{2}}dt+\frac{1}{2}\left|f'(a)\right|^q\int_0^1 t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}dt \\ &\leq \frac{1}{2}M^q\frac{3\pi}{8}+\frac{1}{2}M^q\frac{\pi}{8}=\frac{\pi}{4}M^q\end{aligned}$$

and

$$\int_0^1 t\left|f'(tx+(1-t)b)\right|^q dt \leq \int_0^1\left[\frac{t\sqrt{t}}{2\sqrt{1-t}}\left|f'(x)\right|^q+\frac{t\sqrt{1-t}}{2\sqrt{t}}\left|f'(b)\right|^q\right]dt \leq \frac{\pi}{4}M^q$$

Therefore, we have

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \leq M\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{\pi}{4}\right)^{\frac{1}{q}}\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right]$$

which is required. $\square$

Remark 4. *In Theorem 4, if we choose $x = (a+b)/2$, then we get*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \leq M.\pi^{\frac{1}{q}}.\left(\frac{1}{2}\right)^{2+\frac{1}{q}}(b-a). \tag{2.7}$$

Corollary 1. *In Theorem 4, if we choose $x = a$, then we get*

$$\left| f(a) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \leq M.\left(\frac{1}{2}\right)^{1+\frac{1}{q}}.\pi^{\frac{1}{q}}.(b-a), \tag{2.8}$$

if wee choose $x = b$, then we get

$$\left| f(b) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \leq M.\left(\frac{1}{2}\right)^{1+\frac{1}{q}}.\pi^{\frac{1}{q}}.(b-a), \tag{2.9}$$

adding above inequalities (2.8) and (2.9), we get

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)\,du \right| \leq M.\left(\frac{1}{2}\right)^{1+\frac{1}{q}}.\pi^{\frac{1}{q}}.(b-a). \tag{2.10}$$

Now, We want to give a few example for MT-convex function, preparatory to not to pass to the section of applications, as follows:

Example 1. *i- $f : (0,1] \to \mathbb{R}$, $f(x) = -\ln x^k$, $k \in (0,\infty)$ is $MT-$convex function.*
ii- $f, g, h : \mathbb{R} \to \mathbb{R}^+$, $f(x) = x^n$, $g(x) = \left(\frac{x}{n}\right)^n$, $h(x) = \frac{1}{nx^n}$, $n \in \mathbb{R}$ are MT-convex functions. The details are left to the interested reader.

3. Applications to special means

We consider the means for arbitrary positive numbers a, b $(a \neq b)$ as follows:
The arithmetic mean: $A = A(a,b) := \frac{a+b}{2}$,
The Identric mean: $I = I(a,b) := \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}$,
The p-logarithmic mean: $L_p = L_p(a,b) := \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{1/p}$, $p \in \mathbb{R}\setminus\{-1,0\}$.
Now, using the results of Section 2, we give some applications to special means of real numbers.

Proposition 1. *Let $0 < a < b$ and $q \geq 1$. Then we have*

$$\left|A^n(a,b) - L_n^n(a,b)\right| \leq M.\pi^{\frac{1}{q}}.\left(\frac{1}{2}\right)^{2+\frac{1}{q}}(b-a)$$

Proof. The inequality follows from (2.7) applied to the MT-convex function $f : \mathbb{R} \to \mathbb{R}^+$, $f(x) = x^n$, $n \in \mathbb{R}$. The details are omitted. □

Proposition 2. *Let $0 < a < b$ and $q \geq 1$. Then we have*

$$\left|\ln I(a,b) - \ln A(a,b)\right| \leq M.\pi^{\frac{1}{q}}.\left(\frac{1}{2}\right)^{2+\frac{1}{q}}(b-a)$$

Proof. The inequality follows from (2.7) applied to the MT-convex function $f : (0,1] \to \mathbb{R}$, $f(x) = -\ln x$. The details are omitted. □

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Proposition 3. *Let $0 < a < b$ and $q \geq 1$. Then we have*

$$\left| A\left(a^n, b^n\right) - L_n^n\left(a, b\right) \right| \leq M.\pi^{\frac{1}{q}}.\left(\frac{1}{2}\right)^{2+\frac{1}{q}}\left(b-a\right)$$

Proof. The inequality follows from (2.10) applied to the MT-convex function $f : \mathbb{R} \to \mathbb{R}^+$, $f(x) = x^n$, $n \in \mathbb{R}$. The details are omitted. □

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Direct and fixed point approaches to the stability of an AQ-functional equation in non-Archimedean normed spaces

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Abstract. In this paper, using the direct and the fixed point methods, we prove the Hyers-Ulam stability of the following mixed additive and quadratic functional equation $f(kx+y)+f(kx-y) = f(x+y)+f(x-y)+(k-1)\left[(k+2)f(x) + kf(-x)\right]$ ($k \geq 2$ is an integer) in complete non-Archimedean normed spaces.

Keywords: Hyers-Ulam stability; direct method; fixed point method; non-Archimedean normed space; additive-quadratic functional equation.

MR(2000) Subject Classification. 39B82, 39B52, 46S40.

1 Introduction

In 1897, Hensel discovered the p-adic numbers as a number theoretical analogue of power series in complex analysis. The most important examples of non-Archimedean spaces are p-adic numbers. A key property of p-adic numbers is that they do not satisfy the Archimedean axiom: for all $x, y > 0$, there exists an integer n such that $x < ny$. It turned out that non-Archimedean spaces have many nice applications [1, 2]. During the last three decades, theory of non-Archimedean spaces has gained the interest of physicists for their research in particular in problems coming from quantum physics, p-adic strings and superstrings [1].

The study of stability problems for functional equations is related to a question of Ulam [3] concerning the stability of group homomorphisms and affirmatively answered for Banach spaces by Hyers [4]. The result of Hyers was generalized by Aoki [5] for approximate additive mappings and by Rassias [6] for approximate linear mappings by allowing the Cauchy difference operator $CDf(x, y) = f(x + y) - [f(x) + f(y)]$ to be controlled by $\epsilon(\|x\|^p + \|y\|^p)$. In 1994, a further generalization was obtained by Găvruţa [7], who replaced $\epsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x, y)$. The reader is referred to the following books and research papers which provide an extensive account of progress made on Ulam's problem during the last seventy years (see for instance [8–21]). The quadratic functional equation and several other functional equations are useful to characterize inner product spaces (see [10, 11, 13]).

Now we consider a mapping $f : X \to Y$ satisfies the following additive-quadratic (AQ) functional equation, which is introduced by Eskandani et al. (see [8]),

$$f(kx + y) + f(kx - y) = f(x + y) + f(x - y) + (k - 1)\left[(k + 2)f(x) + kf(-x)\right] \tag{1.1}$$

for a fixed integer with $k \geq 2$. It is easy to see that the function $f(x) = ax^2 + bx$ is a solution of Eq.(1.1). Eskandani et al. [8] have established the general solution and investigated the generalized Hyers-Ulam stability of Eq.(1.1) in quasi-β-normed spaces.

The main purpose of the present paper is to prove the generalized Hyers-Ulam stability of an AQ-functional equation (1.1) in complete non-Archimedean normed spaces using the direct and the fixed point methods.

Throughout this paper, we will assume that X is a linear space over $\mathbb{Q}$ or a non-Archimedean field of characteristic different from 2 and k (i.e. $|2| \neq 0, |k| \neq 0$), Y is a complete non-Archimedean normed space over a non-Archimedean field of characteristic different from $2, 3$, and k ($k \geq 2$ is a fixed integer).

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2 Hyers-Ulam stability of the functional equation (1.1): a direct method

Throughout this section, using direct method, we prove the generalized Hyers-Ulam stability for the mixed AQ-functional equation in non-Archimedean Banach space. For convenience, we use the following abbreviation for a given function $f : X \to Y$:

$$Df(x,y) := f(kx+y) + f(kx-y) - f(x+y) - f(x-y) - (k-1)\left[(k+2)f(x) + kf(-x)\right]$$

for all $x, y \in X$.

Theorem 2.1 *Let $\varphi : X \times X \to [0,\infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{\varphi(k^n x, k^n y)}{|k|^n} = 0 \tag{2.1}$$

for all $x, y \in X$ and let for each $x \in X$ the limit

$$\lim_{n\to\infty} \max\left\{\frac{\varphi(k^j x, 0)}{|k|^j} : \quad 0 \le j < n\right\}, \tag{2.2}$$

denoted by $\tilde{\varphi}_a(x)$, exists. Suppose that $f : X \to Y$ is an odd mapping such that

$$\|Df(x,y)\| \le \varphi(x,y) \tag{2.3}$$

for all $x, y \in X$. Then there exists an additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{1}{|2k|}\tilde{\varphi}_a(x) \tag{2.4}$$

for all $x \in X$. Moreover, if

$$\lim_{i\to\infty}\lim_{n\to\infty} \max\left\{\frac{\varphi(k^j x, 0)}{|k|^j} : \quad i \le j < n+i\right\} = 0,$$

then A is the unique additive mapping satisfying (2.4).

Proof. Setting $y = 0$ in (2.3), we get

$$\|f(kx) - kf(x)\| \le \frac{1}{|2|}\varphi(x,0) \tag{2.5}$$

for all $x \in X$. Replacing x by $k^{n-1}x$ in (2.5), we get

$$\left\|\frac{f(k^n x)}{k^n} - \frac{f(k^{n-1}x)}{k^{n-1}}\right\| \le \frac{1}{|2\cdot k^n|}\varphi(k^{n-1}x, 0) \tag{2.6}$$

for all $x \in X$. It follows from (2.6) and (2.1) that the sequence $\left\{\frac{f(k^n x)}{k^n}\right\}$ is Cauchy. Since Y is complete, we conclude that $\left\{\frac{f(k^n x)}{k^n}\right\}$ is convergent. Set $A(x) = \lim\limits_{n\to\infty} \frac{1}{k^n} f(k^n x)$. Using induction one can show that

$$\left\|\frac{f(k^n x)}{k^n} - f(x)\right\| \le \frac{1}{|2k|}\max\left\{\frac{\varphi(k^j x, 0)}{|k|^j} : \quad 0 \le j < n\right\} \tag{2.7}$$

for all $x \in X$ and all $n \in \mathbb{N}$. Letting $n \to \infty$ in (2.7) and using (2.2) one obtains (2.4). By (2.1) and (2.3), we obtain

$$\|DA(x,y)\| = \lim_{n\to\infty} \frac{1}{|k|^n}\|Df(k^n x, k^n y)\| \le \lim_{n\to\infty} \frac{1}{|k|^n}\varphi(k^n x, k^n y) = 0$$

for all $x, y \in X$. Therefore, the mapping $A : X \to Y$ satisfies (1.1). By [8, Lemma 2.2] we get that the mapping A is additive. To prove the uniqueness property of A, let A' be another additive mapping satisfying (2.4). Then

$$\begin{aligned}
\|A(x) - A'(x)\| &\le \lim_{i\to\infty} \frac{1}{|k|^i}\|A(k^i x) - A'(k^i x)\| \\
&\le \lim_{i\to\infty} \frac{1}{|k|^i}\max\{\|A(k^i x) - f(k^i x)\|, \|f(k^i x) - A'(k^i x)\|\} \\
&\le \frac{1}{|2k|}\lim_{i\to\infty}\lim_{n\to\infty}\max\left\{\frac{\varphi(k^j x, 0)}{|k|^j} : \quad i \le j < n+i\right\}
\end{aligned}$$

for all $x \in X$. If

$$\lim_{i\to\infty}\lim_{n\to\infty} \max\left\{\frac{\varphi(k^j x, 0)}{|k|^j} : \quad i \le j < n+i\right\} = 0,$$

then $A = A'$. This completes the proof. $\square$

Corollary 2.2 *Let $\alpha : [0, \infty) \to [0, \infty)$ be a function satisfying*
(i) $\alpha(|k|t) \leq \alpha(|k|)\alpha(t)$ *for all* $t > 0$,
(ii) $\alpha(|k|) < |k|$.
Let $\delta > 0$, X be a normed space and let $f : X \to Y$ be an odd mapping such that

$$\|Df(x, y)\| \leq \delta[\alpha(\|x\|) + \alpha(\|y\|)]$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{1}{|2k|}\,\delta\alpha(\|x\|)$$

for all $x \in X$.

Proof. Let $\varphi : X \times X \to [0, \infty)$ be defined by $\varphi(x, y) = \delta\,[\alpha(\|x\|) + \alpha(\|y\|)]$ for all $x, y \in X$. Then we have

$$\lim_{n\to\infty} \frac{\varphi(k^n x, k^n y)}{|k|^n} \leq \lim_{n\to\infty} \left(\frac{\alpha(|k|)}{|k|}\right)^n \varphi(x, y) = 0 \quad (x, y \in X),$$

$$\tilde{\varphi}_a(x) = \lim_{n\to\infty} \max\left\{\frac{\varphi(k^j x, 0)}{|k|^j} : \quad 0 \leq j < n\right\} = \varphi(x, 0),$$

$$\lim_{i\to\infty} \lim_{n\to\infty} \max\left\{\frac{\varphi(k^j x, 0)}{|k|^j} : \quad i \leq j < n + i\right\} = \lim_{i\to\infty} \frac{\varphi(k^i x, k^i y)}{|k|^i} = 0.$$

The corollary follows from Theorem 2.1. $\square$

Remark 2.3 The classical example of the function α is the mapping $\alpha(t) = t^r$ for all $t \geq 0$, where $r > 1$ with the further assumption that $|k| < 1$. The assumption $|k| < 1$ cannot be omitted (see Example 3.4).

Remark 2.4 We can formulate similar statements to Theorem 2.1 in which we can define the sequence $A(x) := \lim_{n\to\infty} k^n f(\frac{x}{k^n})$ under suitable conditions on the function φ and obtain similar result to Corollary 2.2 for $r < 1$.

Theorem 2.5 *Let $\varphi : X \times X \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{\varphi(k^n x, k^n y)}{|k|^{2n}} = 0 \tag{2.8}$$

for all $x, y \in X$ and let for each $x \in X$ the limit

$$\lim_{n\to\infty} \max\left\{\frac{\varphi(k^j x, 0)}{|k|^{2j}} : \quad 0 \leq j < n\right\}, \tag{2.9}$$

denoted by $\tilde{\varphi}_q(x)$, exists. Suppose that $f : X \to Y$ is an even mapping such that

$$\|Df(x, y)\| \leq \varphi(x, y) \tag{2.10}$$

for all $x, y \in X$. Then there exists a quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{1}{|2k^2|}\tilde{\varphi}_q(x) \tag{2.11}$$

for all $x \in X$. Moreover, if

$$\lim_{i\to\infty} \lim_{n\to\infty} \max\left\{\frac{\varphi(k^j x, 0)}{|k|^{2j}} : \quad i \leq j < n + i\right\} = 0,$$

then Q is the unique quadratic mapping satisfying (2.11).

Proof. Setting $y = 0$ in (2.10), we get

$$\|f(kx) - k^2 f(x)\| \leq \frac{1}{|2|}\varphi(x, 0) \tag{2.12}$$

for all $x \in X$. Replacing x by $k^{n-1}x$ in (2.12), we get

$$\left\|\frac{f(k^n x)}{k^{2n}} - \frac{f(k^{n-1}x)}{k^{2(n-1)}}\right\| \leq \frac{1}{|2 \cdot k^{2n}|}\varphi(k^{n-1}x, 0) \tag{2.13}$$

for all $x \in X$. It follows from (2.13) and (2.8) that the sequence $\left\{\frac{f(k^n x)}{k^{2n}}\right\}$ is Cauchy. Since Y is complete, we conclude that $\left\{\frac{f(k^n x)}{k^{2n}}\right\}$ is convergent. Set $Q(x) = \lim_{n\to\infty} \frac{1}{k^{2n}} f(k^n x)$. Using induction one can show that

$$\left\|\frac{f(k^n x)}{k^{2n}} - f(x)\right\| \leq \frac{1}{|2k^2|} \max\left\{\frac{\varphi(k^j x, 0)}{|k|^{2j}} : \quad 0 \leq j < n\right\} \tag{2.14}$$

for all $x \in X$ and all $n \in \mathbb{N}$. By taking n to approach infinity in (2.14) and using (2.9) one can obtain (2.11). By (2.8) and (2.10), we obtain

$$\|DQ(x,y)\| = \lim_{n\to\infty} \frac{1}{|k|^{2n}} \|Df(k^n x, k^n y)\| \le \lim_{n\to\infty} \frac{1}{|k|^{2n}} \varphi(k^n x, k^n y) = 0$$

for all $x, y \in X$. Therefore, the mapping Q satisfies (1.1), by [8, Lemma 2.1] we get that the mapping Q is quadratic. Let now

$$\lim_{i\to\infty} \lim_{n\to\infty} \max\left\{ \frac{\varphi(k^j x, 0)}{|k|^{2j}} : \quad i \le j < n+i \right\} = 0$$

and let Q' be another quadratic mapping satisfying (2.11). Then

$$\begin{aligned} \|Q(x) - Q'(x)\| &\le \lim_{i\to\infty} \frac{1}{|k|^{2i}} \left\|Q(k^i x) - Q'(k^i x)\right\| \\ &\le \lim_{i\to\infty} \frac{1}{|k|^{2i}} \max\{\|Q(k^i x) - f(k^i x)\|, \|f(k^i x) - Q'(k^i x)\|\} \\ &\le \frac{1}{|2k^2|} \lim_{i\to\infty} \lim_{n\to\infty} \max\left\{ \frac{\varphi(k^j x, 0)}{|k|^{2j}} : \quad i \le j < n+i \right\} = 0 \end{aligned}$$

for all $x \in X$. Hence $Q = Q'$. This completes the proof. □

Corollary 2.6 *Let $\beta : [0,\infty) \to [0,\infty)$ be a function satisfying*
(i) *$\beta(|k|t) \le \beta(|k|)\beta(t)$ for all $t > 0$,*
(ii) *$\beta(|k|) < |k|^2$.*
Let $\delta > 0$, X be a normed space and let $f : X \to Y$ be an even mapping such that

$$\|Df(x,y)\| \le \delta[\beta(\|x\|) + \beta(\|y\|)]$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{1}{|2k^2|} \delta\beta(\|x\|)$$

for all $x \in X$.

Proof. Let $\varphi : X \times X \to [0,\infty)$ be defined by $\varphi(x,y) = \delta\left[\beta(\|x\|) + \beta(\|y\|)\right]$ for all $x, y \in X$. Then we have

$$\lim_{n\to\infty} \frac{\varphi(k^n x, k^n y)}{|k|^{2n}} \le \lim_{n\to\infty} \left(\frac{\beta(|k|)}{|k|^2}\right)^n \varphi(x,y) = 0 \quad (x, y \in X),$$

$$\tilde{\varphi}_q(x) = \lim_{n\to\infty} \max\left\{ \frac{\varphi(k^j x, 0)}{|k|^{2j}} : \quad 0 \le j < n \right\} = \varphi(x, 0),$$

$$\lim_{i\to\infty} \lim_{n\to\infty} \max\left\{ \frac{\varphi(k^j x, 0)}{|k|^{2j}} : \quad i \le j < n+i \right\} = \lim_{i\to\infty} \frac{\varphi(k^i x, k^i y)}{|k|^{2i}} = 0.$$

The corollary follows from Theorem 2.5. □

Remark 2.7 The classical example of the function β is the mapping $\beta(t) = t^r$ for all $t \ge 0$, where $r > 2$ with the further assumption that $|k| < 1$. The assumption $|k| < 1$ cannot be omitted (see Example 3.10).

Remark 2.8 We can formulate similar statements to Theorem 2.5 in which we can define the sequence $Q(x) := \lim\limits_{n\to\infty} k^{2n} f\left(\frac{x}{k^n}\right)$ under suitable conditions on the function φ and obtain similar result to Corollary 2.6 for $r < 2$.

Theorem 2.9 *Let $\varphi : X \times X \to [0,\infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{\varphi(k^n x, k^n y)}{|k|^{2n}} = 0$$

for all $x, y \in X$ and let for each $x \in X$ the limit

$$\lim_{n\to\infty} \max\left\{ \frac{\varphi(k^j x, 0)}{|k|^j} : \quad 0 \le j < n \right\},$$

denoted by $\tilde{\varphi}_a(x)$, and

$$\lim_{n\to\infty} \max\left\{ \frac{\varphi(k^j x, 0)}{|k|^{2j}} : \quad 0 \le j < n \right\},$$

denoted by $\tilde{\varphi}_q(x)$*, exist. Suppose that* $f : X \to Y$ *is a mapping such that*

$$\|Df(x,y)\| \le \varphi(x,y)$$

for all $x, y \in X$*. Then there exist an additive mapping* $A : X \to Y$ *and a quadratic mapping* $Q : X \to Y$ *such that*

$$\|f(x) - A(x) - Q(x)\| \le \frac{1}{|4k|} \max\left\{\max\{\tilde{\varphi}_a(x), \tilde{\varphi}_a(-x)\}, \frac{1}{|k|}\max\{\tilde{\varphi}_q(x), \tilde{\varphi}_q(-x)\}\right\}$$

for all $x \in X$*. Moreover, if*

$$\lim_{i\to\infty}\lim_{n\to\infty}\max\left\{\frac{\varphi(k^j x, 0)}{|k|^j} : \quad i \le j < n+i\right\} = \lim_{i\to\infty}\lim_{n\to\infty}\max\left\{\frac{\varphi(k^j x, 0)}{|k|^{2j}} : \quad i \le j < n+i\right\} = 0,$$

then A *is the unique additive mapping and* Q *is the unique quadratic mapping.*

Proof. If we decompose f into the even and the odd parts by putting

$$f_e(x) = \frac{f(x)+f(-x)}{2} \quad \text{and} \quad f_o(x) = \frac{f(x)-f(-x)}{2}$$

for all $x \in X$, then $f(x) = f_e(x) + f_o(x)$, and

$$\|Df_o(x,y)\| \le \frac{1}{|2|}\max\{\varphi(x,y), \varphi(-x,-y)\}, \quad \|Df_e(x,y)\| \le \frac{1}{|2|}\max\{\varphi(x,y), \varphi(-x,-y)\}$$

for all $x, y \in X$. Hence by Theorems 2.1 and 2.5, there exist an additive mapping $A : X \to Y$ and a quadratic mapping $Q : X \to Y$ satisfying

$$\|f_o(x) - A(x)\| \le \frac{1}{|4k|}\max\left\{\tilde{\varphi}_a(x), \tilde{\varphi}_a(-x)\right\}, \quad \|f_e(x) - Q(x)\| \le \frac{1}{|4k^2|}\max\left\{\tilde{\varphi}_q(x), \tilde{\varphi}_q(-x)\right\}$$

for all $x \in X$. Therefore

$$\begin{aligned}
\|f(x) - A(x) - Q(x)\| &\le \max\{\|f_o(x) - A(x)\|, \ \|f_e(x) - Q(x)\|\} \\
&\le \frac{1}{|4k|}\max\left\{\max\{\tilde{\varphi}_a(x), \tilde{\varphi}_a(-x)\}, \frac{1}{|k|}\max\{\tilde{\varphi}_q(x), \tilde{\varphi}_q(-x)\}\right\}
\end{aligned}$$

for all $x \in X$. □

3 Hyers-Ulam stability of the functional equation (1.1): a fixed point method

Throughout this section, we establish the generalized Hyers-Ulam stability for the mixed AQ-functional equation (1.1) in non-Archimedean Banach space, using the fixed point method introduced by Radu in [16](see also [12]).

Let Ω be a set. A function $d : \Omega \times \Omega \to [0, \infty]$ is called a generalized metric on Ω if d satisfies

(1) $d(x,y) = 0$ if and only if $x = y$; (2) $d(x,y) = d(y,x)$, $x, y \in \Omega$; (3) $d(x,y) \le d(x,z) + d(y,z)$, $x, y, z \in \Omega$.

For explicitly later use, we recall the following result by Diaz and Margolis [22].

Proposition 3.1 *Let* (Ω, d) *be a complete generalized metric space and* $J : \Omega \to \Omega$ *be a strictly contractive mapping with Lipschitz constant* $L < 1$*, that is*

$$d(Jx, Jy) \le Ld(x,y), \qquad \forall\, x, y \in \Omega.$$

Then, for each given $x \in \Omega$*, either*

$$d\left(J^n x, J^{n+1}x\right) = \infty, \qquad \forall\, n \ge 0,$$

or there exists a non-negative integer n_0 *such that*

(1) $d\left(J^n x, J^{n+1}x\right) < \infty$ *for all* $n \ge n_0$;

(2) *the sequence* $\{J^n x\}$ *is converges to a fixed point* y^* *of* J;

(3) y^* *is the unique fixed point of* J *in the set* $\Omega^* = \{y \in \Omega \mid \ d\left(J^{n_0}x, y\right) < \infty\}$;

(4) $d(y, y^*) \le \frac{1}{1-L}d(y, Jy)$ *for all* $y \in \Omega^*$.

Now we are going to investigate the stability problem of the mixed AQ-functional equation (1.1) in non-Archimedean Banach space.

Theorem 3.2 *Let* $\varphi : X \times X \to [0, \infty)$ *be a function such that there exists an* $0 < L < 1$ *with*

$$\varphi(kx, ky) \le |k|L\varphi(x,y) \tag{3.1}$$

for all $x, y \in X$*. Let* $f : X \to Y$ *be an odd mapping such that*

$$\|Df(x,y)\| \le \varphi(x,y) \tag{3.2}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{1}{|2k|(1-L)}\,\varphi(x,0) \tag{3.3}$$

for all $x \in X$.

Proof. Letting $y = 0$ in (3.2), we get

$$\|f(kx) - kf(x)\| \leq \frac{1}{|2|}\varphi(x,0) \tag{3.4}$$

for all $x \in X$. Consider the set $\Omega := \{g \mid \ g : X \to Y, g(0) = 0\}$ and introduce the generalized metric on Ω:

$$d(g,h) = \inf\{C \in (0,\infty) \mid \ \|g(x) - h(x)\| \leq C\varphi(x,0), \ \ \forall x \in X\}. \tag{3.5}$$

It is easy to show that (Ω, d) is a complete generalized metric space. We now define a function $J : \Omega \to \Omega$ by

$$(Jg)(x) = \frac{1}{k}g(kx), \quad \forall g \in \Omega, x \in X. \tag{3.6}$$

Let $g, h \in \Omega$ and $C \in [0,\infty]$ be an arbitrary constant with $d(g,h) < C$, by the definition of d, it follows

$$\|g(x) - h(x)\| \leq C\varphi(x,0), \quad \forall x \in X. \tag{3.7}$$

By the last inequality, one has

$$\left\|\frac{1}{k}g(kx) - \frac{1}{k}h(kx)\right\| \leq CL\,\varphi(x,0), \quad \forall x \in X. \tag{3.8}$$

Hence, $d(Jg, Jh) \leq L\,d(g,h)$. It follows from (3.4) that $d(Jf, f) \leq 1/|2k| < \infty$. Therefore, by Proposition 3.1, J has a unique fixed point $A : X \to Y$ in the set $\Omega^* = \{g \in \Omega \mid \ d(f,g) < \infty\}$ such that

$$A(x) := \lim_{n\to\infty} (J^n f)(x) = \lim_{n\to\infty} \frac{1}{k^n} f(k^n x) \tag{3.9}$$

and $A(kx) = kA(x)$ for all $x \in X$. Also,

$$d(A,f) \leq \frac{1}{1-L} d(Jf,f) \leq \frac{1}{|2k|(1-L)}. \tag{3.10}$$

This means that (3.3) holds for all $x \in X$.

Now we show that A is additive. By (3.1), (3.2) and (3.9), we obtain

$$\|DA(x,y)\| = \lim_{n\to\infty} \frac{1}{|k|^n}\,\|Df\,(k^n x, k^n y)\| \leq \lim_{n\to\infty} \frac{1}{|k|^n}\,\varphi\,(k^n x, k^n y) \leq \lim_{n\to\infty} L^n \varphi(x,y) = 0$$

for all $x, y \in X$. Therefore, by [8, Lemma 2.2] we get that the mapping A is additive. □

Corollary 3.3 *Let $\delta > 0$, $r > 1$, $|k| < 1$ and $f : X \to Y$ be an odd mapping satisfying*

$$\|Df(x,y)\| \leq \delta\left(\|x\|^r + \|y\|^r\right)$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{1}{|2|(|k| - |k|^r)}\,\delta\|x\|^r$$

for all $x \in X$.

Proof. Let $\varphi : X \times X \to [0,\infty)$ be defined by $\varphi(x,y) = \delta\left(\|x\|^r + \|y\|^r\right)$ for all $x, y \in X$. Then the corollary follows from Theorem 3.2 by $L = |k|^{r-1} < 1$. □

The following example shows that the assumption $|k| < 1$ cannot be omitted in Corollary 3.3. This example is a modification of the example of [19].

Example 3.4 Let $p > 2$ be a prime number and $f : \mathbb{Q}_p \to \mathbb{Q}_p$ be defined by $f(x) = x^3$. Then for $\delta = 1$ and $r = 3$,

$$|Df(x,y)|_p \leq \max\left\{|x|_p^3, |y|_p^3\right\} \leq |x|_p^3 + |y|_p^3 \quad (x, y \in \mathbb{Q}_p).$$

However, for $k = 2$ we have $|k|_p = |2|_p = 1$ and

$$\left|\frac{1}{2^n}f(2^n x) - \frac{1}{2^{n+1}}f(2^{n+1}x)\right|_p = \left|2^{2n}\right|_p |3|_p\, |x|_p^3 = |3|_p |x|_p^3$$

for all $x \in \mathbb{Q}_p$ and $n \in \mathbb{N}$. Hence $\left\{\frac{1}{2^n}f(2^n x)\right\}$ is not a Cauchy sequence for each nonzero $x \in \mathbb{Q}_p$.

Similar to Theorem 3.2, one can obtain the following theorem.

Theorem 3.5 *Let $\varphi : X \times X \to [0, \infty)$ be a function such that there exists an $0 < L < 1$ with*

$$\varphi(x, y) \le \frac{L}{|k|}\varphi(kx, ky)$$

for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping such that

$$\|Df(x, y)\| \le \varphi(x, y)$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{L}{|2k|(1-L)}\,\varphi(x, 0)$$

for all $x \in X$.

As an application for Theorem 3.5, one can get the following corollary.

Corollary 3.6 *Let $\delta > 0$, $0 \le r < 1$, $|k| < 1$ and $f : X \to Y$ be an odd mapping satisfying*

$$\|Df(x, y)\| \le \delta\left(\|x\|^r + \|y\|^r\right)$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{1}{|2|(|k|^r - |k|)}\,\delta\|x\|^r$$

for all $x \in X$.

Proof. Let $\varphi : X \times X \to [0, \infty)$ be defined by $\varphi(x, y) = \delta\left(\|x\|^r + \|y\|^r\right)$ for all $x, y \in X$. Then the corollary follows from Theorem 3.5 by $L = |k|^{1-r} < 1$. $\square$

The following example shows that the assumption $|k| < 1$ cannot be omitted in Corollary 3.6.

Example 3.7 Let $p > 2$ be a prime number and $f : \mathbb{Q}_p \to \mathbb{Q}_p$ be defined by $f(x) = x^{1/3}$. Then for $\delta = 1$ and $r = 1/3$,

$$|Df(x, y)|_p \le \max\left\{|x|_p^r, |y|_p^r\right\} \le |x|_p^r + |y|_p^r \quad (x, y \in \mathbb{Q}_p).$$

However, for $k = 2$ we have $|k|_p = |2|_p = 1$ and

$$\left|2^n f(\frac{x}{2^n}) - 2^{n+1} f(\frac{x}{2^{n+1}})\right|_p = \left|2^{2n/3}\right|_p \left|1 - 2^{2/3}\right|_p |x|_p^{1/3} = \left|1 - 2^{2/3}\right|_p |x|_p^{1/3}$$

for all $x \in \mathbb{Q}_p$ and $n \in \mathbb{N}$. Hence $\left\{2^n f(\frac{x}{2^n})\right\}$ is not a Cauchy sequence for each nonzero $x \in \mathbb{Q}_p$.

Theorem 3.8 *Let $\varphi : X \times X \to [0, \infty)$ be a function such that there exists an $0 < L < 1$ with*

$$\varphi(kx, ky) \le |k|^2 L\varphi(x, y) \tag{3.11}$$

for all $x, y \in X$. Let $f : X \to Y$ be an even mapping such that

$$\|Df(x, y)\| \le \varphi(x, y) \tag{3.12}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{1}{|2k^2|(1-L)}\,\varphi(x, 0) \tag{3.13}$$

for all $x \in X$.

Proof. Letting $y = 0$ in (3.12), we get

$$\|f(kx) - k^2 f(x)\| \le \frac{1}{|2|}\varphi(x, 0) \tag{3.14}$$

for all $x \in X$. Consider the set $\Omega := \{g \mid g : X \to Y\}$ and introduce the generalized metric on Ω:

$$d(g, h) = \inf\{C \in (0, \infty) \mid \ \|g(x) - h(x)\| \le C\varphi(x, 0), \quad \forall x \in X\}. \tag{3.15}$$

It is easy to show that (Ω, d) is a complete generalized metric space. We now define a function $J : \Omega \to \Omega$ by

$$(Jg)(x) = \frac{1}{k^2} g(kx), \quad \forall g \in \Omega, x \in X. \tag{3.16}$$

Let $g, h \in \Omega$ and $C \in [0, \infty]$ be an arbitrary constant with $d(g, h) < C$, by the definition of d, it follows

$$\|g(x) - h(x)\| \le C\varphi(x, 0), \quad \forall x \in X. \tag{3.17}$$

By the given hypothesis and the last inequality, one has

$$\left\| \frac{1}{k^2} g(kx) - \frac{1}{k^2} h(kx) \right\| \leq CL\,\varphi(x,0), \quad \forall x \in X. \tag{3.18}$$

Hence, $d(Jg, Jh) \leq L\,d(g,h)$. It follows from (3.14) that $d(Jf,f) \leq 1/|2k^2| < \infty$. Therefore, by Proposition 3.1, J has a unique fixed point $Q : X \to Y$ in the set $\Omega^* = \{g \in \Omega \mid \ d(f,g) < \infty\}$ such that

$$Q(x) := \lim_{n\to\infty} (J^n f)(x) = \lim_{n\to\infty} \frac{1}{k^{2n}} f(k^n x) \tag{3.19}$$

and $Q(kx) = k^2 Q(x)$ for all $x \in X$. Also,

$$d(Q,f) \leq \frac{1}{1-L} d(Jf,f) \leq \frac{1}{|2k^2|(1-L)}. \tag{3.20}$$

This means that (3.13) holds for all $x \in X$. By (3.11), (3.12) and (3.19) we obtain

$$\|DQ(x,y)\| = \lim_{n\to\infty} \frac{1}{|k^{2n}|} \|Df\,(k^n x, k^n y)\| \leq \lim_{n\to\infty} \frac{1}{|k^{2n}|} \varphi\,(k^n x, k^n y) \leq \lim_{n\to\infty} |k|^n \varphi(x,y) = 0,$$

for all $x, y \in X$, and so by [8, Lemma 2.1] we get that the mapping Q is quadratic. □

Corollary 3.9 *Let $\delta > 0$, $r > 2$, $|k| < 1$ and $f : X \to Y$ be an even mapping satisfying*

$$\|Df(x,y)\| \leq \delta\,(\|x\|^r + \|y\|^r)$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{1}{|2|(|k|^2 - |k|^r)}\,\delta\|x\|^r$$

for all $x \in X$.

Proof. Let $\varphi : X \times X \to [0,\infty)$ be defined by $\varphi(x,y) = \delta\,(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Then the corollary follows from Theorem 3.8 by $L = |k|^{r-2} < 1$. □

The following example shows that the assumption $|k| < 1$ cannot be omitted in Corollary 3.9.

Example 3.10 Let $p > 2$ be a prime number and $f : \mathbb{Q}_p \to \mathbb{Q}_p$ be defined by $f(x) = x^4$. Then for $\delta = 1$ and $r = 4$,

$$|Df(x,y)|_p \leq \max\left\{|x|_p^4, |y|_p^4\right\} \leq |x|_p^4 + |y|_p^4 \quad (x, y \in \mathbb{Q}_p).$$

However, for $k = 2$ we have $|k|_p = |2|_p = 1$ and

$$\left| \frac{1}{2^{2n}} f(2^n x) - \frac{1}{2^{2(n+1)}} f(2^{n+1} x) \right|_p = \left|2^{2n}\right|_p |3|_p\,|x|_p^4 = |3|_p |x|_p^4$$

for all $x \in \mathbb{Q}_p$ and $n \in \mathbb{N}$. Hence $\left\{\frac{1}{2^{2n}} f(2^n x)\right\}$ is not a Cauchy sequence for each nonzero $x \in \mathbb{Q}_p$.

Similar to Theorem 3.8, one can obtain the following theorem.

Theorem 3.11 *Let $\varphi : X \times X \to [0,\infty)$ be a function such that there exists an $0 < L < 1$ with*

$$\varphi(x,y) \leq \frac{L}{|k|^2} \varphi\left(\frac{x}{k}, \frac{y}{k}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be an even mapping such that

$$\|Df(x,y)\| \leq \varphi(x,y)$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{L}{|2k^2|(1-L)}\,\varphi(x,0)$$

for all $x \in X$.

Corollary 3.12 *Let $\delta > 0$, $0 \leq r < 2$, $|k| < 1$ and $f : X \to Y$ be an even mapping satisfying*

$$\|Df(x,y)\| \leq \delta\,(\|x\|^r + \|y\|^r)$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{1}{|2|(|k|^2 - |k|^r)}\,\delta\|x\|^r$$

for all $x \in X$.

Proof. Let $\varphi : X \times X \to [0, \infty)$ be defined by $\varphi(x, y) = \delta\left(\|x\|^r + \|y\|^r\right)$ for all $x, y \in X$. Then the corollary follows from Theorem 3.11 by $L = |k|^{r-2} < 1$. □

The following example shows that the assumption $|k| < 1$ cannot be omitted in Corollary 3.12.

Example 3.13 Let $p > 2$ be a prime number and $f : \mathbb{Q}_p \to \mathbb{Q}_p$ be defined by $f(x) = 2$. For $k = 2$, $\delta = 1$ and $r = 0$,

$$|Df(x, y)|_p = |12|_p \le 1 = \delta \quad (x, y \in \mathbb{Q}_p).$$

Note that if $p > 2$, then $|2^n|_p = 1$ for each integer n, we have

$$\left|2^n f(2^{-n}x) - 2^{n+1} f(2^{-(n+1)}x)\right|_p = \left|2^{n+1}\right|_p = 1$$

for all $x \in \mathbb{Q}_p$ and $n \in \mathbb{N}$. Hence $\{2^n f(2^{-n}x)\}$ is not a Cauchy sequence for $x \in \mathbb{Q}$.

We now prove our main theorem in this section.

Theorem 3.14 *Let $\varphi : X \times X \to [0, \infty)$ be a function such that there exists an $0 < L < 1$ with*

$$\varphi(kx, ky) \le |k|^2 L\varphi(x, y) \tag{3.21}$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping such that

$$\|Df(x, y)\| \le \varphi(x, y) \tag{3.22}$$

for all $x, y \in X$. Then there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - A(x) - Q(x)\| \le \frac{1}{|4k^2|(1 - L)} \max\{\varphi(x, 0), \varphi(-x, 0)\} \tag{3.23}$$

for all $x \in X$.

Proof. If we decompose f into the even and the odd parts by putting

$$f_e(x) = \frac{f(x) + f(-x)}{2} \quad \text{and} \quad f_o(x) = \frac{f(x) - f(-x)}{2} \tag{3.24}$$

for all $x \in X$. Then $f(x) = f_e(x) + f_o(x)$. Let $\psi(x, y) = \frac{1}{|2|} \max\{\varphi(x, y), \varphi(-x, -y)\}$, then by (3.21), (3.22) and (3.24) we have

$$\psi(kx, ky) \le |k|^2 L\, \psi(x, y) \le |k| L\, \psi(x, y), \quad \|Df_o(x, y)\| \le \psi(x, y), \quad \|Df_e(x, y)\| \le \psi(x, y).$$

Hence by Theorems 3.2 and 3.8, there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that

$$\|f_o(x) - A(x)\| \le \frac{1}{|2k|(1 - L)}\, \psi(x, 0), \quad \|f_e(x) - Q(x)\| \le \frac{1}{|2k^2|(1 - L)}\, \psi(x, 0)$$

for all $x \in X$. Therefore

$$\begin{aligned}
\|f(x) - A(x) - Q(x)\| &\le \max\{\|f_o(x) - A(x)\|, \ \|f_e(x) - Q(x)\|\} \\
&\le \max\left\{\frac{1}{|2k|(1 - L)}\, \psi(x, 0), \ \frac{1}{|2k^2|(1 - L)}\, \psi(x, 0)\right\} \\
&\le \frac{1}{|4k^2|(1 - L)} \max\{\varphi(x, 0), \varphi(-x, 0)\}
\end{aligned}$$

for all $x \in X$. □

Corollary 3.15 *Let $\delta > 0$, $r > 2$, $|k| < 1$ and $f : X \to Y$ be a mapping for which*

$$\|Df(x, y)\| \le \delta\left(\|x\|^r + \|y\|^r\right)$$

for all $x, y \in X$. Then there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - A(x) - Q(x)\| \le \frac{1}{|4|(|k|^2 - |k|^r)}\, \delta\|x\|^r$$

for all $x \in X$.

Similar to Theorem 3.14, one can obtain the following theorem.

Theorem 3.16 *Let $\varphi : X \times X \to [0, \infty)$ be a function such that there exists an $0 < L < 1$ with*

$$\varphi(x, y) \le \frac{L}{|k|}\varphi(kx, ky)$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping such that

$$\|Df(x, y)\| \le \varphi(x, y)$$

for all $x, y \in X$. Then there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - A(x) - Q(x)\| \le \frac{L}{|4k^2|(1 - L)} \max\{\varphi(x, 0), \varphi(-x, 0)\}$$

for all $x \in X$.

Corollary 3.17 *Let $\delta > 0$, $|k| < 1$, $0 \le r < 1$ and $f : X \to Y$ be a mapping for which*

$$\|Df(x, y)\| \le \delta (\|x\|^r + \|y\|^r)$$

for all $x, y \in X$. Then there exist a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - A(x) - Q(x)\| \le \frac{1}{|4k|(|k|^r - |k|)} \delta\|x\|^r$$

for all $x \in X$.

Remark 3.18 The assumption $|k| < 1$ cannot be omitted in Corollaries 3.15 and 3.17.

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The parameter reduction of soft decision information systems and its algorithm *

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Abstract: In this paper, we introduced a general soft decision information system and reveal that every soft decision information system may be seen as a [0,1]-valued information system. We investigate the parameter reduction of soft decision information systems and give its algorithm.

Keywords: Soft sets; Information systems; Soft decision information systems; Parameter reductions; Importance; Decision rules; Algorithms.

1 Introduction

In 1999, Molodtsov [6] proposed soft set theory as a new mathematical tool for dealing with uncertainties which is free from the difficulties affecting existing method. As reported in [6, 7], a wide range of applications of soft sets have been developed in many different fields, including the smoothness of functions, game theory, operations research, Riemann integration, Perron integration, probability theory and measurement theory.

Rough set theory was initiated by Pawlak [10] for dealing with vagueness and granularity in information systems.

A decision information system means an information system whose attribute set is divided into a condition attribute set and a decision attribute set. In soft set theory, we can consider soft decision information systems.

The parameter reduction of soft decision information systems is a very important problem in soft set theory. To solve decision making problems by using this theory, Gong et al. [3] proposed a bijective soft set. Based on it, authors introduced bijective soft decision information systems and studied their parameter reductions. Xiao et al. [11] presented an exclusive disjunctive soft set by means of bijective soft sets. Based on it, authors introduced exclusive disjunctive soft decision information systems and investigated their parameter reductions. It is worthwhile to mention that methods of parameter reductions in [3, 11] are similar.

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The purpose of this paper is to investigate the parameter reduction of soft decision information systems.

2 Preliminaries

2.1 Soft sets

Definition 2.1 ([6]). *Let U be an initial universe and let A be a set of parameters. A pair (f, A) is called a soft set over U, if f is a map given by $f : A \to 2^U$ where 2^U is the power set of U. We also denote (f, A) by f_A.*

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $e \in A$, $f(e)$ may be considered as the set of e-approximate elements of the soft set f_A.

Example 2.2. *Let $U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$ be a universe consisting of six stores. Let $A = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$ be is a set of status of stores where a_1, a_2, a_3, a_4, a_5, a_6 and a_7 represent respectively the parameters "high empowerment of sales personnel", "medium empowerment of sales personnel", "low empowerment of sales personnel", "good perceived quality of merchandise", "average perceived quality of merchandise", "high traffic location" and "low traffic location", respectively. We define f_A by*

$$f(a_1) = \{h_1, h_6\},\ f(a_2) = \{h_2, h_3, h_5\},\ f(a_3) = \{h_4\},\ f(a_4) = \{h_1, h_2, h_3\},$$
$$f(a_5) = \{h_4, h_5, h_6\},\ f(a_6) = \{h_1, h_2, h_3, h_6\},\ f(a_7) = \{h_4, h_5\}.$$

f_A can be described as the following Table 1. If $h_i \in f(a_j)$, then $h_{ij} = 1$; otherwise $h_{ij} = 0$, where h_{ij} are the entries in the table 1.

Table 1: Tabular representation of the soft set f_A

	a_1	a_2	a_3	a_4	a_5	a_6	a_7
h_1	1	0	0	1	0	1	0
h_2	0	1	0	1	0	1	0
h_3	0	1	0	1	0	1	0
h_4	0	0	1	0	1	0	1
h_5	0	1	0	0	1	0	1
h_6	1	0	0	0	1	1	0

Definition 2.3. *Let f_A be a soft set over U. f_A is called non-trivial, if for any $a \in A$, $f(a) \neq \emptyset$ and $f(a) \neq U$.*

2.2 Information systems

Definition 2.4 ([12]). *Let U be a finite set of objects and let A be a finite set of attributes. The pair (U, A, V, g) is called an information system, if g is an*

information function from $U \times A$ *to* $V = \bigcup_{a \in A} V_a$ *where every* $V_a = \{g(x,a) : a \in A$ *and* $x \in U\}$ *is the values of the attribute* a*.*

If $A = C \cup D$ *and* $C \cap D = \emptyset$*, then* (U, A, V, g) *is called a decision information system where* C *is called a condition attribute set and* D *is called a decision attribute set. Sometimes the decision information system* $(U, C \cup D, V, g)$ *denotes by* (U, C, D, V, g)*.*

If $V \subseteq \{0, 1\}$*, then* (U, A, V, g) *is called a 2-valued information system; If* $V \subseteq [0, 1]$*, then* (U, A, V, g) *is called a [0,1]-valued information system.*

2.3 The relationship between soft sets and information systems

Definition 2.5. *Let* $S = (f, A)$ *be a soft set over* U*. Then* $I_S = (U, A, V, g_s)$ *is called a 2-value information system induced by* S*, where*

$$g_s : U \times A \to V.$$

For any $x \in U$ *and* $a \in A$*,*

$$g_s(x,a) = \begin{cases} 1, & x \in f(a), \\ 0, & x \notin f(a). \end{cases}$$

Definition 2.6. *Let* $I = (U, A, V, g)$ *be a 2-value information system. Then* $S_I = (f_I, A)$ *is called a soft set over* U *induced by* I*, where* $f_I : A \to 2^U$ *and for any* $x \in U$ *and* $a \in A$*,*

$$f_I(a) = \{x \in U : g(x,a) = 1\}.$$

Lemma 2.7. *Let* $S = f_A$ *be a soft set over* U*, let* $I_S = (U, A, V, g_s)$ *be a 2-value information system induced by* S *and let* $S_{I_S} = (f_{I_S}, A)$ *be a soft set over* U *induced by* I_S*. Then* $S = S_{I_S}$*.*

Proof. By Definition 2.6, for any $a \in A$, $f_{I_S}(a) = \{x \in U : g_s(x,a) = 1\}$.

By Definition 2.5, for For any $x \in U$ and $a \in A$,

$$g_s(x,a) = \begin{cases} 1, & x \in f(a), \\ 0, & x \notin f(a). \end{cases}$$

This implies that $g_s(x,a) = 1 \Leftrightarrow x \in f(a)$. So for any $x \in U, a \in A$, $f(a) = f_{I_S}(a)$. Hence $f_A = (f_{I_S}, A)$. This implies that $S = S_{I_S}$. □

Lemma 2.8. *Let* $I = (U, A, V, g)$ *be a 2-value information system, Let* $S_I = (f_{I,A})$ *be a soft set over* U *induced by* I *and let* $I_{S_I} = (U, A, V, g_{s_I})$ *be a 2-value information system induced by* S_I*. Then* $I = I_{S_I}$*.*

Proof. By Definition 2.5, for any $x \in U$ and $a \in A$,

$$g_{s_I}(x,a) = \begin{cases} 1, & x \in f_I(a), \\ 0, & x \notin f_I(a). \end{cases}$$

3

For any $x \in U$ and $a \in A$, by Definition 2.6, $f_I(a) = \{x \in U : g(x,a) = 1\}$. Since $I = (U, A, V, g)$ is a 2-value information system, $g(x,a) = 0$ for $x \notin f_I(a)$, This implies that

$$g(x,a) = \begin{cases} 1, & x \in f_I(a), \\ 0, & x \notin f_I(a). \end{cases}$$

So for any $x \in U$ and $a \in A$, $g_{s_I}(x,a) = g(x,a)$. Hence $g_{s_I} = g$. This implies that $I = I_{S_I}$. □

Theorem 2.9. *Let*

$$\Sigma = \{S : \ S = f_A \ is \ a \ soft \ set \ over \ U\}$$

and

$$\Gamma = \{I : I = (U, A, V, g) \ \textit{is a 2-value information system}\}.$$

Then there exists a one-to-one correspondence between Σ and Γ.

Proof. Two maps $F : \Sigma \to \Gamma$ and $G : \Gamma \to \Sigma$ are defined as follows:

$$F(S) = I_S, \quad for \ \forall S \in \Sigma,$$

$$G(I) = S_I, \quad for \ \forall I \in \Gamma.$$

By Lemma 2.7,

$$G \circ F = i_\Sigma,$$

where $G \circ F$ is the composition of F and G, and i_Σ is the identity map on Σ.

By Lemma 2.8,

$$F \circ G = i_\Gamma,$$

where $G \circ F$ is the composition of G and F, and i_Γ is the identity map on Γ.

Hence F and G are both a one-to-one correspondence. This prove that there exists a one-to-one correspondence between Σ and Γ. □

3 The parameter reduction of soft sets

Soft set itself has classification ability. The parameter reduction of soft sets means reducing the number of parameters to the minimum without distorting its original classification ability.

Since there exists a one-to-one correspondence between "the set of all soft sets" and "the set of all 2-value information systems", we can do the parameter reduction of soft sets with the help of the knowledge reduction in rough set theory.

Definition 3.1. *Let f_A be a soft set over U and let (U, A, V, g) be a 2-value information system induced by f_A. For any $P \subseteq A$, $ind(P)$ is defined as follows:*

$$ind(P) = \{(x,y) \in U \times U : g(x,a) = g(y,a)(\forall a \in P)\}.$$

Proposition 3.2. *Let f_A be a soft set over U and let $P \subseteq A$. Then the following properties hold.*

(1) $ind(P) = \{(x, y) \in U \times U : \{x, y\} \subseteq f(a)$ *or* $\{x, y\} \cap f(a) = \emptyset\ (\forall a \in P)\}$.

(1) $ind(P)$ *is an equivalence relation.*

(2) *If* $P_1 \subseteq P_2 \subseteq A$, *then* $ind(P_1) \supseteq ind(P_2) \supseteq ind(A)$.

Proof. Obviously. □

Sometimes, we replace respectively $ind(P)$ and $U/ind(P)$ by $\mathbf{P}$ and $U/\mathbf{P}$ where

$$U/ind(P) = \{[x]_{ind(P)} : x \in U\}.$$

Specially, we replace $ind(\{a\})$ by $\mathbf{a}$ for $a \in A$.

Proposition 3.3. *Let f_A be a soft set over U and let (U, A, V, g) be a 2-value information system induced f_A. Then for any $a \in A$,*

$$U/\mathbf{a} = \{f(a),\ U - f(a)\}.$$

Proof. Obviously. □

Definition 3.4. *Let f_A be a soft set over U.*

(1) $A^* \subseteq A$ *is called a parameter reduction of f_A (brief. a f_A-parameter reduction), if* $ind(A) = ind(A^*)$ *and* $ind(A) \neq ind(P)$ *for any* $P \subsetneq A^*$.

(2) *The intersection set of all f_A-parameter reductions is called the core of f_A. We denote it by core(f_A).*

In this paper, we denote the set of all f_A-parameter reductions by $pr(f_A)$. Then

$$core(f_A) = \cap\ pr(f_A).$$

Proposition 3.5. *Let f_A be a soft set over U. Then $pr(f_A) \neq \emptyset$.*

Proof. (1) If $ind(A) \neq ind(A - \{a\})$ for any $a \in A$, then $A \in pr(f_A)$.

(2) If $ind(A) = ind(A - \{a\})$ for some $a \in A$, then we consider $P_1 = A - \{a\}$. If $ind(A) \neq ind(P_1 - \{p_1\})$ for any $p_1 \in P_1$, then $P_1 \in pr(f_A)$. Otherwise, we consider $P_1 - \{p_1\}$ again.

Repeat the above process. Since A is a finite set, we can find at least a f_A-parameter reduction.

Thus, $pr(f_A) \neq \emptyset$. □

Example 3.6. *Let $U = \{h_1, h_2, h_3, h_4, h_5\}$, $A = \{a_1, a_2, a_3, a_4\}$ and let f_A be a soft set over U, defined as follows*

$$f(a_1) = \{h_1, h_2, h_5\},\quad f(a_2) = \emptyset,\ f(a_3) = \{h_3\},\ f(a_4) = \{h_3, h_4\}.$$

By Proposition 3.3, we have

$$U/\mathbf{a_1} = \{\{h_1, h_2, h_5\}, \{h_3, h_4\}\},\ U/\mathbf{a_2} = \{\{h_1, h_2, h_3, h_4, h_5\}\},$$

$U/\mathbf{a_3} = \{\{h_3\}, \{h_1, h_2, h_4, h_5\}\}$, $U/\mathbf{a_4} = \{f(a_4), U - f(a_4)\} = \{\{h_3, h_4\}, \{h_1, h_2, h_5\}\}$.

And
$U/\mathbf{A} = \{\{h_1, h_2, h_5\}, \{h_3\}, \{h_4\}\}$.

$U/ind(A - \{a_1\}) = \{\{h_1, h_2, h_5\}, \{h_3\}, \{h_4\}\} = U/ind(A)$.
$U/ind(A - \{a_2\}) = \{\{h_1, h_2, h_5\}, \{h_3\}, \{h_4\}\} = U/ind(A)$.
$U/ind(A - \{a_3\}) = \{\{h_1, h_2, h_5\}, \{h_3, h_4\}\} \neq U/ind(A)$.
$U/ind(A - \{a_4\}) = \{\{h_1, h_2, h_5\}, \{h_3\}, \{h_4\}\} = U/ind(A)$.
This implies that
$U/ind(\{a_2, a_3, a_4\}) = U/ind(\{a_1, a_3, a_4\}) = U/ind(\{a_1, a_2, a_3\}) = U/ind(A)$.

Since $U/ind(\{a_2, a_3, a_4\}) = U/ind(\{a_3, a_4\})$, $U/ind(\{a_3, a_4\}) \neq U/ind(\{a_3\})$ *and* $U/ind(\{a_3, a_4\}) \neq U/ind(\{a_4\})$, $\{a_3, a_4\}$ *is a* f_A*-parameter reduction.*

Since $U/ind(\{a_1, a_3, a_4\}) = U/ind(\{a_1, a_3\})$, $U/ind(\{a_1, a_3\}) \neq U/ind(\{a_1\})$ *and* $U/ind(\{a_1, a_3\}) \neq U/ind(\{a_4\})$, $\{a_1, a_3\}$ *also is a* f_A*-parameter reduction.*

Thus,

$$pr(f_A) = \{\{a_3, a_4\}, \{a_1, a_3\}\}, \quad core(f_A) = \{a_3, a_4\} \cap \{a_1, a_3\} = \{a_3\}.$$

Definition 3.7. *Let* f_A *be a soft set over* U *and let* $pr(f_A) = \{C_i : 1 \leq i \leq n\}$. *Then*

(1) $a \in A$ *is called core, if* $a \in \bigcap_{i=1}^{n} C_i = core(f_A)$.

(2) $a \in A$ *is called relative indispensable, if* $a \in \bigcup_{i=1}^{n} C_i - core(f_A)$.

(3) $a \in A$ *is called absolutely dispensable, if* $a \in A - \bigcup_{i=1}^{n} C_i$.

(4) $a \in A$ *is called dispensable, if* $a \in A - core(f_A)$.

Obviously, $a \in A$ *is dispensable if and only if* a *is relative indispensable or absolutely dispensable.*

Example 3.8. *In Example 3.6, we have*
(1) a_3 *is core.*
(2) a_1 *and* a_4 *are relative indispensable.*
(3) a_2 *is absolutely dispensable.*
(4) a_1, a_2 *and* a_4 *are dispensable.*

Definition 3.9. *Let* $\mathcal{A}, \mathcal{B} \subset 2^U$. $\mathcal{A}$ *is called a refinement of* $\mathcal{B}$, *if for any* $A \in \mathcal{A}$, *there exists* $B \in \mathcal{B}$ *such that* $A \subseteq B$. *We denote it by* $\mathcal{A} \leq \mathcal{B}$.

Lemma 3.10. *Let* R *and* ρ *be two equivalence relations on* U. *If* $R \subseteq \rho$, *then* $U/R \leq U/\rho$.

Proof. Suppose $A \in X/R$. Since R is an equivalence relation over X, there exists $x \in X$, such that $A = [x]_R$.

Suppose $y \in [x]_R$. Then xRy. This implies that $(x, y) \in R$. Since $R \subseteq \rho$, $(x, y) \in \rho$. This implies that $y \in [x]_\rho$. Then $[x]_R \subseteq [x]_\rho$.

Pick $B = [x]_\rho$, then $A \subseteq B$, thus $X/R \leq X/\rho$. $\square$

Theorem 3.11. *Let f_A be a soft set over U. Then*
(1) $|pr(f_A)| = 1$ *if and only if* $core(f_A) \in pr(f_A)$.
(2) $a \in core(f_A)$ *if and only if* $U/ind(A) \neq U/ind(A - \{a\})$.
(3) $a \in A$ *is dispensable* *if and only if* $U/ind(A) = U/ind(A - \{a\})$.

Proof. (1) Sufficiency. Let $core(f_A) \in pr(f_A)$. Note that $pr(f_A) = \{C_i : 1 \leq i \leq n\}$. We only need to prove that $n = 1$.

Suppose $n = 2$. Then there are only two different f_A-parameter reductions C_1 and C_2. Suppose $C_1 \subsetneq C_2$. Since $C_2 \in pr(f_A)$, $ind(A) \neq ind(C_1)$. This implies that $C_1 \notin pr(f_A)$, a contradiction; Suppose $C_2 \subsetneq C_1$. Similarly, this implies a contradiction. Suppose that $C_1 \nsubseteq C_2$ and $C_2 \nsubseteq C_1$. Obviously, $core(f_A) = C_1 \cap C_2$ and $core(f_A) \subsetneq C_1$. Since $C_1 \in pr(f_A)$, $ind(A) \neq ind(core(f_A))$. This implies that $core(f_A) \notin pr(f_A)$, a contradiction.

Suppose $n \geq 3$. Similarly, this also implies a contradiction.

Thus, $|pr(f_A)| = 1$.

Necessity. This is obvious.

(2) Sufficiency. Suppose $U/ind(A) \neq U/ind(A - \{a\})$. We claim that $a \in C_i$ for any $1 \leq i \leq n$.

Otherwise. $a \notin C_{i_0}$ for some C_{i_0}. This implies that $U/ind(A) = U/ind(C_{i_0})$. Since $ind(C_{i_0}) \supseteq ind(A - \{a\}) \supseteq ind(A)$, $U/ind(C_{i_0}) \geq U/ind(A - \{a\}) \geq U/ind(A)$, by Lemma 3.7. So $U/ind(A) = U/ind(A - \{a\})$, a contradiction.

This implies that $a \in core(f_A)$.

Necessity. Suppose $U/ind(A) = U/ind(A - \{a\})$. Since $pr(f_A) \neq \emptyset$, we can find at least a $B_1' \subseteq A - \{a\}$ such that $B_1' \in pr(f_A)$. So $a \notin core(f_A)$, a contradiction.

Thus $U/ind(A) \neq U/ind(A - \{a\})$.

(3) Sufficiency. Suppose $U/ind(A) = U/ind(A - \{a\})$. Since $A - \{a\}$ is a finite set, there exists $B_2 \subseteq A - \{a\}$ such that $B_2 \in pr(f_A)$. So $a \notin core(f_A)$. This implies that $a \in A - core(f_A)$.

Thus, a is a dispensable parameter.

Necessity. Suppose $U/ind(A) \neq U/ind(A - \{a\})$. Similar to (2), we have $a \in core(f_A)$. Then $a \notin A - core(f_A)$. Since a is a dispensable parameter, $a \in A - core(f_A)$, a contradiction.

Thus, $U/ind(A) = U/ind(A - \{a\})$. $\square$

4 A soft decision information system and its parameter reduction

4.1 The concept of soft decision information systems

Definition 4.1. *Let U be an initial universe, let A and B be two sets of parameters. (f_A, g_B, U) is called a soft decision information system, if (f_A, g_B, U) satisfies*

(1) *f_A and g_B are two soft sets over U;*

(2) $\bigcup\limits_{a\in A} f(a) = \bigcup\limits_{b\in B} g(b) = U$;

(3) $A \cap B = \emptyset$.

In this case, f_A is called the condition parameter set, g_B is called the decision parameter set.

Example 4.2. *Let $U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$ be a universe consisting of six stores. Let $A = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$ be is a set of status of stores where a_1, a_2, a_3, a_4, a_5, a_6 and a_7 represent respectively the parameters "high empowerment of sales personnel", "medium empowerment of sales personnel", "low empowerment of sales personnel", "good perceived quality of merchandise", "average perceived quality of merchandise", "high traffic location" and "low traffic location", respectively. And let $B = \{b_1, b_2\}$ represent respectively the parameters "store profit" and "store loss", respectively. We define f_A and g_B as follows*

$$f(a_1) = \{h_1, h_6\},\ f(a_2) = \{h_2, h_3, h_5\},\ f(a_3) = \{h_4\},\ f(a_4) = \{h_1, h_2, h_3\},$$
$$f(a_5) = \{h_4, h_5, h_6\},\ \ f(a_6) = \{h_1, h_2, h_3, h_6\},\ \ f(a_7) = \{h_4, h_5\}.$$

$$g(b_1) = \{h_1, h_3, h_6\},\ \ g(b_2) = \{h_2, h_4, h_5\}.$$

Then (f_A, g_B, U) is a soft decision system over U.

Remark 4.3. *If f and g are the same corresponding law, then the soft decision information system (f_A, g_B, U) may be seen as the soft set $f_{A\cup B}$.*

Proposition 4.4. *Every soft decision information system may be seen as a [0,1]-valued information systems.*

Proof. Let (f_A, g_B, U) be a soft decision information system. We define

$$g(x, e) = \begin{cases} |f(e)|/|U|, & x \in U,\ e \in A, \\ |g(e)|/|U|, & x \in U,\ e \in B. \end{cases}$$

Then $(U, A \cup B, V, g)$ is a [0,1]-valued information system.

Thus, (f_A, g_B, U) may be seen as the [0,1]-valued information system $(U, A\cup B, V, g)$. □

4.2 The parameter reduction of soft decision information systems

The parameter reduction of soft decision information systems means reducing the number of condition parameters to the minimum without distorting its

original classification ability of knowledge discovering.

By Remark 4.3, soft sets and soft decision systems are closely related. So we can do the parameter reduction of soft decision information systems by using the parameter reduction of soft sets.

Definition 4.5. *Let (f_A, g_B, U) be a soft decision information system and let $C \in pr(f_A)$. Then (f_C, g_B, U) is called a soft decision information system after C.*

Definition 4.6. *Let (f_A, g_B, U) be a soft decision information system and let $C \in pr(f_A)$. Then $Pos_C B$ is called C-positive region of B, where*

$$Pos_C B = \bigcup_{X \in U/\boldsymbol{B}} \underline{\boldsymbol{C}}X = \bigcup \{[x]_{\boldsymbol{C}} : \exists y \in U, s.t.[x]_{\boldsymbol{C}} \subseteq [y]_{\boldsymbol{B}}\}.$$

Definition 4.7. *Let (f_A, g_B, U) be a soft decision information system and let $C \in pr(f_A)$. If*

$$Pos_C B = Pos_{C-\{a\}} B, \ for \ some \ a \in C,$$

then a is called B-dispensable in C. Or a is called B-indispensable in C.

Definition 4.8. *Let (f_A, g_B, U) be a soft decision information system, let $C \in pr(f_A)$ and let (f_C, g_B, U) be a soft decision information system after C.*

(1) *$C^* \subseteq C$ is called a B-reduction of C, if $Pos_C B = Pos_{C^*} B$ and $Pos_C B \neq Pos_D B$ for any $D \subsetneq C$.*

(2) *The intersection set of all B-reductions of C is called the B-core of C. We denote it by $core_C(f_A, g_B, U)$.*

In this paper, we denote the set of all B-reductions of C by $pr_C(f_A, g_B, U)$. Then

$$core_C(f_A, g_B, U) = \cap \ pr_C(f_A, g_B, U).$$

Proposition 4.9. *Let (f_A, g_B, U) be a soft decision information system, let $C \in pr(f_A)$ and let (f_C, g_B, U) be a soft decision information system after C. Then $pr(f_C, g_B, U) \neq \emptyset$.*

Proof. (1) If $Pos_C B = Pos_{C-\{a\}} B$ for any $a \in A$, then C itself is a B-reduction of C.

(2) If $ind(C) = ind(C - \{a\})$ for some $a \in C$, then we consider $B_1 = C - \{a\}$. If $ind(C) \neq ind(B_1 - \{b_1\})$ for any $b_1 \in B_1$, B_1 is a B-reduction of C. Otherwise, we consider $B_1 - \{b_1\}$ again.

Repeat the above process. Since C is a finite set, we can find at least a B-reduction of C.

Thus, $pr(f_C, g_B, U) \neq \emptyset$. □

Example 4.10. *In Example 4.2, similar to the process of Example 3.6, we obtain four f_A-parameter reductions: C_1, C_2, C_3 and C_4, where $C_1 = \{a_2, a_3, a_4\}$, $C_2 = \{a_2, a_4, a_6\}$, $C_3 = \{a_1, a_3, a_4\}$ and $C_4 = \{a_1, a_2, a_4\}$.*

Moreover, a_4 is core parameter; a_1, a_2, a_3 and a_6 are relative indispensable parameters. There is no absolutely dispensable parameter.

Obviously,

$$U/\mathbf{A} = U/\mathbf{C_i} = \{\{h_1\}, \{h_2, h_3\}, \{h_4\}, \{h_5\}, \{h_6\}\}(i = 1, 2, 3, 4).$$

$$U/\mathbf{B} = \{\{h_1, h_3, h_6\}, \{h_2, h_4, h_5\}\}$$

and

$$Pos_{C_i}B = \{h_1, h_4, h_5, h_6\}\ (i = 1, 2, 3, 4).$$

(1) *We consider $C_1 = \{a_2, a_3, a_4\}$. Then*

$Pos_{C_1-\{a_2\}}B = Pos_{ind\{a_3,a_4\}}B = \{h_4\} \neq Pos_{C_1}B,$
$Pos_{C_1-\{a_3\}}B = Pos_{ind\{a_2,a_4\}}B = \{h_1, h_5\} \neq Pos_{C_1}B,$
$Pos_{C_1-\{a_4\}}B = Pos_{ind\{a_2,a_3\}}B = \{h_1, h_4, h_6\} \neq Pos_{C_1}B.$

Since $Pos_{C_1}B = Pos_{ind\{a_2,a_3,a_4\}}B \neq Pos_{ind\{a_3,a_4\}}B$, $Pos_{ind\{a_2,a_3,a_4\}}B \neq Pos_{ind\{a_2,a_4\}}B$ and $Pos_{ind\{a_2,a_3,a_4\}}B \neq Pos_{ind\{a_2,a_3\}}B$, $\{a_2, a_3, a_4\}$ is a B-reduction.

There is only a B-reduction of C_1.

(2) *We consider $C_2 = \{a_2, a_4, a_6\}$. Then*

$Pos_{C_2-\{a_4\}}B = Pos_{ind\{a_2,a_6\}}B = \{h_1, h_4, h_5, h_6\} = Pos_{ind\{a_4,a_6\}}B,$
$Pos_{ind\{a_2\}}B = \emptyset \neq Pos_{C_2}B,$
$Pos_{ind\{a_6\}}B = \{h_4, h_5\} \neq Pos_{ind\{a_4,a_6\}}B.$

Since $Pos_{C_2}B = Pos_{ind\{a_2,a_4,a_6\}}B = Pos_{ind\{a_2,a_6\}}B$, $Pos_{ind\{a_2,a_6\}}B \neq Pos_{ind\{a_2\}}B$ and $Pos_{ind\{a_2,a_6\}}B \neq Pos_{ind\{a_6\}}B$, $\{a_2, a_6\}$ is a B-reduction.

There is only a B-reduction of C_2.

(3) *We consider $C_3 = \{a_1, a_3, a_4\}$. Then*

$Pos_{C_3-\{a_3\}}B = Pos_{ind\{a_1,a_4\}}B = \{h_1, h_4, h_5, h_6\} = Pos_{C_3}B,$
$Pos_{ind\{a_1\}}B = \{h_1, h_6\} \neq Pos_{C_3}B,$
$Pos_{ind\{a_4\}}B = \emptyset \neq Pos_{C_3}B.$

Since $Pos_{C_3}B = Pos_{ind\{a_1,a_3,a_4\}}B = Pos_{ind\{a_1,a_4\}}B$, $Pos_{ind\{a_1,a_4\}}B \neq Pos_{ind\{a_1\}}B$ and $Pos_{ind\{a_1,a_4\}}B \neq Pos_{ind\{a_4\}}B$, $\{a_1, a_4\}$ is a B-reduction.

There is only a B-reduction of C_3.

(4) *We consider $C_4 = \{a_1, a_2, a_4\}$.*

$\{a_1, a_4\}$ is only a B-reduction of C_4.

Therefore, $pr_{C_1}(f_A, g_B, U) = \{\{a_2, a_3, a_4\}\}$, $pr_{C_2}(f_A, g_B, U) = \{\{a_2, a_6\}\}$, $pr_{C_3}(f_A, g_B, U) = pr_{C_4}(f_A, g_B, U) = \{\{a_1, a_4\}\}$.

$core_{C_1}(f_A, g_B, U) = \{a_2, a_3, a_4\}$, $core_{C_2}(f_A, g_B, U) = \{a_2, a_6\}$,
$core_{C_3}(f_A, g_B, U) = core_{C_4}(f_A, g_B, U) = \{a_1, a_4\}$.

Definition 4.11. *Let (f_A, g_B, U) be a soft decision information system, let $C \in pr(f_A)$ and let (f_C, g_B, U) be a soft decision information system after C. Denote $pr_C(f_A, g_B, U) = \{D_i : 1 \leq i \leq l\}$. Then*

(1) $a \in C$ *is called B-core of C, if* $a \in \bigcap_{i=1}^{l} D_i = core_C(f_A, g_B, U)$.

(2) $a \in A$ *is called B-relative indispensable of C, if* $a \in \bigcup_{i=1}^{l} D_i - core_C(f_A, g_B, U)$.

(3) $a \in C$ *is called B-absolutely dispensable of C, if* $a \in C - \bigcup_{i=1}^{l} D_i$.

(4) $a \in C$ *is called B-dispensable of C, if* $a \in C - core_C(f_A, g_B, U)$.

Obviously, $a \in C$ is dispensable of C if and only if a is B-relative indispensable of C or B-absolutely dispensable of C.

Example 4.12. *In Example 4.10, for $C_1 = \{a_2, a_3, a_4\}$, Since there is only a B-reduction of C_1, a_2,a_3 and a_4 are B-core of C_1. There is no B-relative indispensable parameter of C, B-absolutely dispensable parameter of C and B-dispensable parameter of C.*

Let $pr(f_A) = \{C_i : 1 \le i \le n\}$. Put

$$pr(f_A, g_B, U) = \bigcup_{i=1}^{n} pr_{C_i}(f_A, g_B, U).$$

Definition 4.13. *Let (f_A, g_B, U) be a soft decision information system. Define the core of (f_A, g_B, U) by*

$$core(f_A, g_B, U) = \cap\, pr(f_A, g_B, U).$$

Obviously, $core(f_A, g_B, U) = \bigcap_{i=1}^{n} core_{Ci}(f_A, g_B, U)$.

Example 4.14. *In Example 4.10, we have*

$pr(f_A, g_B, U)$
$= pr_{C_1}(f_A, g_B, U) \cup pr_{C_2}(f_A, g_B, U) \cup pr_{C_3}(f_A, g_B, U) \cup pr_{C_4}(f_A, g_B, U)$
$= \{\{a_2, a_3, a_4\}, \{a_2, a_6\}, \{a_1, a_4\}\}$.

$core(f_A, g_B, U) = \{a_2, a_3, a_4\} \cap \{a_2, a_6\} \cap \{a_1, a_4\} \cap \{a_1, a_4\} = \emptyset$.

Definition 4.15. *Let (f_A, g_B, U) be a soft decision information system, let $pr(f_A) = \{C_i : 1 \le i \le n\}$ and let (f_{C_i}, g_B, U) be a soft decision information system after C_i $(1 \le i \le n)$. Then*

(1) $a \in A$ *is called B-core, if* $a \in \bigcap_{i=1}^{n} core_{C_i}(f_A, g_B, U) = core(f_A, g_B, U)$.

(2) $a \in A$ *is called B-relative indispensable, if* $a \in \bigcup_{i=1}^{n} C_i - core(f_A, g_B, U)$.

(3) $a \in A$ *is called B-absolutely dispensable, if* $a \in A - \bigcup_{i=1}^{n} C_i$.

(4) $a \in A$ *is called B-dispensable, if* $a \in A - core(f_A, g_B, U)$.

Obviously, $a \in A$ is dispensable if and only if a is B-relative indispensable or B-absolutely dispensable.

Example 4.16. *In Example 4.14, we have*

(1) *There is no B-core parameter in A.*

(2) a_1,a_2,a_3,a_4 *and* a_6 *are B-relative indispensable.*

(3) a_5 *and* a_7 *are B-absolutely dispensable.*

(4) a_1,a_2,a_3,a_4,a_5,a_6 *and* a_7 *are B-dispensable.*

Lemma 4.17. *Let X be a set and let R, λ and ρ be three equivalence relations on X. If $R \subseteq \rho$, then $Pos_R\lambda \supseteq Pos_\rho\lambda$.*

Proof. Suppose $x \in Pos_\rho\varrho$. Then there exists $[y]_\varrho$ such that $[x]_\rho \subseteq [y]_\varrho$. Note that $X/R \leq X/\rho$. By Lemma 3.7, $[x]_R \subseteq [x]_\rho$. So $[x]_R \subseteq [x]_\varrho$. This implies that $x \in Pos_R\varrho$. Thus $Pos_R\varrho \supseteq Pos_\rho\varrho$. □

Proposition 4.18. *Let (f_A, g_B, U) be a soft decision information system, let $C \in pr(f_A)$ and let (f_C, g_B, U) be a soft decision information system after C. Denote $pr_C(f_A, g_B, U) = \{D_i : 1 \leq i \leq l\}$. Then*

$$a \in core_C(f_A, g_B, U) \ \ if\ and\ only\ if \ \ Pos_{C-\{a\}}B \neq Pos_C B.$$

Proof. Sufficiency. Suppose $Pos_{C-\{a\}}B \neq Pos_C B$. Then $a \in D_i$ for any $1 \leq i \leq l$.

Otherwise. $a \notin D_{i0}$ for some D_{i0}. This implies that $Pos_{D_{i0}}B = Pos_C B$. Since $ind(D_{i0}) \supseteq ind(C - \{a\}) \supseteq ind(C)$, $Pos_{D'_{i_0}}B \subseteq Pos_{C-\{a\}}B \subseteq Pos_C B$ by Lemma 4.17. Then $Pos_{C-\{a\}}B = Pos_C B$, a contradiction.

This implies that $a \in core_C(f_A, g_B, U)$.

Necessity. Let $a \in core_C(f_A, g_B, U)$. Suppose $Pos_{C-\{a\}}B = Pos_C B$. Since $C - \{a\}$ is a finite set, we can find at least a $F \subseteq C - \{a\}$ such that $F \in pr_c(f_A, g_B, U)$. So $a \notin core_C(f_A, g_B, U)$, a contradiction.

Thus $Pos_{C-\{a\}}B \neq Pos_C B$. □

Proposition 4.19. *Let (f_A, g_B, U) be a soft decision information system, let $pr(f_A) = \{C_i : 1 \leq i \leq n\}$ and let (f_{C_i}, g_B, U) be a soft decision information system after C_i $(1 \leq i \leq n)$. Then*

$$core(f_A, g_B, U) \subseteq core(f_A).$$

Proof. Suppose $a \in core(f_A, g_B, U)$. By Proposition 4.18, for any $1 \leq i \leq n$, $Pos_{C_i-\{a\}}B \neq Pos_{C_i}B$. This implies that $U/ind(C_i - \{a\}) \neq U/ind(C_i)$ for any $1 \leq i \leq n$. Since (f_{C_i}, g_B, U) are a soft decision information system after C_i, $U/ind(A) = U/ind(C_i)$ for any $1 \leq i \leq n$. This implies that for any $1 \leq i \leq n$,

$$\{(x, y) \in U \times U : \forall a \in A, g_s(x, a) = g_s(y, a)\}$$

$$= \{(x, y) \in U \times U : \forall c \in C_i, g_s(x, c) = g_s(y, c)\}.$$

Then

$$\{(x, y) \in U \times U : \forall a' \in A - \{a\}, g_s(x, a') = g_s(y, a')\}$$

$$= \{(x, y) \in U \times U : \forall c' \in C_i - \{a\} \ and \ g_s(x, c') = g_s(y, c')\}$$

and

$$U/ind(A - \{a\}) = U/ind(C_i - \{a\}).$$

So $U/ind(A) \neq U/ind(A - \{a\})$. By Theorem 3.8, $a \in core(f_A)$.
Hence, $core(f_A, g_B, U) \subseteq core(f_A)$. □

Example 4.20. *In Example 4.2, $a_4 \in core(f_A)$ and $a_4 \notin core(f_A, g_B, U)$. Thus*

$$core(f_A, g_B, U) \not\supseteq core(f_A).$$

5 The importance of condition parameters and decision rules in soft decision information systems

Definition 5.1. *Let (f_A, g_B, U) be a soft decision information system, let $C \in pr(f_A)$ and let (f_C, g_B, U) be a soft decision information system after C. f_C is said to depend on g_B to a degree $k (0 \leq k \leq 1)$, denote by $f_C \Rightarrow_k g_B$, and k is called soft decision information system dependency of (f_C, g_B, U), if*

$$k = \gamma_C(B) = \frac{|Pos_C B|}{|U|}.$$

If $k = 1$, we say f_C is full depended on g_B.
If $k = 0$, we say f_C is not depended on g_B.

Proposition 5.2. *Let (f_A, g_B, U) be a soft decision information system, let $C \in pr(f_A)$ and let (f_C, g_B, U) be a soft decision information system after C. If there is a $C' \subseteq C$, s.t.*

$$\gamma_{C'}(B) = \frac{|Pos_{C'} B|}{|U|} = k,$$

then $Pos_{C'} B = Pos_C B$.

Proof. Since $C' \subseteq C$, $ind(C') \supseteq ind(C)$. By Lemma 4.17, $Pos_{C'} B \subseteq Pos_C B$. This implies that $\gamma_{C'}(B) \leq k$. Since $\gamma_{C'}(B) = k$, $Pos_{C'} B = Pos_C B$. □

Example 5.3. *Let (f_A, g_B, U) be a soft decision information system with $U = \{h_1, h_2, h_3, h_4, h_5, h_6, h_7\}$, $A = \{a_1, a_2, a_3, a_4, a_5\}$ and $B = \{b_1, b_2\}$. f_A and g_B are defined as follows*

$f(a_1) = \{h_1, h_2, h_3\}$, $f(a_2) = \{h_2, h_5, h_7\}$, $f(a_3) = \{h_4, h_7\}$,
$f(a_4) = \{h_1, h_4, h_5\}$, $f(a_5) = \{h_1, h_2, h_3, h_7\}$.

$g(b_1) = \{h_1, h_2, h_3\}$, $g(b_2) = \{h_4, h_5, h_6, h_7\}$.

Similar to Example 3.6, we obtain $C_1 = \{a_1, a_2, a_4\}$ is a f_A-parameter reduction and $U/\mathbf{A} = U/\mathbf{C_1} = \{\{h_1\}, \{h_2\}, \{h_3\}, \{h_4\}, \{h_5\}, \{h_6\}, \{h_7\}\}$.

Similar to Example 4.10, we obtain $D = \{a_1\}$ *is a* B*-reduction of* C_1 *and* $Pos_{C_1}B = Pos_D B = \{h_1, h_2, h_3, h_4, h_5, h_6, h_7\}$.

Let $C' = \{a_1, a_2\} \subset C_1$. *Obviously,* C' *is not a* B*-reduction. However,* $Pos_{C'}B = \{h_1, h_2, h_3, h_4, h_5, h_6, h_7\} = Pos_{C_1}B$.

Definition 5.4. *Let* (f_A, g_B, U) *be a soft decision information system, let* $C \in pr(f_A)$ *and let* (f_C, g_B, U) *be a soft decision information system after* C. *For* $C' \subseteq C$,

$$\sigma_{CB}(C') = \gamma_C(B) - \gamma_{C-C'}(B)$$

is called the importance of parameter subset C' *on the decision parameter set* B.

Especially, when $C' = \{a\}$, *the importance of parameter* $a \in C$ *on the decision-making parameter set* B *is* $\sigma_{CB}(a)$, *where*

$$\sigma_{CB}(a) = \gamma_C(B) - \gamma_{C-\{a\}}(B).$$

Example 5.5. *In Example 4.10, for* B*-reduction* C_2, *we have*

$\sigma_{C_2B}(a_2) = \gamma_{C_2}(B) - \gamma_{C-\{a_2\}}(B) = \frac{4}{6} - \frac{3}{6} = \frac{1}{6}$.
$\sigma_{C_2B}(a_4) = \gamma_{C_2}(B) - \gamma_{C-\{a_4\}}(B) = \frac{4}{6} - \frac{4}{6} = 0$.
$\sigma_{C_2B}(a_6) = \gamma_{C_2}(B) - \gamma_{C-\{a_6\}}(B) = \frac{4}{6} - \frac{2}{6} = \frac{2}{6}$.

So, in the soft decision information system of Example 4.10, for B*-reduction* C_2. a_6*(high traffic location) is the most important parameter,* a_2*(medium empowerment of sales personnel) is in the second place and* a_4 *(good perceived quality of merchandise) is not important for store profit or loss.*

Definition 5.6. *Let* (f_A, g_B, U) *be a soft decision information system.* X_i *and* Y_i *represent the equivalence classes of* U/A *and* U/B *respectively,* $des(X_i)$ *represents the condition parameter values for* X_i *and* $des(Y_j)$ *represents the decision parameter values for* Y_j. *Decision rules are defined as follows:*

$r_{ij} : des(X_i) \to des(Y_j)$, *where* $Y_j \cap X_i \neq \emptyset$,

The decisive factor of rule: $\mu(X_i, Y_j) = |Y_j \cap X_i|/|X_i|$.

If $\mu(X_i, Y_j) = 1$, *then* r_{ij} *is decisive; if* $0 < \mu(X_i, Y_j) < 1$, *then* r_{ij} *is indecisive.*

Example 5.7. *Let* $U = \{h_1, h_2, h_3, h_4\}$ *be a set of patients under diagnosis. Let* $A = \{a_1, a_2\}$ *be a set of symptoms about patients where* a_1 *and* a_2 *represent respectively parameters " headache" and " nasal obstruction". Let* $B = \{b_1, b_2\}$ *be a set of the probable diagnosis results where* b_1 *and* b_2 *represent parameters " flu" and " nasitis" respectively. Then* (f_A, g_B, U) *be a soft decision information system.*

By Proposition 3.3, we have

$U/\mathbf{A} = \{X_1, X_2, X_3\}$, *where* $X_1 = \{h_1, h_2\}, X_2 = \{h_3\}, X_3 = \{h_4\}$.
$U/\mathbf{B} = \{Y_1, Y_2, Y_3, Y_4\}$, *where* $Y_1 = \{h_1\}, Y_2 = \{h_2\}, Y_3 = \{h_3\}, Y_4 = \{h_4\}$.
Hence

(1) *We have the following deterministic rule*

$(a_1, 1) \wedge (a_2, 0) \Longrightarrow (b_1, 1) \wedge (b_3, 0)$;

(This means that if " headache", then "flu".)

(2) *We have the following deterministic rules*

$(a_1, 1) \wedge (a_2, 1) \Longrightarrow (b_1, 1) \wedge (b_3, 1)$ *and the factor of rule is 0.5;*

(This means that if " headache" and "nasal obstruction", then flu and nasitis(0.5)".)

$(a_1, 1) \wedge (a_2, 1) \Longrightarrow (b_1, 0) \wedge (b_3, 0)$ *and the factor of rule is 0.5.*

(This means that if " headache" and "nasal obstruction", then nasitis(0.5)".)

6 Algorithms

Algorithms 6.1. *Let (f_A, g_B, U) be a soft decision information system. Denote $pr(f_A, g_B, U) = \{H_j : 1 \le j \le h\}$. The algorithm on its parameter reduction is shown as follows:*

Input: The soft decision information system (f_A, g_B, U).
Output: The soft decision information systems (f_{H_j}, g_B, U) after B-reduction, the importance of every parameter in H_j and decision rules.

Step 1. *Calculate f_A-parameter reductions and obtain the soft decision information systems (f_{C_i}, g_B, U);*
Step 2. *Calculate B-reductions and obtain the soft decision information systems (f_{H_j}, g_B, U);*
Step 3. *Calculate the importance of every parameter in H_j.*
Step 4. *Output the soft decision information systems (f_{H_j}, g_B, U) after B-reduction, the importance of every parameter in H_j and decision rules.*

Example 6.2. *In Example 4.10, we have*

In Step 1. *We can calculate four f_A-parameter reductions: C_1, C_2, C_3 and C_4, where $C_1 = \{a_2, a_3, a_4\}$, $C_2 = \{a_2, a_4, a_6\}$, $C_3 = \{a_1, a_3, a_4\}$ and $C_4 = \{a_1, a_2, a_4\}$ respectively.*
And obtain soft decision information systems $(f_{C_i}, g_B, U)(i = 1, 2, 3, 4)$.

In Step 2. *We can calculate three B-reductions: H_1, H_2 and H_3 . Where $H_1 = \{a_2, a_3, a_4\}$, $H_2 = \{a_2, a_6\}$ and $H_3 = \{a_1, a_4\}$ respectively.*
And soft decision information systems (f_{H_1}, g_B, U), (f_{H_2}, g_B, U) and (f_{H_3}, g_B, U).

In Step 3. *For H_1, we have*

$\sigma_{H_1 B}(a_2) = \gamma_{H_1}(B) - \gamma_{H_1 - \{a_2\}}(B) = \frac{4}{6} - \frac{1}{6} = \frac{3}{6}$.
$\sigma_{H_1 B}(a_3) = \gamma_{H_1}(B) - \gamma_{H_1 - \{a_3\}}(B) = \frac{4}{6} - \frac{2}{6} = \frac{2}{6}$.
$\sigma_{H_1 B}(a_4) = \gamma_{H_1}(B) - \gamma_{H_1 - \{a_4\}}(B) = \frac{4}{6} - \frac{3}{6} = \frac{1}{6}$.

So, in soft decision information system of Example 4.10, for B*-reduction* H_1. a_2 *(medium empowerment of sales personnel) is the most important parameter,* a_3*(low empowerment of sales personnel) is in the second place and* a_4*(good perceived quality of merchandise)is in the third place for store profit or loss.*

For H_2*, we have*

$\sigma_{H_2B}(a_2) = \gamma_{H_2}(B) - \gamma_{H_2-\{a_2\}}(B) = \frac{4}{6} - \frac{2}{6} = \frac{2}{6}.$
$\sigma_{H_2B}(a_6) = \gamma_{H_2}(B) - \gamma_{H_2-\{a_6\}}(B) = \frac{4}{6} - 0 = \frac{4}{6}.$

So, in soft decision information system of Example 4.10, for B*-reduction* H_2. a_6*(high traffic location) is the most important parameter and* a_2*(medium empowerment of sales personnel) is in the second place for store profit or loss.*

For H_3*, we have*

$\sigma_{H_3B}(a_1) = \gamma_{H_3}(B) - \gamma_{H_3-\{a_1\}}(B) = \frac{4}{6} - 0 = \frac{4}{6}.$
$\sigma_{H_3B}(a_4) = \gamma_{H_3}(B) - \gamma_{H_3-\{a_4\}}(B) = \frac{4}{6} - \frac{2}{6} = \frac{2}{6}.$

So, in soft decision information system of Example 4.10, for B*-reduction* H_3. a_1*(high empowerment of sales personnel) is the most important parameter and* a_4*(good perceived quality of merchandise) is in the second place for store profit or loss.*

In Step 4*. We can induce decision rules by Definition 5.2 as follows*

(1) *We have the following deterministic rules*

$(a_1, 1) \wedge (a_2, 0) \wedge (a_3, 0) \wedge (a_4, 1) \wedge (a_5, 0) \wedge (a_6, 1) \wedge (a_7, 0) \Longrightarrow (b_1, 0) \wedge (b_2, 0);$

(This means that if "high empowerment of sales personnel", "good perceived quality of merchandise" and "high traffic location", then "store profit".)

$(a_1, 0) \wedge (a_2, 0) \wedge (a_3, 1) \wedge (a_4, 0) \wedge (a_5, 1) \wedge (a_6, 0) \wedge (a_7, 1) \Longrightarrow (b_1, 0) \wedge (b_2, 1);$

(This means that if "low empowerment of sales personnel", "average perceived quality of merchandise" and "low traffic location", then "store loss".)

$(a_1, 0) \wedge (a_2, 1) \wedge (a_3, 0) \wedge (a_4, 0) \wedge (a_5, 1) \wedge (a_6, 0) \wedge (a_7, 1) \Longrightarrow (b_1, 0) \wedge (b_2, 1);$

(This means that if "medium empowerment of sales personnel", "average perceived quality of merchandise" and "low traffic location", then "store loss".)

$(a_1, 1) \wedge (a_2, 0) \wedge (a_3, 0) \wedge (a_4, 0) \wedge (a_5, 1) \wedge (a_6, 1) \wedge (a_7, 0) \Longrightarrow (b_1, 0) \wedge (b_2, 1);$

(This means that if "high empowerment of sales personnel", "average perceived quality of merchandise" and"high traffic location", then "store profit".)

(2) *We have the following deterministic rules*

$(a_1, 0) \wedge (a_2, 1) \wedge (a_3, 0) \wedge (a_4, 1) \wedge (a_5, 0) \wedge (a_6, 1) \wedge (a_7, 0) \Longrightarrow (b_1, 0) \wedge (b_2, 1)$, *and the factor of rule is 0.5;*

(This means that if "medium empowerment of sales personnel", "good perceived quality of merchandise" and "high traffic location", then "stores loss(0.5)".)

$(a_1, 0) \wedge (a_2, 1) \wedge (a_3, 0) \wedge (a_4, 1) \wedge (a_5, 0) \wedge (a_6, 1) \wedge (a_7, 0) \Longrightarrow (b_1, 1) \wedge (b_2, 0)$, *and the factor of rule is 0.5;*

(This means that if "medium empowerment of sales personnel", "good perceived quality of merchandise" and "high traffic location", then "stores profit(0.5)".)

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SOME NEW SPACES OF DOUBLE SEQUENCES AND THE STATISTICAL CORE ON $\ell_2^\infty(p)$

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Abstract: In this paper, we introduce the double sequence spaces $(\overline{c}(p))_2^P$, $({}_0\overline{c}(p))_2^P$, $(\overline{c}(p))_2^{PB}$, $({}_0\overline{c}(p))_2^{PB}$, $\overline{\ell}_2^\infty(p)$ and $(\overline{c^*}(p))_2^P$. The inclusion relations among these spaces are obtained. We also characterize the matrix classes $(\ell_2^\infty(p), {}_0c_2^{PB}), (\ell_2^\infty(p), c_2^{PB}), (\ell_2^\infty(p), c_2^P)$, $((\overline{c}(p))_2^{PB}, c_2^{PB})$ and $((\overline{c}(p))_2^{PB}, c_2^{PB})_{reg}$. Furthermore, we define the statistical core of a real-valued double sequence belonging to the more general class $\ell_2^\infty(p)$ and study the statistical core inequalities related to this new type of statistical core by using our matrix classes $((\overline{c}(p))_2^{PB}, c_2^{PB})$ and $((\overline{c}(p))_2^{PB}, c_2^{PB})_{reg}$.
Keyword: Double Sequence, Double Series, Statistical Double limit.

1. Introduction

The notion of convergence for double sequence was presented by Pringsheim [9]. Thus, by the convergence of a double sequence we mean the convergence in Pringsheim's sense. A double sequence $x = (x_{jk})_{j,k=0}^\infty$ has Pringsheim limit (or double limit) L provided that given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_{jk} - L| < \varepsilon$ whenever $j, k > N$. A sequence which is Pringsheim's sense convergent to zero is called as a null double sequence.

A double sequence x is bounded if there exists a positive number M such that $|x_{jk}| < M$ for all j and k, i.e., if $\|x\|_\infty = \sup_{j,k} |x_{jk}| < \infty$. c_2^P, ${}_0c_2^P$ and ℓ_2^∞ will denote the sets of all convergent, null and bounded double sequences, respectively. It is clear that the convergence of x in Pringsheim′s sense does not guarantee the boundedness of x. So, bounded and convergent in Pringsheim's sense, bounded and null in Pringsheim's sense double sequences are defined by $c_2^{PB} = c_2^P \cap \ell_2^\infty$ and ${}_0c_2^{PB} = {}_0c_2^P \cap \ell_2^\infty$, respectively.

Gökhan and Çolak defined the following sequence spaces in [2], [3] and [4]:

$$\ell_2^\infty(p) = \left\{ x = (x_{jk}) \in w^2 : \sup_{j,k} |x_{jk}|^{p_{jk}} < \infty \right\}$$

$$c_2^P(p) = \left\{ x = (x_{jk}) \in w^2 : \lim_{j,k\to\infty} |x_{jk} - L|^{p_{jk}} = 0 \text{ for some } L \right\}$$

$${}_0c_2^P(p) = \left\{ x = (x_{jk}) \in w^2 : \lim_{j,k\to\infty} |x_{jk}|^{p_{jk}} = 0 \right\}$$

$$c_2^{PB}(p) = c_2^P(p) \cap \ell_2^\infty(p) \text{ and } {}_0c_2^{PB}(p) = {}_0c_2^P(p) \cap \ell_2^\infty(p) ,$$

where $p = (p_{jk})$ is a double sequence of strictly positive real numbers p_{jk} and w^2 is the space of all complex double sequences. It is well known that w^2 is a linear space under the coordinatewise addition and scalar multiplication. The double series with real terms are

defined in the same way as single series. Given a double sequence $(x_{jk}), j,k = 1,2,...,$ we define its partial sum by the formula

$$S_{mn} = \sum_{j,k=1}^{m,n} x_{jk}.$$

The sum of a double series $\sum_{j,k=1}^{\infty} x_{jk}$ is defined as

$$\lim_{m,n\to\infty} S_{mn}.$$

The double limit may or may not exist. If the double limit exists, we say that the double series converges; otherwise it diverges.

Statistical convergence of double sequences was first introduced by Mursaleen and Edely [7], Tripathy [11] and Móricz [6] independently. Furthermore, Móricz [6] defined the statistical bounded double sequence. Çakan and Altay [1] introduced statistical limit superior and inferior for any real double sequence and investigated statistical core for double sequences.

Recall that a subset K of the set $\mathbb{N}\times\mathbb{N}$ is said to have "double natural density" $\delta_2(K)$ if

$$\delta_2(K) = \lim_{m,n\to\infty} \frac{1}{mn} \left|\{(j,k),\ j \le m \text{ and } k \le n :\ (j,k) \in K\}\right|,$$

where the vertical bars denote the cardinality of the enclosed set. Clearly we have $\delta_2(K^c) = \delta_2(\mathbb{N}\times\mathbb{N}-K) = 1-\delta_2(K)$. It is obvious that all finite subsets of $\mathbb{N}\times\mathbb{N}$ and all subsets which have finite rows (or columns) have zero density. For example

$$K = \{(m,n) :\ m \in \mathbb{N} \text{ and } n_0 \le n \le n_1, \text{ for fixed } n_0 \text{ and } n_1\}$$

has zero density.

Now, the concept of statistical convergence can be reformulated in terms of natural density as follows:

A double sequence $x = (x_{jk})$ is said to be statistically convergent to a number L, in symbol: $st_2 - \lim x_{jk} = L$, if for any given $\varepsilon > 0$,
$\delta_2\left(\{(j,k),\ j \le m \text{ and } k \le n :\ |x_{jk} - L| \ge \varepsilon\}\right) = 0$. The number L is called statistical double limit or statistical Pringsheim limit.

If a double sequence x is statistically convergent to 0, then it is said to be statistically null. $(\overline{c})_2^P$ and $({}_0\overline{c})_2^P$ will denote the sets of all statistically convergent and statistically null double sequences, respectively. Usual convergence (in Pringsheim$'$s sense) implies statistical convergence to the same limit; that is, $c_2^P \subset (\overline{c})_2^P$ and ${}_0c_2^P \subset ({}_0\overline{c})_2^P$.

It is clear that the spaces $(\overline{c})_2^P$ and $({}_0\overline{c})_2^P$ contain some unbounded sequences. Hence we introduce the following sequence spaces:

$$(\overline{c})_2^{PB} = (\overline{c})_2^P \cap \ell_2^\infty \text{ and } ({}_0\overline{c})_2^{PB} = ({}_0\overline{c})_2^P \cap \ell_2^\infty.$$

A double sequence $x = (x_{jk})$ is said to be statistically bounded if there exists a constant K such that $\delta_2\left(\{(j,k),\ j \le m \text{ and } k \le n :\ |x_{jk}| > K\}\right) = 0$. $\overline{\ell}_2^\infty$ will denote the sets of all statistically bounded sequences. It is clear that $\ell_2^\infty \subset \overline{\ell}_2^\infty$ and $(\overline{c})_2^P \subset \overline{\ell}_2^\infty$.

2. Extensions of the spaces $(\overline{c})_2^P, (\overline{c})_2^{PB}, ({}_0\overline{c})_2^P, ({}_0\overline{c})_2^{PB}$ and $\overline{\ell}_2^\infty$.

In this section, our goal is to extend the double sequence spaces $(\overline{c})_2^P, (\overline{c})_2^{PB}$, $({}_0\overline{c})_2^P, ({}_0\overline{c})_2^{PB}$ and $\overline{\ell}_2^\infty$, known in the literature.

Definition 2.1. For a given double sequence $p = (p_{jk})$ of strictly positive numbers p_{jk}, $(\overline{c}(p))_2^P$ is the set of all real double sequences $x = (x_{jk})$ such that

$$\delta_2\left(\{(j,k),\ j \leq m \text{ and } k \leq n:\ |x_{jk} - L|^{p_{jk}} \geq \varepsilon\}\right) = 0$$

for some L and $({}_0\overline{c}(p))_2^P$ is the set of $x = (x_{jk})$ such that

$$\delta_2\left(\{(j,k),\ j \leq m \text{ and } k \leq n:\ |x_{jk}|^{p_{jk}} \geq \varepsilon\}\right) = 0.$$

It is trivial that $({}_0\overline{c}(p))_2^P \subset (\overline{c}(p))_2^P$. When all terms of the double sequence $p = (p_{jk})$ are constant and all equal to $p > 0$, then we obtain the set $(\overline{c})_2^P$ of statistically convergent double sequence and $({}_0\overline{c})_2^P$ of the statistically null in Pringsheim's sense. Furthermore, when all terms of (p_{jk}), excluding the first finite number of j and k are constant and all are equal to p>0, we obtain $(\overline{c}(p))_2^P = (\overline{c})_2^P$ and $({}_0\overline{c}(p))_2^P = ({}_0\overline{c})_2^P$.
Now, let us consider the following example:

Example 2.1. Define $x = (x_{jk})$ as

$$x_{jk} = \begin{cases} (1+jk)^{1/p_{jk}}, & \text{if } j \in [3^p, 3^p + p) \text{ and } k \in [3^q, 3^q + q),\ p, q = 1, 2, ... \\ (\frac{1}{j+k})^{1/p_{jk}}, & \text{otherwise.} \end{cases}$$

It is easy to see that $x \in (\overline{c}(p))_2^P$ but $x \notin \ell_2^\infty(p)$ and $x \notin c_2^P(p)$. So we define

$$(\overline{c}(p))_2^{PB} = (\overline{c}(p))_2^P \cap \ell_2^\infty(p) \text{ and } ({}_0\overline{c}(p))_2^{PB} = ({}_0\overline{c}(p))_2^P \cap \ell_2^\infty(p).$$

From the above definitions, it is clear that $c_2^P(p) \subset (\overline{c}(p))_2^P$, ${}_0c_2^P(p) \subset ({}_0\overline{c}(p))_2^P$, $c_2^{PB}(p) \subset (\overline{c}(p))_2^{PB}$ and ${}_0c_2^{PB}(p) \subset ({}_0\overline{c}(p))_2^{PB}$ and the inclusion relations are proper.

Definition 2.2. For $p = (p_{jk})_{j,k\in\mathbb{N}}$ with $p_{jk} > 0$, $\overline{\ell}_2^\infty(p)$ is the set of all $x = (x_{jk})$ such that

$$\delta_2\left(\{(j,k),\ j \leq m \text{ and } k \leq n:\ |x_{jk}|^{p_{jk}} > K\}\right) = 0$$

for a $K > 0$.
Let $p_{jk} = const = p$ for all $j,\ k \in \mathbb{N}$ then we get $\overline{\ell}_2^\infty$. We have $\ell_2^\infty(p) \subset \overline{\ell}_2^\infty(p)$ and from example 2.1, the inclusion relation is proper. Furthermore, one can easily prove that if $p = (p_{jk}) \in \ell_2^\infty$ then $(\overline{c}(p))_2^P \subset \overline{\ell}_2^\infty(p)$.

Definition 2.3. Let $p = (p_{jk})$ with $p_{jk} > 0$ for all $j, k \in \mathbb{N}$. $st_2 - \limsup |x_{jk}|^{p_{jk}}$ and $st_2 - \liminf |x_{jk}|^{p_{jk}}$ are defined as follows:

(i) Let $K_x = \{M \in \mathbb{R} : \delta_2(\{(j,k) : |x_{jk}|^{p_{jk}} > M\}) \neq 0\}$. Then

$$st_2 - \limsup |x_{jk}|^{p_{jk}} = \begin{cases} \sup K_x, & \text{if } K_x \neq \phi \\ -\infty, & \text{if } K_x = \phi. \end{cases}$$

(ii) Let $L_x = \{N \in \mathbb{R} : \delta_2(\{(j,k) : |x_{jk}|^{p_{jk}} < N\}) \neq 0\}$. Then

$$st_2 - \liminf |x_{jk}|^{p_{jk}} = \begin{cases} \inf L_x, & \text{if } L_x \neq \phi \\ +\infty, & \text{if } L_x = \phi. \end{cases}$$

It is well known that $\delta_2(K) \neq 0$ means either $\delta_2(K) > 0$ or K does not have double natural density.

Example 2.2. Let the double sequence $x = (x_{jk})$ be given by

$$x_{jk} = \begin{cases} (j+k)^{1/p_{jk}}, & \text{if } j = 1,2 \text{ and } k = 1,2,..., \\ 1, & \text{if } j \geq 3 \text{ and } k = 2p,\ p = 1,2,... \\ 2^{1/p_{jk}} & \text{if } j \geq 3 \text{ and } k = 2p-1,\ p = 1,2,..., \end{cases}$$

It is easy to see that $st_2 - \limsup |x_{jk}|^{p_{jk}} = 2$ and $st_2 - \liminf |x_{jk}|^{p_{jk}} = 1$ since $K_x = (-\infty, 2]$ and $L_x = [1, +\infty)$.

Theorem 2.1. For a given double sequence $p = (p_{jk})$ of strictly positive numbers p_{jk},
(i) $st_2 - \limsup |x_{jk}|^{p_{jk}} = \beta$ if and only if for every $\varepsilon > 0$

$$\delta_2(\{(j,k) : |x_{jk}|^{p_{jk}} > \beta - \varepsilon\}) \neq 0 \text{ and } \delta_2(\{(j,k) : |x_{jk}|^{p_{jk}} > \beta + \varepsilon\}) = 0.$$

(ii) $st_2 - \liminf |x_{jk}|^{p_{jk}} = \alpha$ if and only if for every $\varepsilon > 0$

$$\delta_2(\{(j,k) : |x_{jk}|^{p_{jk}} < \alpha + \varepsilon\}) \neq 0 \text{ and } \delta_2(\{(j,k) : |x_{jk}|^{p_{jk}} < \alpha - \varepsilon\}) = 0.$$

Proof: The proof is easily obtained from the Theorem 2.4 in [1].

The proof of the following proposition is the same as the proofs of the Theorem 2.5 and the inequality (2.2) in [1] and therefore it is left to the reader.

Proposition 2.1. Let $p_{jk} > 0$ for every $j,k \in N$. Then we have
(i) $st_2 - \liminf |x_{jk}|^{p_{jk}} \leq st_2 - \limsup |x_{jk}|^{p_{jk}}$,
(ii) $\liminf |x_{jk}|^{p_{jk}} \leq st_2 - \liminf |x_{jk}|^{p_{jk}}$
(iii) $st_2 - \limsup |x_{jk}|^{p_{jk}} \leq \limsup |x_{jk}|^{p_{jk}}$.

Definition 2.4. For $p = (p_{jk})_{j,k\in\mathbb{N}}$ with $p_{jk} > 0$, $(\overline{c^*}(p))_2^P$ is the set of all real double sequences $x = (x_{jk})$ such that

$$\delta_2\left(\{(j,k),\ j \leq m \text{ and } k \leq n :\ ||x_{jk}|^{p_{jk}} - L| \geq \varepsilon\}\right) = 0$$

for some L. It is trivial that $(\overline{c^*}(p))_2^P \subset \overline{\ell}_2^{\infty}(p)$ but $\ell_2^{\infty}(p)$ does not include $(\overline{c^*}(p))_2^P$ from example 2.1.

Example 2.3. We define a double sequence $x = (x_{jk})$ as follows

$$x_{jk} = \begin{cases} \frac{1}{j+k}, & \text{if } j \text{ and } k \text{ are squares} \\ L + \frac{1}{k}, & \text{otherwise,} \end{cases}$$

where $L>1$ is a fixed real number.

Now, let $p_{jk}=2+\frac{(-1)^j jk}{(j+1)(k+1)}$ for $j,k\in\mathbb{N}$. It is easy to see that (p_{jk}) is bounded. We assert that $st_2-\lim|x_{jk}-L|^{p_{jk}}$ exists but $st_2-\lim|x_{jk}|^{p_{jk}}$ does not exist. In fact, we have

$$|x_{jk}-L|^{p_{jk}}=\begin{cases}\left|\frac{1}{j+k}-L\right|^{2+\left[(-1)^j jk/(j+1)(k+1)\right]}, & \text{if } j \text{ and } k \text{ are squares}\\ (\frac{1}{k})^{2+\ (jk/(j+1)(k+1))}, & \text{if } j \text{ and } k \text{ are non squares and } j \text{ is even}\\ (\frac{1}{k})^{2-\ (jk/(j+1)(k+1))}, & \text{if } j \text{ and } k \text{ are non squares and } j \text{ is odd.}\end{cases}$$

and

$$|x_{jk}|^{p_{jk}}=\begin{cases}(\frac{1}{j+k})^{2+((-1)^j jk/(j+1)(k+1))}, & \text{if } j \text{ and } k \text{ are squares}\\ (L+\frac{1}{k})^{2+\ (jk/(j+1)(k+1))}, & \text{if } j \text{ and } k \text{ are non squares and } j \text{ is even}\\ (L+\frac{1}{k})^{2-\ (jk/(j+1)(k+1))}, & \text{if } j \text{ and } k \text{ are non squares and } j \text{ is odd.}\end{cases}$$

Hence we have $st_2-\lim|x_{jk}-L|^{p_{jk}}=0$. But $st_2-\lim|x_{jk}|^{p_{jk}}$ does not exist since it has two disjoint subsequences of positive double density that converge (Pringsheim's sense) to L^3 and L. Thus $st_2-\limsup|x_{jk}|^{p_{jk}}=L^3$ and $st_2-\liminf|x_{jk}|^{p_{jk}}=L$, therefore $st_2-\lim|x_{jk}|^{p_{jk}}$ does not exist.

As an immediate consequence of exercise, we may say that the existence of $st_2-\lim|x_{jk}-L|^{p_{jk}}$ does not imply the existence of $st_2-\lim|x_{jk}|^{p_{jk}}$, in general. However, even if these limits exist, they may not be equal, in general. For example, let $K\subset\mathbb{N}\times\mathbb{N}$ with $\delta_2(K)=0$ and $p_{jk}=2+\frac{k+j}{kj}$ for all $j,k\in\mathbb{N}$. Let us define the sequence (x_{jk}) as follows:

$$x_{jk}=\begin{cases}(-1)^{j+k}, & \text{if } j,\ k\in K\\ L+\frac{k+j}{kj}, & \text{if } j,\ k\notin K\end{cases}$$

where $L\neq 1$ is a fixed real number. Notice that (p_{jk}) is a convergent and bounded double sequence. It is easy to see that $st_2-\lim|x_{jk}-L|^{p_{jk}}=0$ and $st_2-\lim|x_{jk}|^{p_{jk}}=L^2$, i.e. $L\neq L^2$ for $L\neq 1$.

Lemma 2.1. Let $p=(p_{jk})\in c_2^P$. Then the statistical limit $st_2-\lim\alpha^{p_{jk}}$ exists for any real number $\alpha>0$.

Proof: The proof is trivial. Therefore we omit it.

Theorem 2.2. Let $p=(p_{jk})\in c_2^{PB}$. Then $(\bar{c}(p))_2^P\subset(\overline{c^*}(p))_2^P$.

Proof: Let $x\in(\bar{c}(p))_2^P$ and $M=\max(1,\sup p_{jk})$. Then there exists an $L\in\mathbb{R}$ and $\alpha>0$ such that

$$\delta_2\left(\{(j,k),\ j\leq m \text{ and } k\leq n:\ |x_{jk}-L|^{p_{jk}}\geq\varepsilon\}\right)=0 \tag{1}$$

and there exists an $\alpha>0$ by Lemma 2.1 such that

$$\delta_2\left(\left\{(j,k),\ j\leq m \text{ and } k\leq n:\ \left||L|^{p_{jk}/M}-\alpha^{1/M}\right|\geq\varepsilon\right\}\right)=0.$$

By the inequality $\left||x_{jk}|^{p_{jk}/M}-|L|^{p_{jk}/M}\right| \leq |x_{jk}-L|^{p_{jk}/M}$,we can say that

$$\begin{aligned}
&\delta_2\left(\left\{(j,k),\ j\leq m \text{ and } k\leq n:\ \left||x_{jk}|^{p_{jk}/M}-|L|^{p_{jk}/M}\right|\geq\varepsilon\right\}\right)\\
&\leq \delta_2\left(\left\{(j,k),\ j\leq m \text{ and } k\leq n:\ |x_{jk}-L|^{p_{jk}/M}\geq\varepsilon\right\}\right).
\end{aligned}$$

Using this inequality and (1), we obtain that

$$\delta_2\left(\left\{(j,k),\ j\leq m \text{ and } k\leq n:\ \left||x_{jk}|^{p_{jk}/M}-|L|^{p_{jk}/M}\right|\geq\varepsilon\right\}\right)=0. \tag{2}$$

Now, from (2), we have

$$\begin{aligned}
&\delta_2\left(\left\{(j,k),\ j\leq m \text{ and } k\leq n:\ \left||x_{jk}|^{p_{jk}/M}-\alpha^{1/M}\right|\geq\varepsilon\right\}\right)\\
&\leq \delta_2\left(\left\{(j,k),\ j\leq m \text{ and } k\leq n:\ \left||x_{jk}|^{p_{jk}/M}-|L|^{p_{jk}/M}\right|\geq\frac{\varepsilon}{2}\right\}\right)\\
&+\delta_2\left(\left\{(j,k),\ j\leq m \text{ and } k\leq n:\ \left||L|^{p_{jk}/M}-\alpha^{1/M}\right|\geq\frac{\varepsilon}{2}\right\}\right)\\
&=0.
\end{aligned}$$

Hence $x\in(\overline{c^{*}}(p))_2^P$.

3. Some matrix transformations

Let X and Y be two double sequence spaces. By (X,Y), we denote the class of all four-dimensional matrices $A=(a_{jk}^{mn})$ such that $Ax=(A_{mn}(x))\in Y$ whenever $x=(x_{jk})\in X$, where $A_{mn}(x)$ is the $A-$transform of x given by $A_{mn}(x)=\sum\limits_{j,k=1}^{\infty}a_{jk}^{mn}x_{jk}$, $(m,n=1,2,...)$. If a four-dimensional matrix A maps every bounded convergent double sequence into a convergent sequence with the same limit, then the transformation is said to be regular. The class of regular matrix transformations from X into Y is denoted by $(X,Y)_{reg}$.

Robison [10] characterized some four-dimensional matrix transformations between the double sequence ℓ_2^{∞}, c_2^P and c_2^{PB} and presented the necessary and sufficient conditions of regularity for double sequences. In [5] Gökhan, Çolak and Mursaleen extended the conditions of Robison to the class of regular matrix transformations between the double sequence spaces $c_2^{PB}(p)$ and c_2^{PB} and characterized the matrix class ($c_2^{PB}(p)$, c_2^{PB}).

In this section, we characterized the matrix classes $(\ell_2^{\infty}(p),_0c_2^{PB}),(\ell_2^{\infty}(p),c_2^{PB})$ $(\ell_2^{\infty}(p),c_2^{P}),((\overline{c}(p))_2^{PB},c_2^{PB})$ and $((\overline{c}(p))_2^{PB},c_2^{PB})_{reg}$.

Theorem 3.1. Let $p=(p_{jk})$ be any sequence of strictly positive real numbers. Then $A\in(\ell_2^{\infty}(p),_0c_2^{PB})$ if and only if

(i) $\lim\limits_{m,n\to\infty}\sum\limits_{j,k=1}^{\infty}\left|a_{jk}^{mn}\right|B^{1/p_{jk}}=0$ for some $B>1$,

(ii) $\sup\limits_{m,n\geq1}\sum\limits_{j,k=1}^{\infty}\left|a_{jk}^{mn}\right|B^{1/p_{jk}}=C$ for some $C>0$.

Proof: If we take $q_{jk} = 1$ for all $j, k \in \mathbb{N}$ in Theorem 2.5 [5], then we can easily obtain the conditions (i) and (ii).

The matrix class $(\ell_2^\infty, c_2^{PB})$ was characterized by Robison in Theorem XII in [10]. The conditions of this matrix class may be restated as follows:

Theorem 3.2. $A \in (\ell_2^\infty, c_2^{PB})$ if and only if
(i) $\lim_{m,n\to\infty} a_{jk}^{mn} = a_{jk}$ for fixed j and k,
(ii) $\lim_{m,n\to\infty} \sum_{j,k=1}^{\infty} \left| a_{jk}^{mn} - a_{jk} \right| = 0$,
(iii) $\sup_{m,n\geq 1} \sum_{j,k=1}^{\infty} \left| a_{jk}^{mn} \right| = C < \infty$ for some $C > 0$.

Proof: The proof is easy. Therefore we omit it.

Example 3.1: Let us define the matrix $A = (a_{jk}^{mn})$ as $a_{jk}^{mn} = \frac{mn}{(mn+1)2^{j+k}}$. Then $A \in (\ell_2^\infty, c_2^{PB})$ since
(i) $\lim_{m,n\to\infty} \frac{mn}{(mn+1)2^{j+k}} = \frac{1}{2^{j+k}}$ for fixed j and k,
(ii) $\lim_{m,n\to\infty} \sum_{j,k=1}^{\infty} \left| \frac{mn}{(mn+1)2^{j+k}} - \frac{1}{2^{j+k}} \right| = \lim_{m,n\to\infty} \frac{1}{mn+1} \sum_{j,k=1}^{\infty} \frac{1}{2^{j+k}} = 0$,
(iii) $\sup_{m,n\geq 1} \sum_{j,k=1}^{\infty} \left| \frac{mn}{(mn+1)2^{j+k}} \right| = 1$.

Theorem 3.3. Let $p_{jk} > 0$ for every j, k. Then $A \in (\ell_2^\infty(p), c_2^{PB})$ if and only if
(i) $\lim_{m,n\to\infty} a_{jk}^{mn} = a_{jk}$ for fixed j and k,
(ii) $\lim_{m,n\to\infty} \sum_{j,k=1}^{\infty} \left| a_{jk}^{mn} - a_{jk} \right| B^{1/p_{jk}} = 0$ for some $B > 1$,
(iii) $\sup_{m,n\geq 1} \sum_{j,k=1}^{\infty} \left| a_{jk}^{mn} \right| B^{1/p_{jk}} = C < \infty$ for every $B > 1$.

Proof: $(\Leftarrow)$ Let $x \in \ell_2^\infty(p)$. Then there is a real number $B > 1$ such that $B > \max(1, \sup_{j,k\geq 1} |x_{jk}|^{p_{jk}})$. Using conditions (i) and (iii), we have

$$\sum_{j,k=1}^{\infty} |a_{jk}x_{jk}| \leq \sum_{j,k=1}^{\infty} \lim_{m,n\to\infty} \left| a_{jk}^{mn} \right| |x_{jk}| \leq \sup_{m,n\geq 1} \sum_{j,k=1}^{\infty} \left| a_{jk}^{mn} \right| B^{1/p_{jk}} < \infty,$$

i.e $\sum_{j,k=1}^{\infty} a_{jk}x_{jk}$ converges. From (ii) and (iii), we obtain that $\lim_{m,n\to\infty} \sum_{j,k=1}^{\infty} a_{jk}^{mn} x_{jk} = \sum_{j,k=1}^{\infty} a_{jk}x_{jk}$ and $|A_{mn}(x)| \leq C < \infty$ for all m, n; i.e $Ax \in c_2^{PB}$.
$(\Rightarrow)$ Let $A \in (\ell_2^\infty(p), c_2^{PB})$.
(i) Since $e_{jk} = (e_{jk}^{mn})$, where

$$e_{jk}^{mn} = \begin{cases} 1, & \text{if } j = m \text{ and } k = n \\ 0, & \text{if } j \neq m \text{ and } k \neq n \end{cases},$$

e_{jk} belongs to $\ell_2^\infty(p)$, then the necessity of (i) holds.

(ii) Assume that condition (ii) is not satisfied. By Theorem 3.2, we have $(a_{jk}^{mn}B^{1/p_{jk}}) \notin (\ell_2^\infty, c_2^{PB})$ for some integer $B > 1$ and so there exists an $x \in \ell_2^\infty$ with $\sup\limits_{j,k\geq 1} |x_{jk}| = 1$ such that $(\sum\limits_{j,k=1}^{\infty} a_{jk}^{mn} x_{jk} B^{1/p_{jk}}) \notin c_2^{PB}$. Hence, although $z = (B^{1/p_{jk}} x_{jk}) \in \ell_2^\infty(p)$, the sequence $(A_{mn}(z)) \notin c_2^{PB}$ and this contradicts the fact that $A \in (\ell_2^\infty(p), c_2^{PB})$.

(iii) Because of $c_2^{PB}(p) \subset \ell_2^\infty(p)$, we have $A \in (c_2^{PB}(p), c_2^{PB})$. Thus, the condition (iii) is satisfied from Theorem 2.1 (vi) in [5].

Example 3.2. Let $p_{jk} > 0$ for every $j, k \in \mathbb{N}$ and $a_{jk}^{mn} = (1+\frac{1}{mn})\frac{1}{2^{j+k}B^{1/p_{jk}}}$, where $B > 1$ is any real number. Then, $A \in (\ell_2^\infty(p), c_2^{PB})$ since

(i) $\lim\limits_{m,n\to\infty} a_{jk}^{mn} = \frac{1}{2^{j+k}B^{1/p_{jk}}}$ for fixed j and k,

(ii) $\lim\limits_{m,n\to\infty} \sum\limits_{j,k=1}^{\infty} \left|a_{jk}^{mn} - a_{jk}\right| B^{1/p_{jk}} = \lim\limits_{m,n\to\infty} \sum\limits_{j,k=1}^{\infty} \frac{1}{mn2^{j+k}} = 0,$

(iii) $\sup\limits_{m,n\geq 1} \sum\limits_{j,k=1}^{\infty} \left|a_{jk}^{mn}\right| B^{1/p_{jk}} = \sup\limits_{m,n\geq 1} (1+\frac{1}{mn}) \sum\limits_{j,k=1}^{\infty} \frac{1}{2^{j+k}} = 2.$

Notice that if we consider the condition (iii) as follows:

" $\sum\limits_{j,k=1}^{\infty} \left|a_{jk}^{mn}\right| B^{1/p_{jk}}$ converges for every $B > 1$ and for each m and n",

then we obtain that $A \in (\ell_2^\infty(p), c_2^{P})$.

Theorem 3.4. Let $0 < p_{jk} \leq \sup\limits_{j,k} p_{jk} = H < \infty$ and $M = \max(1, H)$. Then $A \in ((\bar{c}(p))_2^{PB}, c_2^{PB})$ if and only if

(i) $A \in (c_2^{PB}(p), c_2^{PB})$,

(ii) $\lim\limits_{m,n\to\infty} \sum\limits_{(j,k)\in K} \left|a_{jk}^{mn} - a_{jk}\right| B^{1/p_{jk}} = 0$ for some $B > 1$ and for every $K \subset \mathbb{N}\times\mathbb{N}$ with $\delta_2(K) = 0$.

Proof: $(\Rightarrow)$ Let $A \in ((\bar{c}(p))_2^{PB}, c_2^{PB})$.

(i) Because of $c_2^{PB}(p) \subset (\bar{c}(p))_2^{PB}$, we have $A \in (c_2^{PB}(p), c_2^{PB})$.

(ii) Let $x = (x_{jk}) \in \ell_2^\infty(p)$. Now, we define the sequence $z = (z_{jk})$ via a sequence x as follows:

$$z_{jk} = \begin{cases} x_{jk}, & \text{if } (j,\, k) \in K \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that $st_2 - \lim |z_{jk} - 0|^{p_{jk}} = 0$ and $z \in \ell_2^\infty(p)$, i.e. $z \in (\bar{c}(p))_2^{PB}$. Hence, $Az \in c_2^{PB}$. Now, define the matrix $B = (b_{jk}^{mn})$ by

$$b_{jk}^{mn} = \begin{cases} a_{jk}^{mn}, & \text{if } (j,\, k) \in K \\ 0, & \text{otherwise} \end{cases}$$

for all $m, n \in \mathbb{N}$. Then, we obtain that $Bx = \left(\sum\limits_{(j,k)\in K} a_{jk}^{mn} x_{jk}\right)_{m,n\in\mathbb{N}}$ is in the class $(\ell_2^\infty(p), c_2^{PB})$. Therefore, the condition (ii) follows Theorem 3.3.

($\Leftarrow$) Let $x \in (\overline{c}(p))_2^{PB}$. Define $K = \{(j,k) \in \mathbb{N} \times \mathbb{N} : |x_{jk} - L|^{p_{jk}} \geq \varepsilon\}$, so that $\delta_2(K) = 0$ and $|x_{jk} - L|^{p_{jk}} < \varepsilon$ whenever $(j,k) \notin K$. This yields $|x_{jk} - L| < \varepsilon^{1/p_{jk}} < \varepsilon^{1/M}$ for every $\varepsilon < 1$ and for $(j,k) \notin K$. Furthermore, we obtain a real number $B = [R^{1/M} + \max(1, |L|)]^M > 1$ such that $|x_{jk} - L| \leq B^{1/p_{jk}}$ for all $j,k \in \mathbb{N}$, where $|x_{jk}|^{p_{jk}} \leq R$ for all $j,k \in \mathbb{N}$, since $x \in \ell_2^{\infty}(p)$. From condition Theorem 2.1 (ii) in [5], we may write

$$\sum_{j,k=1}^{\infty} a_{jk}^{mn} + r_{mn} = a,$$

where $\lim_{m,n\to\infty} r_{mn} = 0$. Whence

$$\begin{aligned}
\left| \sum_{j,k=1}^{\infty} a_{jk}^{mn} x_{jk} - aL - \sum_{j,k=1}^{\infty} a_{jk}(x_{jk} - L) \right| &\leq \sum_{j,k=1}^{\infty} \left| a_{jk}^{mn} - a_{jk} \right| |x_{jk} - L| + |L r_{mn}| \\
&\leq \sum_{(j,k)\in K} \left| a_{jk}^{mn} - a_{jk} \right| |x_{jk} - L| \\
&+ \sum_{(j,k)\notin K} \left| a_{jk}^{mn} - a_{jk} \right| |x_{jk} - L| + |L r_{mn}| \\
&\leq \sum_{(j,k)\in K} \left| a_{jk}^{mn} - a_{jk} \right| B^{1/p_{jk}} \\
&+ \varepsilon^{1/M} 2C + \varepsilon |L|
\end{aligned}$$

for every $\varepsilon > 1$.

Since $\delta_2(K) = 0$ and ε is arbitrary, property (ii) implies that $\lim_{m,n\to\infty} A_{mn}(x) = aL + \sum_{j,k=1}^{\infty} a_{jk}(x_{jk} - L)$. Furthermore, it is easy to see that $|A_{mn}(x)| \leq C < \infty$ for all $m,n \in \mathbb{N}$. Thus, we obtain that $Ax \in c_2^{PB}$.

Theorem 3.5. Let $0 < p_{jk} \leq \sup_{j,k} p_{jk} = H < \infty$ and $M = \max(1, H)$. Then $A \in ((\overline{c}(p))_2^{PB}, c_2^{PB})_{reg}$ if and only if

(i) $A \in (c_2^{PB}(p), c_2^{PB})_{reg}$,

(ii) $\lim_{m,n\to\infty} \sum_{(j,k)\in K} \left| a_{jk}^{mn} \right| B^{1/p_{jk}} = 0$ for some $B > 1$ and for every $K \subset \mathbb{N} \times \mathbb{N}$ with $\delta_2(K) = 0$.

Proof: Using Theorem 3.1, the proof is easily obtained from the proof of Lemma 3.3. in [1].

4. An Application: Generalized Statistical Core

The concept of Pringsheim's core or $P - core(x)$ of a real bounded double sequence $x = (x_{jk})$ was given by Patterson [8] as the closed interval $[\liminf x_{jk}, \limsup\, x_{jk}]$. In [5] Gökhan,Çolak and Mursallen generalized the Pringsheim's core for $x \in \ell_2^{\infty}(p)$ (or briefly

$st_2 - core(x)$) and established core inequalities by using the matrix classes $(c_2^{PB}(p), c_2^{PB})$ and $(c_2^{PB}(p), c_2^{PB})_{reg}$. We shall prove a similar result for $x \in \overline{\ell}_2^{\infty}(p)$.

Definition 4.1. For any real double sequence $x \in \overline{\ell}_2^{\infty}(p)$, the statistical core of x is the closed interval $[st_2 - \liminf |x_{jk}|^{p_{jk}}, st_2 - \limsup |x_{jk}|^{p_{jk}}]$. We shall denote the statistical core of $x \in \overline{\ell}_2^{\infty}(p)$ by $st_2 - core\{x\}$. If $x \notin \overline{\ell}_2^{\infty}(p)$, $st_2 - core\{x\}$ is defined by either $(-\infty, st_2 - \limsup |x_{jk}|^{p_{jk}}]$, $[st_2 - \liminf |x_{jk}|^{p_{jk}}, +\infty)$ or $(-\infty, +\infty)$.

Lemma 4.1. Let (p_{jk}) be a double sequence of strictly positive real number such that $0 < \inf p_{jk} = h < \sup p_{jk} = H < \infty$ and $A = (a_{jk}^{mn})$ be a four dimensional matrix. If A satisfies the conditions

(i) $A \in ((\overline{c}(p))_2^{PB}, c_2^{PB})_{reg}$,

(ii) $\lim\limits_{m,n\to\infty} \sum\limits_{j,k=1}^{\infty} \left| a_{jk}^{mn} \right| = 1$

then for every $x \in \ell_2^{\infty}(p)$, we have

$$\limsup A_{mn}(x) \leq \max \begin{bmatrix} (st_2 - \limsup |x_{jk}|^{p_{jk}})^{1/h}, \\ (st_2 - \limsup |x_{jk}|^{p_{jk}})^{1/H} \end{bmatrix} \tag{3}$$

Proof: Assume that (i) and (ii) hold. Let $x \in \ell_2^{\infty}(p)$. Then, we may say that $\beta(x) = st_2 - \limsup |x_{jk}|^{p_{jk}}$ is finite and Ax is a bounded sequence. For a given $\varepsilon > 0$, let $K = \{(j,k) \in \mathbb{N} \times \mathbb{N} : |x_{jk}|^{p_{jk}} > \beta(x) + \varepsilon\}$. Then we have $\delta_2(K) = 0$ and also it is clear that $|x_{jk}|^{p_{jk}} \leq \beta(x) + \varepsilon$, if $(j,k) \notin K$. Furthermore, since $x \in \ell_2^{\infty}(p)$, there is a real number $B > 1$ such that $|x_{jk}|^{p_{jk}} \leq B$ for all $j,k \in \mathbb{N}$. It is well known that for any real number, we write

$$z^+ = \max\{z, 0\} \text{ and } z^- = \max\{-z, 0\},$$

whence

$$z^+ + z^- = |z| \text{ and } z^+ - z^- = z.$$

For two fixed points $K, L > 1$, we obtain the following

$$\begin{aligned}
&\sum_{j,k=1}^{\infty} a_{jk}^{mn} x_{jk} \\
= &\sum_{j,k=1}^{K,L} a_{jk}^{mn} x_{jk} + \sum_{j=K+1}^{\infty} \sum_{k=1}^{L} a_{jk}^{mn} x_{jk} + \sum_{j=1}^{K} \sum_{k=L+1}^{\infty} a_{jk}^{mn} x_{jk} + \sum_{j=K+1}^{\infty} \sum_{k=L+1}^{\infty} a_{jk}^{mn} x_{jk} \\
= &\sum_{j,k=1}^{K,L} a_{jk}^{mn} x_{jk} + \sum_{j=K+1}^{\infty} \sum_{k=1}^{L} a_{jk}^{mn} x_{jk} + \sum_{j=1}^{K} \sum_{k=L+1}^{\infty} a_{jk}^{mn} x_{jk} \\
&+ \sum_{j=K+1}^{\infty} \sum_{k=L+1}^{\infty} \left(a_{jk}^{mn}\right)^+ x_{jk} - \sum_{j=K+1}^{\infty} \sum_{k=L+1}^{\infty} \left(a_{jk}^{mn}\right)^- x_{jk}
\end{aligned}$$

$$\leq \quad B^{1/h}\sum_{j,k=1}^{K,L}\left|a_{jk}^{mn}\right| + \sum_{j=K+1}^{\infty}\sum_{k=1}^{L}\left|a_{jk}^{mn}\right| B^{1/p_{jk}} + \sum_{j=1}^{K}\sum_{k=L+1}^{\infty}\left|a_{jk}^{mn}\right| B^{1/p_{jk}}$$

$$+ \sum_{j=K+1}^{\infty}\sum_{\substack{k=L+1\\(j,k)\notin K}}^{\infty}\left(a_{jk}^{mn}\right)^{+} x_{jk} + \sum_{j=K+1}^{\infty}\sum_{\substack{k=L+1\\(j,k)\in K}}^{\infty}\left(a_{jk}^{mn}\right)^{+} x_{jk}$$

$$+ B^{1/h}\sum_{j=K+1}^{\infty}\sum_{k=L+1}^{\infty}\left(\left|a_{jk}^{mn}\right| - a_{jk}^{mn}\right)$$

$$\leq B^{1/h}\sum_{j,k=1}^{K,L}\left|a_{jk}^{mn}\right| + \sum_{j=K+1}^{\infty}\sum_{k=1}^{L}\left|a_{jk}^{mn}\right| B^{1/p_{jk}} + \sum_{j=1}^{K}\sum_{k=L+1}^{\infty}\left|a_{jk}^{mn}\right| B^{1/p_{jk}}$$

$$+ \max\left[(\beta(x)+\varepsilon)^{1/h}, (\beta(x)+\varepsilon)^{1/H}\right]\sum_{j=K+1}^{\infty}\sum_{\substack{k=L+1\\(j,k)\notin K}}^{\infty}\left|a_{jk}^{mn}\right|$$

$$+ \sum_{j=K+1}^{\infty}\sum_{\substack{k=L+1\\(j,k)\in K}}^{\infty}\left|a_{jk}^{mn}\right| B^{1/p_{jk}} + B^{1/h}\sum_{j=K+1}^{\infty}\sum_{k=L+1}^{\infty}\left(\left|a_{jk}^{mn}\right| - a_{jk}^{mn}\right).$$

Taking the limit superior as $m, n \to \infty$ and using (i) and (ii), we have

$$\limsup A_{mn}(x) \leq \max\left[(\beta(x)+\varepsilon)^{1/h}, (\beta(x)+\varepsilon)^{1/H}\right].$$

This implies (3) since ε is arbitrary.

Lemma 4.2. Let (p_{jk}) be a convergent double sequence of strictly positive real number such that $1 \leq \inf_{j,k\geq 1} p_{jk} = h < \sup_{j,k\geq 1} p_{jk} = H < \infty$ and $A = (a_{jk}^{mn})$ be a four dimensional matrix. If $x = (x_{jk}) \in \ell_2^{\infty}(p)$ and

$$\limsup A_{mn}(x) \leq st_2 - \limsup \left|x_{jk}\right|^{p_{jk}} \tag{4}$$

then

(i) $A \in ((\bar{c}(p))_2^{PB}, c_2^{PB})$,

(ii) $\limsup_{m,n\to\infty} \sum_{j,k=1}^{\infty} \left|a_{jk}^{mn}\right| \leq 1$.

Proof: Suppose that A satisfies (4) and $x \in \ell_2^{\infty}(p)$. Then from Proposition 2.1 (iii), we have

$$st_2 - \limsup \left|x_{jk}\right|^{p_{jk}} \leq \limsup \left|x_{jk}\right|^{p_{jk}}.$$

Also $Ax \in \ell_2^{\infty}$ since $\sup\limits_{m,n} \sum\limits_{j,k=1}^{\infty} \left| a_{jk}^{mn} \right| B^{1/p_{jk}} < \infty$ for every $B > 1$. By (4), we have

$$\begin{aligned} st_2 - \liminf |x_{jk}|^{p_{jk}} &\leq \liminf A_{mn}(x) \\ &\leq \limsup A_{mn}(x) \leq st_2 - \limsup |x_{jk}|^{p_{jk}} . \end{aligned} \tag{5}$$

Since x is arbitrary and $(\bar{c}(p))_2^{PB} \subset \ell_2^{\infty}(p)$, we can consider any $x \in (\bar{c}(p))_2^{PB}$. Then from Theorem 2.2,

$$st_2 - \liminf |x_{jk}|^{p_{jk}} = st_2 - \limsup |x_{jk}|^{p_{jk}} = st_2 - \lim |x_{jk}|^{p_{jk}}$$

exists. So (5) implies that $\lim A_{mn}(x) = st_2 - \lim |x_{jk}|^{p_{jk}}$, i.e. $A \in ((\bar{c}(p))_2^{PB}, c_2^{PB})$. Since $st_2 - \limsup |x_{jk}|^{p_{jk}} \leq \limsup |x_{jk}|^{p_{jk}}$, (4) implies that $\limsup A_{mn}(x) \leq \limsup |x_{jk}|^{p_{jk}}$, and Theorem 3.1 in [5] yields $\limsup\limits_{m,n\to\infty} \sum\limits_{j,k=1}^{\infty} \left| a_{jk}^{mn} \right| \leq 1$.

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Fractional Polya type integral inequality

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Abstract

Here we establish a fractional Polya type integral inequality with the help of generalised right and left fractional derivatives. The amazing fact here is that we do not need any boundary conditions as the classical Polya integral inequality requires.

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Keywords and Phrases: Polya integral inequality, fractional derivative.

1 Introduction

We mention the following famous Polya's integral inequality, see [7], [8, p, 62], [9] and [10, p. 83].

Theorem 1 *Let $f(x)$ be differentiable and not identically a constant on $[a,b]$ with $f(a)=f(b)=0$. Then the exists at least one point $\xi \in [a,b]$ such that*

$$\left|f'(\xi)\right| > \frac{4}{(b-a)^2}\int_a^b f(x)\,dx. \tag{1}$$

In [11], Feng Qi presents the following very interesting Polya type integral inequality (2), which generalizes (1).

Theorem 2 *Let $f(x)$ be differentiable and not identically constant on $[a,b]$ with $f(a)=f(b)=0$ and $M=\sup_{x\in[a,b]}\left|f'(x)\right|$. Then*

$$\left|\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^2}{4}M, \tag{2}$$

where $\frac{(b-a)^2}{4}$ in (2) is the best constant.

In this short note we present a fractional Polya type integral inequality, similar to (2), without the boundary conditions $f(a) = f(b) = 0$.

For the last we need the following fractional calculus background.

Let $\alpha > 0$, $m = [\alpha]$, $\beta = \alpha - m$, $0 < \beta < 1$, $f \in C([a,b])$, $[a,b] \subset \mathbb{R}$, $x \in [a,b]$. The gamma function Γ is given by $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$. We define the left Riemann-Liouville integral

$$\left(J_\alpha^{a+} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt, \tag{3}$$

$a \leq x \leq b$. We define the subspace $C_{a+}^\alpha([a.b])$ of $C^m([a,b])$:

$$C_{a+}^\alpha([a,b]) = \left\{ f \in C^m([a,b]) : J_{1-\beta}^{a+} f^{(m)} \in C^1([a,b]) \right\}. \tag{4}$$

For $f \in C_{a+}^\alpha([a,b])$, we define the left generalized α-fractional derivative of f over $[a,b]$ as

$$D_{a+}^\alpha f := \left(J_{1-\beta}^{a+} f^{(m)} \right)', \tag{5}$$

see [1], p. 24. Canavati first in [3] introduced the above over $[0,1]$.

Notice that $D_{a+}^\alpha f \in C([a,b])$.

We need the following left fractional Taylor's formula, see [1], pp. 8-10, and in [3] the same over $[0,1]$ that appeared first.

Theorem 3 *Let $f \in C_{a+}^\alpha([a,b])$.*

(i) If $\alpha \geq 1$, then

$$f(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2} + \dots + f^{(m-1)}(a)\frac{(x-a)^{m-1}}{(m-1)!} \tag{6}$$

$$+ \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} \left(D_{a+}^\alpha f\right)(t)\, dt, \quad \textit{all } x \in [a,b].$$

(ii) If $0 < \alpha < 1$, we have

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} \left(D_{a+}^\alpha f\right)(t)\, dt, \quad \textit{all } x \in [a,b]. \tag{7}$$

We will use (7).

Notice that

$$\int_a^x (x-t)^{\alpha-1} \left(D_{a+}^\alpha f\right)(t)\, dt = \int_a^x \left(D_{a+}^\alpha f\right)(t)\, d\left(\frac{(x-t)^\alpha}{-\alpha}\right)$$

$$= \left(D_{a+}^\alpha f\right)(\xi_x)\frac{(x-a)^\alpha}{\alpha}, \text{ where } \xi_x \in [a,x], \tag{8}$$

by first integral mean value theorem. Hence, when $0 < \alpha < 1$, we get

$$f(x) = \left(D_{a+}^\alpha f\right)(\xi_x)\frac{(x-a)^\alpha}{\Gamma(\alpha+1)}, \quad \text{all } x \in [a,b]. \tag{9}$$

Furthermore we need:

Let again $\alpha > 0$, $m = [\alpha]$, $\beta = \alpha - m$, $f \in C([a,b])$, call the right Riemann-Liouville fractional integral operator by

$$\left(J_{b-}^{\alpha} f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\, dt, \tag{10}$$

$x \in [a,b]$, see also [2], [4], [5], [6], [12]. Define the subspace of functions

$$C_{b-}^{\alpha}([a,b]) := \left\{ f \in C^m([a,b]) : J_{b-}^{1-\beta} f^{(m)} \in C^1([a,b]) \right\}. \tag{11}$$

Define the right generalized α-fractional derivative of f over $[a,b]$ as

$$D_{b-}^{\alpha} f = (-1)^{m-1} \left(J_{b-}^{1-\beta} f^{(m)} \right)', \tag{12}$$

see [2]. We set $D_{b-}^{0} f = f$. Notice that $D_{b-}^{\alpha} f \in C([a,b])$.

From [2], we need the following right Taylor fractional formula.

Theorem 4 *Let $f \in C_{b-}^{\alpha}([a,b])$, $\alpha > 0$, $m := [\alpha]$. Then*

(i) If $\alpha \geq 1$, we get

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(b)}{k!} (x-b)^k + \left(J_{b-}^{\alpha} D_{b-}^{\alpha} f\right)(x), \quad \text{all } x \in [a,b]. \tag{13}$$

(ii) If $0 < \alpha < 1$, we get

$$f(x) = J_{b-}^{\alpha} D_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} \left(D_{b-}^{\alpha} f\right)(t)\, dt, \quad \text{all } x \in [a,b]. \tag{14}$$

We will use (14).

Notice that

$$\int_x^b (t-x)^{\alpha-1} \left(D_{b-}^{\alpha} f\right)(t)\, dt = \int_x^b \left(D_{b-}^{\alpha} f\right)(t)\, d\left(\frac{(t-x)^{\alpha}}{\alpha}\right)$$

$$= \left(D_{b-}^{\alpha} f\right)(\eta_x) \frac{(b-x)^{\alpha}}{\alpha}, \text{ where } \eta_x \in [x,b], \tag{15}$$

by first integral mean value theorem. Hence, when $0 < \alpha < 1$, we obtain

$$f(x) = \left(D_{b-}^{\alpha} f\right)(\eta_x) \frac{(b-x)^{\alpha}}{\Gamma(\alpha+1)}, \quad \text{all } x \in [a,b]. \tag{16}$$

2 Main Result

We present the following fractional Polya type integral inequality without any boundary conditions.

Theorem 5 *Let* $0 < \alpha < 1$, $f \in C([a,b])$. *Assume* $f \in C_{a+}^{\alpha}\left(\left[a, \frac{a+b}{2}\right]\right)$ *and* $f \in C_{b-}^{\alpha}\left(\left[\frac{a+b}{2}, b\right]\right)$. *Set*

$$M(f) = \max\left\{ \left\| D_{a+}^{\alpha} f \right\|_{\infty,\left[a,\frac{a+b}{2}\right]}, \left\| D_{b-}^{\alpha} f \right\|_{\infty,\left[\frac{a+b}{2},b\right]} \right\}. \tag{17}$$

Then

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx \leq M(f) \frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)\, 2^{\alpha}}. \tag{18}$$

Inequality (18) is sharp, namely it is attained by

$$f_*(x) = \left\{ \begin{array}{ll} (x-a)^{\alpha}, & x \in \left[a, \frac{a+b}{2}\right] \\ (b-x)^{\alpha}, & x \in \left[\frac{a+b}{2}, b\right] \end{array} \right\}, \quad 0 < \alpha < 1. \tag{19}$$

Clearly here non zero constant functions f are excluded.

Proof. By (9) we get

$$|f(x)| \leq \left\| D_{a+}^{\alpha} f \right\|_{\infty,\left[a,\frac{a+b}{2}\right]} \frac{(x-a)^{\alpha}}{\Gamma(\alpha+1)}, \quad \text{for any } x \in \left[a, \frac{a+b}{2}\right]. \tag{20}$$

By (16) we derive

$$|f(x)| \leq \left\| D_{b-}^{\alpha} f \right\|_{\infty,\left[\frac{a+b}{2},b\right]} \frac{(b-x)^{\alpha}}{\Gamma(\alpha+1)}, \quad \text{for any } x \in \left[\frac{a+b}{2}, b\right]. \tag{21}$$

Hence we get

$$\int_a^b |f(x)|\, dx = \int_a^{\frac{a+b}{2}} |f(x)|\, dx + \int_{\frac{a+b}{2}}^b |f(x)|\, dx$$

(by (20), (21))

$$\leq \frac{\left\| D_{a+}^{\alpha} f \right\|_{\infty,\left[a,\frac{a+b}{2}\right]}}{\Gamma(\alpha+1)} \int_a^{\frac{a+b}{2}} (x-a)^{\alpha}\, dx + \frac{\left\| D_{b-}^{\alpha} f \right\|_{\infty,\left[\frac{a+b}{2},b\right]}}{\Gamma(\alpha+1)} \int_{\frac{a+b}{2}}^b (b-x)^{\alpha}\, dx \tag{22}$$

$$= \frac{\left\| D_{a+}^{\alpha} f \right\|_{\infty,\left[a,\frac{a+b}{2}\right]}}{(\Gamma(\alpha+1))(\alpha+1)} \left(\frac{b-a}{2}\right)^{\alpha+1} + \frac{\left\| D_{b-}^{\alpha} f \right\|_{\infty,\left[\frac{a+b}{2},b\right]}}{(\Gamma(\alpha+1))(\alpha+1)} \left(\frac{b-a}{2}\right)^{\alpha+1}$$

$$= \frac{\left(\left\| D_{a+}^{\alpha} f \right\|_{\infty,\left[a,\frac{a+b}{2}\right]} + \left\| D_{b-}^{\alpha} f \right\|_{\infty,\left[\frac{a+b}{2},b\right]} \right)}{\Gamma(\alpha+2)} \left(\frac{b-a}{2}\right)^{\alpha+1}. \tag{23}$$

So we have proved that

$$\int_a^b |f(x)|\,dx \leq \max\left\{\left\|D_{a+}^{\alpha}f\right\|_{\infty,\left[a,\frac{a+b}{2}\right]},\left\|D_{b-}^{\alpha}f\right\|_{\infty,\left[\frac{a+b}{2},b\right]}\right\}\frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)\,2^{\alpha}}, \quad (24)$$

proving (18).

Notice that

$$f_*\left(\left(\frac{a+b}{2}\right)_-\right)=f_*\left(\left(\frac{a+b}{2}\right)_+\right)=\left(\frac{b-a}{2}\right)^{\alpha},$$

so that $f_* \in C([a,b])$.

Here $m=0$. We see that

$$\left(J_{1-\beta}^{\alpha+}(\cdot-a)^{\alpha}\right)(x)=\left(J_{1-\alpha}^{a+}(\cdot-a)^{\alpha}\right)(x)=$$

$$\frac{1}{\Gamma(1-\alpha)}\int_a^x (x-t)^{-\alpha}(t-a)^{\alpha}\,dt=$$

$$\frac{1}{\Gamma(1-\alpha)}\int_a^x (x-t)^{(1-\alpha)-1}(t-a)^{(\alpha+1)-1}\,dt=$$

(by [13], p. 256)

$$\frac{1}{\Gamma(1-\alpha)}\frac{\Gamma(1-\alpha)\Gamma(\alpha+1)}{\Gamma(2)}(x-a)=\Gamma(\alpha+1)(x-a).$$

Hence

$$D_{a+}^{\alpha}(x-a)^{\alpha}=\Gamma(\alpha+1),\quad \text{for all } x\in\left[a,\frac{a+b}{2}\right]. \quad (25)$$

Therefore

$$\left\|D_{a+}^{\alpha}(x-a)^{\alpha}\right\|_{\infty,\left[a,\frac{a+b}{2}\right]}=\Gamma(a+1). \quad (26)$$

Furthermore we have

$$\left(J_{b-}^{1-\alpha}(b-\cdot)^{\alpha}\right)(x)=\frac{1}{\Gamma(1-\alpha)}\int_x^b (t-x)^{-\alpha}(b-t)^{\alpha}\,dt=$$

$$\frac{1}{\Gamma(1-\alpha)}\int_x^b (b-t)^{(\alpha+1)-1}(t-x)^{(1-\alpha)-1}\,dt=$$

(by [13], p. 256)

$$\frac{1}{\Gamma(1-\alpha)}\frac{\Gamma(\alpha+1)\Gamma(1-\alpha)}{\Gamma(2)}(b-x)=\Gamma(\alpha+1)(b-x).$$

Therefore

$$D_{b-}^{\alpha}(b-x)^{\alpha}=\Gamma(\alpha+1),\quad \text{for all } x\in\left[\frac{a+b}{2},b\right], \quad (27)$$

and

$$\left\| D_{b-}^{\alpha} (b-x)^{\alpha} \right\|_{\infty, \left[\frac{a+b}{2}, b\right]} = \Gamma (a+1) . \tag{28}$$

Consequently we find that

$$M (f_*) = \Gamma (\alpha + 1) . \tag{29}$$

Applying f_* into (18) we obtain:

$$\text{R.H.S.(18) for } f_* = \Gamma (\alpha + 1) \frac{(b-a)^{\alpha+1}}{\Gamma (\alpha + 2) 2^{\alpha}} = \frac{(b-a)^{\alpha+1}}{(\alpha + 1) 2^{\alpha}}, \tag{30}$$

while we get the same result from

$$\text{L.H.S. (18) for } f_* = \left| \int_a^b f_* (x) \, dx \right| =$$

$$\int_a^{\frac{a+b}{2}} (x-a)^{\alpha} \, dx + \int_{\frac{a+b}{2}}^b (b-x)^{\alpha} \, dx = \frac{(b-a)^{\alpha+1}}{(\alpha + 1) 2^{\alpha}}, \tag{31}$$

proving sharpness of (18). ■

We make

Remark 6 *When $\alpha \geq 1$, thus $m = [\alpha] \geq 1$, and by assuming that $f^{(k)} (a) = f^{(k)} (b) = 0$, $k = 0, 1, ..., m-1$, we can prove the same statements as in Theorem 5. If we set there $\alpha = 1$ we derive exactly Theorem 2. So we generalize Theorem 2. Again here $f^{(m)}$ cannot be a constant different than zero, equivalently, f cannot be a non-trivial polynomial of degree m.*

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A New Comprehensive Class of Analytic Functions Using Multiplier Transformations

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Abstract

Let $\mathcal{A}(p,n)$ denote the class of normalized analytic functions $f(z)$ in the open unit disc $f(z) = z^p + \sum\limits_{k=p+n}^{\infty} a_k z^k$, $p, n \in \mathbb{N} := \{1,2,3,\ldots\}$. We consider in this paper multiplier transformations, namely $I_p(m,\lambda,l)f(z) := z^p + \sum_{k=p+n}^{\infty} \left[\frac{p+\lambda(k-p)+l}{p+l}\right]^m a_k z^k$ where $m \in \mathbb{N}_0, \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \lambda \in \mathbb{R}$, $\lambda \geq 0$, $l \geq 0$. By making use of the multiplier transformations, a new subclass of p-valent functions in the open unit disc is introduced. The new subclass is denoted by $\mathcal{BI}_p(m,n,\mu,\alpha,\lambda,l)$. Parallel results, for some related classes including the class of starlike and convex functions respectively, are also obtained.

Keywords: Analytic function, $p-$valent starlike function, $p-$valent convex function, multiplier transformations.

2000 Mathematical Subject Classification: 30C45

1 Introduction and definitions

Let $\mathcal{A}(p,n)$ denote the class of functions of the form $f(z) = z^p + \sum\limits_{k=p+n}^{\infty} a_k z^k$, $p, n \in \mathbb{N} := \{1,2,3,\ldots\}$, which are analytic in the open unit disc $U = \{z : |z| < 1\}$. In particular we set $\mathcal{A}(p,1) := \mathcal{A}_p$ and $\mathcal{A}(1,1) := \mathcal{A} = \mathcal{A}_1$. Let $\mathcal{H}(U)$ the space of holomorphic functions in U, $n \in \mathbb{N}$.Let $\mathcal{S}$ denote the subclass of functions that are univalent in U.

By $\mathcal{S}_n^*(p,\alpha)$ we denote a subclass of $\mathcal{A}(p,n)$ consisting of $p-$valently starlike univalent functions of order α in U, $0 \leq \alpha < p$ which satisfies $\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha$, $z \in U$. Further, a function f belonging to $\mathcal{S}$ is said to be $p-$valently convex of order α in U, if and only if $\operatorname{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) > \alpha$, $z \in U$, for some α, $(0 \leq \alpha < p)$. We denote by $\mathcal{K}_n(p,\alpha)$ the class of functions in $\mathcal{S}$ which are $p-$valently convex of order α in U and denote by $\mathcal{R}(p,\alpha)$ the class of functions in $\mathcal{A}(p,n)$ which satisfy $\operatorname{Re} f'(z) > \alpha$, $z \in U$.

It is well known that $\mathcal{K}_n(p,\alpha) \subset \mathcal{S}_n^*(p,\alpha) \subset \mathcal{S}$.

If f and g are analytic functions in U, we say that f is subordinate to g, written $f \prec g$, if there is a function w analytic in U, with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$ such that $f(z) = g(w(z))$ for all $z \in U$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

Definition 1.1 [3] *Let $f \in \mathcal{A}(p,n)$. For $\lambda \in \mathbb{R}$, $\lambda \geq 0$, $l \geq 0$, $m \in \mathbb{N}_0$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ we define the multiplier transformations $I_p(m,\lambda,l)$ on $\mathcal{A}(p,n)$ by the following infinite series*

$$I_p(m,\lambda,l)f(z) := z^p + \sum_{k=p+n}^{\infty} \left[\frac{p+\lambda(k-p)+l}{p+l}\right]^m a_k z^k. \tag{1.1}$$

It follows from (1.1) that

$$I_p(0,\lambda,l)f(z) = f(z)$$

$$(p+l)I_p(2,\lambda,l)f(z) = [p(1-\lambda)+l]I_p(1,\lambda,l)f(z) + \lambda z(I_p(1,\lambda,l)f(z))'$$

$$I_p(m_1,\lambda,l)(I_p(m_2,\lambda,l)f(z)) = I_p(m_2,\lambda,l)(I_p(m_1,\lambda,l)f(z)).$$

For $p=1$, $l=0$, $\lambda \geq 0$, the operator $D_\lambda^m := I_1(m,\lambda,0)$ was introduced and studied by Al-Oboudi [2] which reduces to the Sălăgean differential operator [8] for $\lambda = 1$. The operator $I_l^m := I_1(m,1,l)$ was studied recently by Cho and Srivastava [4] and Cho and Kim [5]. The operator $I_m := I_1(m,1,1)$ was studied by Uralegaddi and Somanatha [10], the operator $D_\lambda^\delta := I_1(\delta,\lambda,0)$, $\delta \in \mathbb{R}$, $\delta \geq 0$ was introduced by Acu and Owa [1] and the operator $I_p(m,l) := I_p(m,1,l)$ was investigated recently by Kumar, Taneja and Ravichandran [9].

If $f \in \mathcal{A}(p,n)$ then we have $I_p(m,\lambda,l)f(z) = (f * \varphi_{p,\lambda,l}^m)(z)$, where $\varphi_{p,\lambda,l}^m(z) = z^p + \sum_{k=p+n}^{\infty} \left[\frac{p+\lambda(k-p)+l}{p+l}\right]^m z^k$.

To prove our main theorem we shall need the following lemma.

Lemma 1.2 [7] *Let u be analytic in U with $u(0)=1$ and suppose that*

$$\mathrm{Re}\left(1+\frac{zu'(z)}{u(z)}\right) > \frac{3\alpha-1}{2\alpha}, \quad z \in U.$$

Then $\mathrm{Re}\, u(z) > \alpha$ for $z \in U$ and $1/2 \leq \alpha < 1$.

2 Main results

Definition 2.1 *We say that a function $f \in \mathcal{A}(p,n)$ is in the class $\mathcal{BI}_p(m,n,\mu,\alpha,\lambda,l)$, $n,m \in \mathbb{N}$, $\mu \geq 0$, $\alpha \in [0,p)$ if*

$$\left|\frac{I_p(m+1,\lambda,l)f(z)}{z^p}\left(\frac{z^p}{I_p(m,\lambda,l)f(z)}\right)^\mu - p\right| < p-\alpha, \qquad z \in U. \tag{2.1}$$

Remark 2.2 *The family $\mathcal{BI}_p(m,n,\mu,\alpha,\lambda,l)$ is a new comprehensive class of analytic functions which includes various new subclasses of analytic univalent functions as well as some very well-known ones. For example, $\mathcal{BI}_1(0,1,1,\alpha,1,0) \equiv \mathcal{S}_1^*(1,\alpha)$, $\mathcal{BI}_1(1,1,1,\alpha,1,0) \equiv \mathcal{K}_1(1,\alpha)$ and $\mathcal{BI}_1(0,1,0,\alpha,1,0) \equiv \mathcal{R}(1,\alpha)$. Another interesting subclass is the special case $\mathcal{BI}_1(0,1,2,\alpha,1,l) \equiv \mathcal{B}(\alpha)$ which has been introduced by Frasin and Darus [1] and also the class $\mathcal{BI}_1(0,1,\mu,\alpha,1,0) \equiv \mathcal{B}(\mu,\alpha)$ which has been introduced by Frasin and Jahangiri [2].*

In this note we provide a sufficient condition for functions to be in the class $\mathcal{BI}_p(m,n,\mu,\alpha,\lambda,l)$. Consequently, as a special case, we show that convex functions of order 1/2 are also members of the above defined family.

Theorem 2.3 *For the function $f \in \mathcal{A}(p,n)$, $n,m \in \mathbb{N}$, $\mu \geq 0$, $1/2 \leq \alpha < 1$ if*

$$\frac{p+l}{\lambda}\frac{I_p(m+2,\lambda,l)f(z)}{I_p(m+1,\lambda,l)f(z)} - \frac{\mu(p+l)}{\lambda}\frac{I_p(m+1,\lambda,l)f(z)}{I_p(m,\lambda,l)f(z)} + + \frac{(\mu-1)(p+l)+\lambda}{\lambda} \prec 1+\beta z, \quad z \in U \tag{2.2}$$

where $\beta = \frac{3\alpha-1}{2\alpha}$, then $f \in \mathcal{BI}_p(m,n,\mu,\alpha,\lambda,l)$.

Proof. If we consider $u(z) = \frac{I_p(m+1,\lambda,l)f(z)}{z^p}\left(\frac{z^p}{I_p(m,\lambda,l)f(z)}\right)^\mu$, then $u(z)$ is analytic in U with $u(0)=1$. A simple differentiation yields

$$\frac{zu'(z)}{u(z)} = \frac{p+l}{\lambda}\frac{I_p(m+2,\lambda,l)f(z)}{I_p(m+1,\lambda,l)f(z)} - \frac{\mu(p+l)}{\lambda}\frac{I_p(m+1,\lambda,l)f(z)}{I_p(m,\lambda,l)f(z)} + \frac{(\mu-1)(p+l)}{\lambda}. \tag{2.3}$$

Using (2.2) we get $\mathrm{Re}\left(1+\frac{zu'(z)}{u(z)}\right) > \frac{3\alpha-1}{2\alpha}$. Thus, from Lemma 1.2 we deduce that

$$\mathrm{Re}\left\{\frac{I_p(m+1,\lambda,l)f(z)}{z^p}\left(\frac{z^p}{I_p(m,\lambda,l)f(z)}\right)^{\mu}\right\} > \alpha.$$

Therefore, $f \in \mathcal{BI}_p(m,n,\mu,\alpha,\lambda,l)$, by Definition 2.1. ■

As a consequence of the above theorem we have the following interesting corollaries.

Corollary 2.4 *If $f \in \mathcal{A}(1,1)$ and $\mathrm{Re}\left\{\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)} - \frac{zf''(z)}{f'(z)}\right\} > -\frac{1}{2},\ z \in U$, then $f \in \mathcal{BI}_1(1,1,1,\frac{1}{2},1,0)$ or $\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \frac{1}{2},\ z \in U$. That is, f is convex of order $\frac{1}{2}$.*

Corollary 2.5 *If $f \in \mathcal{A}(1,1)$ and $\mathrm{Re}\left\{\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}\right\} > -\frac{1}{2},\ z \in U$, then $f \in \mathcal{BI}_1(1,1,0,\frac{1}{2},1,0)$, that is $\mathrm{Re}\{f'(z)+zf''(z)\} > \frac{1}{2},\ z \in U$.*

Corollary 2.6 *If $f \in \mathcal{A}(1,1)$ and $\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \frac{1}{2},\ z \in U$, then $\mathrm{Re}\, f'(z) > \frac{1}{2},\ z \in U$. In another words, if the function f is convex of order $\frac{1}{2}$ then $f \in \mathcal{BI}_1(0,1,0,\frac{1}{2},1,0) \equiv \mathcal{R}\left(1,\frac{1}{2}\right)$.*

Corollary 2.7 *If $f \in \mathcal{A}(1,1)$ and $\mathrm{Re}\left\{\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right\} > -\frac{3}{2},\ z \in U$, then $f \in \mathcal{BI}_1(0,1,1,\frac{1}{2},1,0)$. In another words f is starlike of order $\frac{1}{2}$.*

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A note on special strong differential superordinations using multiplier transformation

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Abstract

In the present paper we establish several strong differential superordinations regarding the extended multiplier transformation $I(m,\lambda,l) : \mathcal{A}^*_{n\zeta} \to \mathcal{A}^*_{n\zeta}$, $I(m,\lambda,l) f(z,\zeta) = z+\sum_{j=n+1}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m a_j(\zeta) z^j$, $z \in U$, $\zeta \in \overline{U}$, and $\mathcal{A}^*_{n\zeta} = \{f \in \mathcal{H}(U \times \overline{U}),\ f(z,\zeta) = z + a_{n+1}(\zeta) z^{n+1} + \dots,\ z \in U, \zeta \in \overline{U}\}$ is the class of normalized analytic functions.

Keywords: strong differential superordination, convex function, best subordinant, extended differential operator.
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1 Introduction

Denote by U the unit disc of the complex plane $U = \{z \in \mathbb{C} : |z| < 1\}$, $\overline{U} = \{z \in \mathbb{C} : |z| \le 1\}$ the closed unit disc of the complex plane and $\mathcal{H}(U \times \overline{U})$ the class of analytic functions in $U \times \overline{U}$.

Let $\mathcal{A}^*_{n\zeta} = \{f \in \mathcal{H}(U \times \overline{U}),\ f(z,\zeta) = z + a_{n+1}(\zeta) z^{n+1} + \dots,\ z \in U, \zeta \in \overline{U}\}$, where $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k \ge 2$, and $\mathcal{H}^*[a,n,\zeta] = \{f \in \mathcal{H}(U \times \overline{U}),\ f(z,\zeta) = a + a_n(\zeta) z^n + a_{n+1}(\zeta) z^{n+1} + \dots,\ z \in U, \zeta \in \overline{U}\}$, for $a \in \mathbb{C}$, $n \in \mathbb{N}$, $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k \ge n$.

Denote by $K_{n\zeta} = \{f \in \mathcal{H}(U \times \overline{U}) : Re\frac{zf''_z(z,\zeta)}{f'_z(z,\zeta)} + 1 > 0\}$ the class of convex function in $U \times \overline{U}$.

As a dual notion of strong differential subordination G.I. Oros has introduced and developed the notion of strong differential superordinations in [3].

Definition 1.1 *[3] Let $f(z,\zeta)$, $H(z,\zeta)$ analytic in $U \times \overline{U}$. The function $f(z,\zeta)$ is said to be strongly superordinate to $H(z,\zeta)$ if there exists a function w analytic in U, with $w(0) = 0$ and $|w(z)| < 1$, such that $H(z,\zeta) = f(w(z),\zeta)$, for all $\zeta \in \overline{U}$. In such a case we write $H(z,\zeta) \prec\prec f(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.*

Remark 1.1 *[3] (i) Since $f(z,\zeta)$ is analytic in $U \times \overline{U}$, for all $\zeta \in \overline{U}$, and univalent in U, for all $\zeta \in \overline{U}$, Definition 1.1 is equivalent to $H(0,\zeta) = f(0,\zeta)$, for all $\zeta \in \overline{U}$, and $H\left(U \times \overline{U}\right) \subset f\left(U \times \overline{U}\right)$.*

(ii) If $H(z,\zeta) \equiv H(z)$ and $f(z,\zeta) \equiv f(z)$, the strong superordination becomes the usual notion of superordination.

Definition 1.2 *We denote by Q^* the set of functions that are analytic and injective on $\overline{U} \times \overline{U} \backslash E(f,\zeta)$, where $E(f,\zeta) = \{y \in \partial U : \lim\limits_{z \to y} f(z,\zeta) = \infty\}$, and are such that $f'_z(y,\zeta) \ne 0$ for $y \in \partial U \times \overline{U} \backslash E(f,\zeta)$. The subclass of Q^* for which $f(0,\zeta) = a$ is denoted by $Q^*(a)$.*

We have need the following lemmas to study the strong differential superordinations.

Lemma 1.1 *Let $h(z,\zeta)$ be a convex function with $h(0,\zeta) = a$ and let $\gamma \in \mathbb{C}^*$ be a complex number with $Re\,\gamma \ge 0$. If $p \in \mathcal{H}^*[a,n,\zeta] \cap Q^*$, $p(z,\zeta) + \frac{1}{\gamma} zp'_z(z,\zeta)$ is univalent in $U \times \overline{U}$ and $h(z,\zeta) \prec\prec p(z,\zeta) + \frac{1}{\gamma} zp'_z(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, then $q(z,\zeta) \prec\prec p(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, where $q(z,\zeta) = \frac{\gamma}{nz^{\frac{\gamma}{n}}}\int_0^z h(t,\zeta) t^{\frac{\gamma}{n}-1} dt$, $z \in U$, $\zeta \in \overline{U}$. The function q is convex and is the best subordinant.*

Lemma 1.2 *Let $q(z,\zeta)$ be a convex function in $U\times\overline{U}$ and let $h(z,\zeta)=q(z,\zeta)+\frac{1}{\gamma}zq'_z(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, where $Re\ \gamma\geq 0$. If $p\in\mathcal{H}^*[a,n,\zeta]\cap Q^*$, $p(z,\zeta)+\frac{1}{\gamma}zp'_z(z,\zeta)$ is univalent in $U\times\overline{U}$ and $q(z,\zeta)+\frac{1}{\gamma}zq'_z(z,\zeta)\prec\prec p(z,\zeta)+\frac{1}{\gamma}zp'_z(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, then $q(z,\zeta)\prec\prec p(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, where $q(z,\zeta)=\frac{\gamma}{nz^{\frac{\gamma}{n}}}\int_0^z h(t,\zeta)\,t^{\frac{\gamma}{n}-1}dt$, $z\in U$, $\zeta\in\overline{U}$. The function q is the best subordinant.*

We extend the differential operators studied in [1] to the new class of analytic functions $\mathcal{A}^*_{n\zeta}$ introduced in [4].

Definition 1.3 *[2] For $n\in\mathbb{N}$, $m\in\mathbb{N}\cup\{0\}$, $\lambda,l\geq 0$, $f\in\mathcal{A}^*_{n\zeta}$, $f(z,\zeta)=z+\sum_{j=n+1}^{\infty}a_j(\zeta)z^j$, the operator $I(m,\lambda,l)f(z,\zeta)$ is defined by the following infinite series*

$I(m,\lambda,l)f(z,\zeta)=z+\sum_{j=n+1}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m a_j(\zeta)z^j$, $z\in U$, $\zeta\in\overline{U}$.

Remark 1.2 *[2] It follows from the above definition that*

$(l+1)I(m+1,\lambda,l)f(z,\zeta)=[l+1-\lambda]I(m,\lambda,l)f(z,\zeta)+\lambda z(I(m,\lambda,l)f(z,\zeta))'_z$, $z\in U$, $\zeta\in\overline{U}$.

2 Main results

Theorem 2.1 *Let $h(z,\zeta)$ be a convex function in $U\times\overline{U}$ with $h(0,\zeta)=1$. Let $m,n\in\mathbb{N}$, $\lambda,l\geq 0$, $f(z,\zeta)\in\mathcal{A}^*_{n\zeta}$, $F(z,\zeta)=I_c(f)(z,\zeta)=\frac{c+2}{z^{c+1}}\int_0^z t^c f(t,\zeta)\,dt$, $z\in U$, $\zeta\in\overline{U}$, $Rec>-2$, and suppose that $(I(m,\lambda,l)f(z,\zeta))'_z$ is univalent in $U\times\overline{U}$, $(I(m,\lambda,l)F(z,\zeta))'_z\in\mathcal{H}^*[1,n,\zeta]\cap Q^*$ and*

$$h(z,\zeta)\prec\prec(I(m,\lambda,l)f(z,\zeta))'_z,\quad z\in U,\ \zeta\in\overline{U},\tag{2.1}$$

then

$$q(z,\zeta)\prec\prec(I(m,\lambda,l)F(z,\zeta))'_z,\quad z\in U,\ \zeta\in\overline{U},$$

where $q(z,\zeta)=\frac{c+2}{nz^{\frac{c+2}{n}}}\int_0^z h(t,\zeta)t^{\frac{c+2}{n}-1}dt$. The function q is convex and it is the best subordinant.

Proof. We have $z^{c+1}F(z,\zeta)=(c+2)\int_0^z t^c f(t,\zeta)\,dt$ and differentiating it, with respect to z, we obtain $(c+1)F(z,\zeta)+zF'_z(z,\zeta)=(c+2)f(z,\zeta)$ and $(c+1)I(m,\lambda,l)F(z,\zeta)+z(I(m,\lambda,l)F(z,\zeta))'_z=(c+2)I(m,\lambda,l)f(z,\zeta)$, $z\in U,\zeta\in\overline{U}$.

Differentiating the last relation with respect to z we have

$$(I(m,\lambda,l)F(z,\zeta))'_z+\frac{1}{c+2}z(I(m,\lambda,l)F(z,\zeta))''_{z^2}=(I(m,\lambda,l)f(z,\zeta))'_z,\ z\in U,\zeta\in\overline{U}.\tag{2.2}$$

Using (2.2), the strong differential superordination (2.1) becomes

$$h(z,\zeta)\prec\prec(I(m,\lambda,l)F(z,\zeta))'_z+\frac{1}{c+2}z(I(m,\lambda,l)F(z,\zeta))''_{z^2}.\tag{2.3}$$

Denote

$$p(z,\zeta)=(I(m,\lambda,l)F(z,\zeta))'_z,\quad z\in U,\ \zeta\in\overline{U}.\tag{2.4}$$

Replacing (2.4) in (2.3) we obtain $h(z,\zeta)\prec\prec p(z,\zeta)+\frac{1}{c+2}zp'_z(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$. Using Lemma 1.1 for $\gamma=c+2$, we have $q(z,\zeta)\prec\prec p(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, i.e. $q(z,\zeta)\prec\prec(I(m,\lambda,l)F(z,\zeta))'_z$, $z\in U$, $\zeta\in\overline{U}$, where $q(z,\zeta)=\frac{c+2}{nz^{\frac{c+2}{n}}}\int_0^z h(t,\zeta)t^{\frac{c+2}{n}-1}dt$. The function q is convex and it is the best subordinant. ■

Corollary 2.2 *Let $h(z,\zeta)=\frac{1+(2\beta-\zeta)z}{1+z}$, where $\beta\in[0,1)$. Let $m,n\in\mathbb{N}$, $\lambda,l\geq 0$, $f(z,\zeta)\in\mathcal{A}^*_{n\zeta}$, $F(z,\zeta)=I_c(f)(z,\zeta)=\frac{c+2}{z^{c+1}}\int_0^z t^c f(t,\zeta)\,dt$, $z\in U$, $\zeta\in\overline{U}$, $Rec>-2$, and suppose that $(I(m,\lambda,l)f(z,\zeta))'_z$ is univalent in $U\times\overline{U}$, $(I(m,\lambda,l)F(z,\zeta))'_z\in\mathcal{H}^*[1,n,\zeta]\cap Q^*$ and*

$$h(z,\zeta)\prec\prec(I(m,\lambda,l)f(z,\zeta))'_z,\quad z\in U,\ \zeta\in\overline{U},\tag{2.5}$$

then

$$q(z,\zeta)\prec\prec(I(m,\lambda,l)F(z,\zeta))'_z,\quad z\in U,\ \zeta\in\overline{U},$$

where q is given by $q(z,\zeta)=2\beta-\zeta+\frac{(c+2)(1+\zeta-2\beta)}{nz^{\frac{c+2}{n}}}\int_0^z\frac{t^{\frac{c+2}{n}-1}}{t+1}dt$, $z\in U$, $\zeta\in\overline{U}$. The function q is convex and it is the best subordinant.

Proof. Following the same steps as in the proof of Theorem 2.1 and considering $p(z,\zeta) = (I(m,\lambda,l)F(z,\zeta))'_z$, the strong differential superordination (2.5) becomes $h(z,\zeta) = \frac{1+(2\beta-\zeta)z}{1+z} \prec\prec p(z,\zeta) + \frac{1}{c+2}zp'_z(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$. By using Lemma 1.1 for $\gamma = c+2$, we have $q(z,\zeta) \prec\prec p(z,\zeta)$, i.e. $q(z,\zeta) = \frac{c+2}{nz^{\frac{c+2}{n}}}\int_0^z h(t,\zeta)t^{\frac{c+2}{n}-1}dt = \frac{c+2}{nz^{\frac{c+2}{n}}}\int_0^z \frac{1+(2\beta-\zeta)t}{1+t}t^{\frac{c+2}{n}-1}dt = 2\beta - \zeta + \frac{(c+2)(1+\zeta-2\beta)}{nz^{\frac{c+2}{n}}}\int_0^z \frac{t^{\frac{c+2}{n}-1}}{t+1}dt \prec\prec (I(m,\lambda,l)F(z,\zeta))'_z$, $z \in U$, $\zeta \in \overline{U}$. The function q is convex and it is the best subordinant. ■

Theorem 2.3 *Let $q(z,\zeta)$ be a convex function in $U\times\overline{U}$ and let $h(z,\zeta) = q(z,\zeta) + \frac{1}{c+2}zq'_z(z,\zeta)$, where $z \in U$, $\zeta \in \overline{U}$, $Re c > -2$. Let $m,n \in \mathbb{N}$, $\lambda, l \geq 0$, $f(z,\zeta) \in \mathcal{A}^*_{n\zeta}$, $F(z,\zeta) = I_c(f)(z,\zeta) = \frac{c+2}{z^{c+1}}\int_0^z t^c f(t,\zeta)\,dt$, $z \in U$, $\zeta \in \overline{U}$, and suppose that $(I(m,\lambda,l)f(z,\zeta))'_z$ is univalent in $U\times\overline{U}$, $(I(m,\lambda,l)F(z,\zeta))'_z \in \mathcal{H}^*[1,n,\zeta]\cap Q^*$ and*

$$h(z,\zeta) \prec\prec (I(m,\lambda,l)f(z,\zeta))'_z, \quad z \in U,\ \zeta \in \overline{U}, \tag{2.6}$$

then

$$q(z,\zeta) \prec\prec (I(m,\lambda,l)F(z,\zeta))'_z, \quad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{c+2}{nz^{\frac{c+2}{n}}}\int_0^z h(t,\zeta)t^{\frac{c+2}{n}-1}dt$. The function q is the best subordinant.

Proof. Following the same steps as in the proof of Theorem 2.1 and considering $p(z,\zeta) = (I(m,\lambda,l)F(z,\zeta))'_z$, $z \in U$, $\zeta \in \overline{U}$, the strong differential superordination (2.6) becomes $h(z,\zeta) = q(z,\zeta) + \frac{1}{c+2}zq'_z(z,\zeta) \prec\prec p(z,\zeta) + \frac{1}{c+2}zp'_z(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$. Using Lemma 1.2 for $\gamma = c+2$, we have $q(z,\zeta) \prec\prec p(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, i.e. $q(z,\zeta) \prec\prec (I(m,\lambda,l)F(z,\zeta))'_z$, $z \in U$, $\zeta \in \overline{U}$, where $q(z,\zeta) = \frac{c+2}{nz^{\frac{c+2}{n}}}\int_0^z h(t,\zeta)t^{\frac{c+2}{n}-1}dt$. The function q is the best subordinant. ■

Theorem 2.4 *Let $h(z,\zeta)$ be a convex function, $h(0,\zeta) = 1$. Let $m,n \in \mathbb{N}$, $\lambda, l \geq 0$, $f(z,\zeta) \in \mathcal{A}^*_{n\zeta}$ and suppose that $(I(m,\lambda,l)f(z,\zeta))'_z$ is univalent and $\frac{I(m,\lambda,l)f(z,\zeta)}{z} \in \mathcal{H}^*[1,n,\zeta]\cap Q^*$. If*

$$h(z,\zeta) \prec\prec (I(m,\lambda,l)f(z,\zeta))'_z, \quad z \in U,\ \zeta \in \overline{U}, \tag{2.7}$$

then

$$q(z,\zeta) \prec\prec \frac{I(m,\lambda,l)f(z,\zeta)}{z}, \quad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1}dt$. The function q is convex and it is the best subordinant.

Proof. Consider $p(z,\zeta) = \frac{I(m,\lambda,l)f(z,\zeta)}{z} = \frac{z+\sum_{j=n+1}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m a_j(\zeta)z^j}{z} = 1 + p_n(\zeta)z^n + p_{n+1}(\zeta)z^{n+1} + \ldots$, $z \in U$, $\zeta \in \overline{U}$. We deduce that $p \in \mathcal{H}^*[1,n,\zeta]$.
Let $I(m,\lambda,l)f(z,\zeta) = zp(z,\zeta)$, $z \in U, \zeta \in \overline{U}$. Differentiating with respect to z we obtain $(I(m,\lambda,l)f(z,\zeta))'_z = p(z,\zeta) + zp'_z(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.

Then (2.7) becomes $h(z,\zeta) \prec\prec p(z,\zeta) + zp'_z(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$. By using Lemma 1.1 for $\gamma = 1$, we have $q(z,\zeta) \prec\prec p(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, i.e. $q(z,\zeta) \prec\prec \frac{I(m,\lambda,l)f(z,\zeta)}{z}$, $z \in U$, $\zeta \in \overline{U}$, where $q(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1}dt$. The function q is convex and it is the best subordinant. ■

Corollary 2.5 *Let $h(z,\zeta) = \frac{1+(2\beta-\zeta)z}{1+z}$ be a convex function in $U\times\overline{U}$, where $0 \leq \beta < 1$. Let $m,n \in \mathbb{N}$, $\lambda, l \geq 0$, $f(z,\zeta) \in \mathcal{A}^*_{n\zeta}$ and suppose that $(I(m,\lambda,l)f(z,\zeta))'_z$ is univalent and $\frac{I(m,\lambda,l)f(z,\zeta)}{z} \in \mathcal{H}^*[1,n,\zeta]\cap Q^*$. If*

$$h(z,\zeta) \prec (I(m,\lambda,l)f(z,\zeta))'_z, \quad z \in U,\ \zeta \in \overline{U}, \tag{2.8}$$

then

$$q(z,\zeta) \prec\prec \frac{I(m,\lambda,l)f(z,\zeta)}{z}, \quad z \in U,\ \zeta \in \overline{U},$$

where q is given by $q(z,\zeta) = 2\beta - \zeta + \frac{1+\zeta-2\beta}{nz^{\frac{1}{n}}}\int_0^z \frac{t^{\frac{1}{n}-1}}{t+1}dt$, $z \in U$. The function q is convex and it is the best subordinant.

Proof. Following the same steps as in the proof of Theorem 2.4 and considering $p(z,\zeta)=\frac{I(m,\lambda,l)f(z,\zeta)}{z}$, the strong differential superordination (2.8) becomes $h(z,\zeta)=\frac{1+(2\beta-\zeta)z}{1+z}\prec\prec p(z,\zeta)+zp'_z(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$. By using Lemma 1.1 for $\gamma=1$, we have $q(z,\zeta)\prec\prec p(z,\zeta)$, i.e. $q(z,\zeta)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1}dt=\frac{1}{nz^{\frac{1}{n}}}\int_0^z\frac{1+(2\beta-\zeta)t}{1+t}t^{\frac{1}{n}-1}dt=2\beta-\zeta+\frac{1+\zeta-2\beta}{nz^{\frac{1}{n}}}\int_0^z\frac{t^{\frac{1}{n}-1}}{t+1}dt\prec\prec\frac{I(m,\lambda,l)f(z,\zeta)}{z}$, $z\in U$, $\zeta\in\overline{U}$. The function q is convex and it is the best subordinant. ∎

Theorem 2.6 *Let $q(z,\zeta)$ be convex in $U\times\overline{U}$ and let h be defined by $h(z,\zeta)=q(z,\zeta)+zq'_z(z,\zeta)$. If $m,n\in\mathbb{N}$, $\lambda,l\geq 0$, $f(z,\zeta)\in\mathcal{A}^*_{n\zeta}$, suppose that $(I(m,\lambda,l)f(z,\zeta))'_z$ is univalent and $\frac{I(m,\lambda,l)f(z,\zeta)}{z}\in\mathcal{H}^*[1,n,\zeta]\cap Q^*$ and satisfies the strong differential superordination*

$$h(z,\zeta)=q(z,\zeta)+zq'_z(z,\zeta)\prec\prec(I(m,\lambda,l)f(z,\zeta))'_z,\quad z\in U,\ \zeta\in\overline{U},\tag{2.9}$$

then

$$q(z,\zeta)\prec\prec\frac{I(m,\lambda,l)f(z,\zeta)}{z},\quad z\in U,\ \zeta\in\overline{U},$$

where $q(z,\zeta)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1}dt$. The function q is the best subordinant.

Proof. Following the same steps as in the proof of Theorem 2.4 and considering $p(z,\zeta)=\frac{I(m,\lambda,l)f(z,\zeta)}{z}$, the strong differential superordination (2.9) becomes $q(z,\zeta)+zq'_z(z,\zeta)\prec\prec p(z,\zeta)+zp'_z(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$. Using Lemma 1.2 for $\gamma=1$, we have $q(z,\zeta)\prec\prec p(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, i.e. $q(z,\zeta)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1}dt\prec\prec\frac{I(m,\lambda,l)f(z,\zeta)}{z}$, $z\in U$, $\zeta\in\overline{U}$, and q is the best subordinant. ∎

Theorem 2.7 *Let $h(z,\zeta)$ be a convex function, $h(0,\zeta)=1$. Let $m,n\in\mathbb{N}$, $\lambda,l\geq 0$, $f(z,\zeta)\in\mathcal{A}^*_{n\zeta}$ and suppose that $\left(\frac{zI(m+1,\lambda,l)f(z,\zeta)}{I(m,\lambda,l)f(z,\zeta)}\right)'_z$ is univalent and $\frac{I(m+1,\lambda,l)f(z,\zeta)}{I(m,\lambda,l)f(z,\zeta)}\in\mathcal{H}^*[1,n,\zeta]\cap Q^*$. If*

$$h(z,\zeta)\prec\prec\left(\frac{zI(m+1,\lambda,l)f(z,\zeta)}{I(m,\lambda,l)f(z,\zeta)}\right)'_z,\quad z\in U,\ \zeta\in\overline{U},\tag{2.10}$$

then

$$q(z,\zeta)\prec\prec\frac{I(m+1,\lambda,l)f(z,\zeta)}{I(m,\lambda,l)f(z,\zeta)},\quad z\in U,\ \zeta\in\overline{U},$$

where $q(z,\zeta)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1}dt$. The function q is convex and it is the best subordinant.

Proof. Consider $p(z,\zeta)=\frac{I(m+1,\lambda,l)f(z,\zeta)}{I(m,\lambda,l)f(z,\zeta)}=\frac{z+\sum_{j=n+1}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m+1}a_j(\zeta)z^j}{z+\sum_{j=n+1}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m}a_j(\zeta)z^j}$. We have $p'_z(z,\zeta)=\frac{(I(m+1,\lambda,l)f(z,\zeta))'_z}{I(m,\lambda,l)f(z,\zeta)}-p(z,\zeta)\cdot\frac{(I(m,\lambda,l)f(z,\zeta))'_z}{I(m,\lambda,l)f(z,\zeta)}$ and we obtain $p(z,\zeta)+z\cdot p'_z(z,\zeta)=\left(\frac{zI(m+1,\lambda,l)f(z,\zeta)}{I(m,\lambda,l)f(z,\zeta)}\right)'_z$.

Relation (2.10) becomes $h(z,\zeta)\prec\prec p(z,\zeta)+zp'_z(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$. By using Lemma 1.1 for $\gamma=1$, we have $q(z,\zeta)\prec\prec p(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, i.e. $q(z,\zeta)\prec\prec\frac{I(m+1,\lambda,l)f(z,\zeta)}{I(m,\lambda,l)f(z,\zeta)}$, $z\in U$, $\zeta\in\overline{U}$, where $q(z,\zeta)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1}dt$. The function q is convex and it is the best subordinant. ∎

Corollary 2.8 *Let $h(z,\zeta)=\frac{1+(2\beta-\zeta)z}{1+z}$ be a convex function in $U\times\overline{U}$, where $0\leq\beta<1$. Let $\lambda,l\geq 0$, $m,n\in\mathbb{N}$, $f(z,\zeta)\in\mathcal{A}^*_{n\zeta}$ and suppose that $\left(\frac{zI(m+1,\lambda,l)f(z,\zeta)}{I(m,\lambda,l)f(z,\zeta)}\right)'_z$ is univalent and $\frac{I(m+1,\lambda,l)f(z,\zeta)}{I(m,\lambda,l)f(z,\zeta)}\in\mathcal{H}^*[1,n,\zeta]\cap Q^*$. If*

$$h(z,\zeta)\prec\prec\left(\frac{zI(m+1,\lambda,l)f(z,\zeta)}{I(m,\lambda,l)f(z,\zeta)}\right)'_z,\quad z\in U,\ \zeta\in\overline{U},\tag{2.11}$$

then

$$q(z,\zeta)\prec\prec\frac{I(m+1,\lambda,l)f(z,\zeta)}{I(m,\lambda,l)f(z,\zeta)},\quad z\in U,\ \zeta\in\overline{U},$$

where q is given by $q(z,\zeta)=2\beta-\zeta+\frac{1+\zeta-2\beta}{nz^{\frac{1}{n}}}\int_0^z\frac{t^{\frac{1}{n}-1}}{t+1}dt$, $z\in U$, $\zeta\in\overline{U}$. The function q is convex and it is the best subordinant.

Proof. Following the same steps as in the proof of Theorem 2.7 and considering $p(z,\zeta)=\frac{I(m+1,\lambda,l)f(z,\zeta)}{I(m,\lambda,l)f(z,\zeta)}$, the strong differential superordination (2.11) becomes $h(z,\zeta)=\frac{1+(2\beta-\zeta)z}{1+z}\prec\prec p(z,\zeta)+zp_z'(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$. By using Lemma 1.1 for $\gamma=1$, we have $q(z,\zeta)\prec\prec p(z,\zeta)$, i.e., $q(z,\zeta)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1}=\frac{1}{nz^{\frac{1}{n}}}\int_0^z\frac{1+(2\beta-\zeta)t}{1+t}t^{\frac{1}{n}-1}dt=2\beta-\zeta+\frac{1+\zeta-2\beta}{nz^{\frac{1}{n}}}\int_0^z\frac{t^{\frac{1}{n}-1}}{t+1}dt\prec\prec\frac{I(m+1,\lambda,l)f(z,\zeta)}{I(m,\lambda,l)f(z,\zeta)}$, $z\in U$, $\zeta\in\overline{U}$. The function q is convex and it is the best subordinant. ■

Theorem 2.9 *Let $q(z,\zeta)$ be a convex function and h be defined by $h(z,\zeta)=q(z,\zeta)+zq_z'(z,\zeta)$. Let $\lambda,l\geq 0$, $m,n\in\mathbb{N}$, $f(z,\zeta)\in\mathcal{A}_{n\zeta}^*$ and suppose that $\left(\frac{zI(m+1,\lambda,l)f(z,\zeta)}{I(m,\lambda,l)f(z,\zeta)}\right)_z'$ is univalent and $\frac{I(m+1,\lambda,l)f(z,\zeta)}{I(m,\lambda,l)f(z,\zeta)}\in\mathcal{H}^*[1,n,\zeta]\cap Q^*$. If*

$$h(z,\zeta)=q(z,\zeta)+zq_z'(z,\zeta)\prec\prec\left(\frac{zI(m+1,\lambda,l)f(z,\zeta)}{I(m,\lambda,l)f(z,\zeta)}\right)_z',\qquad z\in U,\ \zeta\in\overline{U},\tag{2.12}$$

then

$$q(z,\zeta)\prec\prec\frac{I(m+1,\lambda,l)f(z,\zeta)}{I(m,\lambda,l)f(z,\zeta)},\qquad z\in U,\ \zeta\in\overline{U},$$

where $q(z,\zeta)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1}dt$. The function q is the best subordinant.

Proof. Following the same steps as in the proof of Theorem 2.7 and considering $p(z,\zeta)=\frac{I(m+1,\lambda,l)f(z,\zeta)}{I(m,\lambda,l)f(z,\zeta)}$, the strong differential superordination (2.12) becomes $h(z,\zeta)=q(z,\zeta)+zq_z'(z,\zeta)\prec p(z,\zeta)+zp_z'(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$. By using Lemma 1.2 for $\gamma=1$, we have $q(z,\zeta)\prec\prec p(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, i.e. $q(z,\zeta)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1}dt\prec\prec\frac{I(m+1,\lambda,l)f(z,\zeta)}{I(m,\lambda,l)f(z,\zeta)}$, $z\in U$, $\zeta\in\overline{U}$, and q is the best subordinant. ■

Theorem 2.10 *Let $h(z,\zeta)$ be a convex function in $U\times\overline{U}$ with $h(0,\zeta)=1$ and let $\lambda,l\geq 0$, $m,n\in\mathbb{N}$, $f(z,\zeta)\in\mathcal{A}_{n\zeta}^*$, $\frac{l+1}{\lambda}I(m+1,\lambda,l)f(z,\zeta)+\left(2-\frac{l+1}{\lambda}\right)I(m,\lambda,l)f(z,\zeta)$ is univalent and $[I(m,\lambda,l)f(z,\zeta)]_z'\in\mathcal{H}^*[1,n,\zeta]\cap Q^*$. If*

$$h(z,\zeta)\prec\prec\frac{l+1}{\lambda}I(m+1,\lambda,l)f(z,\zeta)+\left(2-\frac{l+1}{\lambda}\right)I(m,\lambda,l)f(z,\zeta),\ z\in U,\ \zeta\in\overline{U},\tag{2.13}$$

holds, then

$$q(z,\zeta)\prec\prec[I(m,\lambda,l)f(z,\zeta)]_z',\qquad z\in U,\ \zeta\in\overline{U},$$

where $q(z,\zeta)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1}$. The function q is convex and it is the best subordinant.

Proof. Let

$$p(z,\zeta)=(I(m,\lambda,l)f(z,\zeta))_z'=1+\sum_{j=n+1}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m ja_j(\zeta)z^{j-1}=1+p_n(\zeta)z^n+p_{n+1}(\zeta)z^{n+1}+\ldots.\tag{2.14}$$

We obtain $p(z,\zeta)+z\cdot p_z'(z,\zeta)=I(m,\lambda,l)f(z,\zeta)+z(I(m,\lambda,l)f(z,\zeta))_z'=I(m,\lambda,l)f(z,\zeta)+\frac{(l+1)I(m+1,\lambda,l)f(z,\zeta)-(l+1-\lambda)I(m,\lambda,l)f(z,\zeta)}{\lambda}=\frac{l+1}{\lambda}I(m+1,\lambda,l)f(z,\zeta)+\left(2-\frac{l+1}{\lambda}\right)I(m,\lambda,l)f(z,\zeta)$.

Using (2.14), the strong differential superordination (2.13) becomes $h(z,\zeta)\prec\prec p(z,\zeta)+zp_z'(z,\zeta)$. By using Lemma 1.1 for $\gamma=1$, we have $q(z,\zeta)\prec\prec p(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, i.e. $q(z,\zeta)\prec\prec(I(m,\lambda,l)f(z,\zeta))_z'$, $z\in U$, $\zeta\in\overline{U}$, where $q(z,\zeta)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1}$. The function q is convex and it is the best subordinant. ■

Corollary 2.11 *Let $h(z,\zeta)=\frac{1+(2\beta-\zeta)z}{1+z}$ be a convex function in $U\times\overline{U}$, where $0\leq\beta<1$. Let $\lambda,l\geq 0$, $m,n\in\mathbb{N}$, $f(z,\zeta)\in\mathcal{A}_{n\zeta}$, suppose that $\frac{l+1}{\lambda}I(m+1,\lambda,l)f(z,\zeta)+\left(2-\frac{l+1}{\lambda}\right)I(m,\lambda,l)f(z,\zeta)$ is univalent in $U\times\overline{U}$ and $[I(m,\lambda,l)f(z,\zeta)]_z'\in\mathcal{H}^*[1,n,\zeta]\cap Q^*$. If*

$$h(z,\zeta)\prec\prec\frac{l+1}{\lambda}I(m+1,\lambda,l)f(z,\zeta)+\left(2-\frac{l+1}{\lambda}\right)I(m,\lambda,l)f(z,\zeta),\ z\in U,\ \zeta\in\overline{U},\tag{2.15}$$

then

$$q(z,\zeta)\prec\prec(I(m,\lambda,l)f(z,\zeta))_z',\qquad z\in U,\ \zeta\in\overline{U},$$

where q is given by $q(z,\zeta)=2\beta-\zeta+\frac{1+\zeta-2\beta}{nz^{\frac{1}{n}}}\int_0^z\frac{t^{\frac{1}{n}-1}}{t+1}dt$, $z\in U$, $\zeta\in\overline{U}$. The function q is convex and it is the best subordinant.

Proof. Following the same steps as in the proof of Theorem 2.10 and considering $p(z,\zeta) = \left(I(m,\lambda,l)f(z,\zeta)\right)'_z$, the strong differential superordination (2.15) becomes $h(z,\zeta) = \frac{1+(2\beta-\zeta)z}{1+z} \prec\prec p(z,\zeta) + zp'_z(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$. By using Lemma 1.1 for $\gamma = 1$, we have $q(z,\zeta) \prec\prec p(z,\zeta)$, i.e., $q(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1} = \frac{1}{nz^{\frac{1}{n}}}\int_0^z \frac{1+(2\beta-\zeta)t}{1+t}t^{\frac{1}{n}-1}dt = 2\beta - \zeta + \frac{1+\zeta-2\beta}{nz^{\frac{1}{n}}}\int_0^z \frac{t^{\frac{1}{n}-1}}{t+1}dt \prec\prec \left(I(m,\lambda,l)f(z,\zeta)\right)'_z$, $z \in U$, $\zeta \in \overline{U}$. The function q is convex and it is the best subordinant. ■

Theorem 2.12 *Let $q(z,\zeta)$ be a convex function in $U \times \overline{U}$ and $h(z,\zeta) = q(z,\zeta) + zq'_z(z,\zeta)$. Let $\lambda, l \geq 0$, $m, n \in \mathbb{N}$, $f(z,\zeta) \in \mathcal{A}^*_{n\zeta}$, suppose that $\frac{l+1}{\lambda}I(m+1,\lambda,l)f(z,\zeta) + \left(2 - \frac{l+1}{\lambda}\right)I(m,\lambda,l)f(z,\zeta)$ is univalent in $U \times \overline{U}$ and $[I(m,\lambda,l)f(z,\zeta)]'_z \in \mathcal{H}^*[1,n,\zeta] \cap Q^*$. If*

$$h(z,\zeta) = q(z,\zeta) + zq'_z(z,\zeta) \prec\prec \tag{2.16}$$

$$\frac{l+1}{\lambda}I(m+1,\lambda,l)f(z,\zeta) + \left(2 - \frac{l+1}{\lambda}\right)I(m,\lambda,l)f(z,\zeta), \quad z \in U,\ \zeta \in \overline{U},$$

then

$$q(z,\zeta) \prec\prec \left(I(m,\lambda,l)f(z,\zeta)\right)'_z, \quad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1}$. The function q is the best subordinant.

Proof. Following the same steps as in the proof of Theorem 2.10 and considering $p(z,\zeta) = \left(I(m,\lambda,l)f(z,\zeta)\right)'_z$, the strong differential superordination (2.16) becomes $h(z,\zeta) = q(z,\zeta) + zq'_z(z,\zeta) \prec\prec p(z,\zeta) + zp'_z(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$. By using Lemma 1.2 for $\gamma = 1$, we have $q(z,\zeta) \prec\prec p(z,\zeta)$, i.e. $q(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1} \prec\prec \left(I(m,\lambda,l)f(z,\zeta)\right)'_z$, $z \in U$, $\zeta \in \overline{U}$. The function q is the best subordinant. ■

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ON THE FUZZY STABILITY PROBLEMS OF GENERALIZED QUINTIC MAPPINGS

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Abstract. We introduce a quasi fuzzy (β, p)-norm and generalized quintic mapping and then investigate the Hyers-Ulam-Rassias stability in quasi Banach space and the fuzzy stability by using a fixed point in quasi fuzzy Banach space for the generalized quintic function.

1. Introduction

The concept of stability problem of a functional equation was first posed by Ulam [34] concerning the stability of group homomorphisms. In the next year, Hyers [15] gave a partial answer to the question of Ulam. Hyers' theorem was generalized in various directions. The very first author who generalized Hyers' theorem to the case of unbounded control functions was Aoki [1]. Rassias [30] succeeded in extending the result of Hyers' theorem by weakening the condition for the Cauchy difference. Rassias' paper [30] has provided a lot of influence in the development of Hyers-Ulam stability or Hyers-Ulam-Rassias stability of functional equations. In 1996, Isac and Rassias [17] were first to provide applications of new fixed point theorems for the proof of stability theory of functional equations. By using fixed point methods the stability problems of several functional equations have been extensively investigated by a number of authors; see [5], [6], [26] and [28]. Recently, the stability problem of functional equations was investigated by using shadowing properties; see [20] and [33].

During the last three decades, several stability problems of a large variety of functional equations have been extensively studied and generalized by a number of authors [10], [13], [16], [30], and [2]. In particular, Cho and et al. [8] introduced the quintic functional equation

$$2f(2x+y)+2f(2x-y)+f(x+2y)+f(x-2y) = 20[f(x+y)+f(x-y)]+90f(x)\,. \tag{1.1}$$

It is easy to see that $f(x) = x^5$ is a solution of (1.1) by virtue of the identity

$$2(2x+y)^5 + 2(2x-y)^5 + (x+2y)^5 + (x-2y)^5 = 20[(x+y)^5 + (x-y)^5] + 90x^5\,.$$

For this reason, (1.1) is called a quintic functional equation. Also Xu and et al. [37], Gordji and et al. [14] and Park [27] introduced a quintic mapping and sextic mapping.

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Definition 1.1. *Let X and Y be real linear spaces. A mapping $f : X \to Y$ is called generalized quintic if the quintic functional equation*

$$\begin{aligned} af(ax+y) + af(ax-y) + f(x+ay) + f(x-ay) \\ = a^2(a^2+1)[f(x+y)+f(x-y)] + 2(a^2-1)(a^4-1)f(x) \end{aligned} \tag{1.2}$$

holds for all $x, y \in X$ and all $a \in \mathbb{Z}\,(a \neq 0, \pm 1)$.

Note that the mapping f is called *generalized quintic* because the following algebraic identity

$$\begin{aligned} a(ax+y)^5 + a(ax-y)^5 + (x+ay)^5 + (x-ay)^5 \\ = a^2(a^2+1)[(x+y)^5 + (x-y)^5] + 2(a^2-1)(a^4-1)x^5 \end{aligned}$$

holds for all $x, y \in X$.

We will use the following definition to prove Hyers-Ulam-Rassias stability for the generalized quintic functional equation in the quasi β-normed space. Let β be a real number with $0 < \beta \leq 1$ and $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$.

Definition 1.2. *Let X be a linear space over a field $\mathbb{K}$. A quasi β-norm $||\cdot||$ is a real-valued function on X satisfying the following statements:*

(1) $||x|| \geq 0$ for all $x \in X$ and $||x|| = 0$ if and only if $x = 0$.

(2) $||\lambda x|| = |\lambda|^\beta \cdot ||x||$ for all $\lambda \in \mathbb{K}$ and all $x \in X$.

(3) There is a constant $K \geq 1$ such that $||x+y|| \leq K(||x||+||y||)$ for all $x, y \in X$.

The pair $(X, ||\cdot||)$ is called a *quasi β-normed space* if $||\cdot||$ is a quasi β-norm on X. The smallest possible K is called the *modulus of concavity* of $||\cdot||$. A *quasi β-Banach space* is a complete quasi-β-normed space.

A quasi β-norm $||\cdot||$ is called a (β, p)-norm $(0 < p \leq 1)$ if (3) takes the form $||x+y||^p \leq ||x||^p + ||y||^p$ for all $x, y \in X$. In this case, a quasi β-Banach space is called a (β, p)-Banach space; see [4], [31] and [29].

In 1984, Katsaras [18] and Wu and Fang [35] independently introduced a notion of a fuzzy norm. Since then some mathematicians have defined fuzzy metrics and norms on a linear space from various points of view; see [3], [12], [19], [36] and [24]. In 2003, Bag and Samanta [3] modified the definition of Cheng and Mordeson [7]. Bag and Samanta [3] introduced the following definition of fuzzy normed spaces. The notion of fuzzy stability of functional equations was given in the paper [25].

We will use the definition of fuzzy normed spaces to investigate a fuzzy version of Hyers-Ulam-Rassias stability in the fuzzy normed algebra setting.

Definition 1.3. *Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0, 1]$ is called a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,*

(N_1) $N(x, t) = 0$ for $t \leq 0$;

(N_2) $x = 0$ if and only if $N(x, t) = 1$ for all $t > 0$;

(N_3) $N(cx, t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;

(N_4) $N(x+y, s+t) \geq$ min $\{N(x, s), N(y, t)\}$;

(N_5) $N(x, \cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x, t) = 1$;

(N_6) for $x \neq 0$, $N(x, \cdot)$ is continuous on $\mathbb{R}$.

The pair (X, N) is called a fuzzy normed vector space.

Mirmostafaee [23] introduced a notion for a quasi fuzzy p-normed space as follows:

Definition 1.4. *By a quasi fuzzy norm, we mean a real vector space X, with a fuzzy subset N of $X \times \mathbb{R}$ and some $K \geq 1$ such that all axioms of fuzzy normed space in Definition 1.3 except (N_4) and*

$$(N_4') \quad N(x+y,\, K(s+t)) \geq \min\,\{N(x,\, s),\, N(y,\, t)\} \;(x,\, y \in X\,, s,\, t > 0)\,.$$

hold.

A quasi fuzzy normed space $(X,\, N)$ which satisfies

$$(N_4'') \quad N(x+y,\, \sqrt[p]{s+t}) \geq \min\,\{N(x,\, \sqrt[p]{s}),\, N(y,\, \sqrt[p]{t})\} \;(x,\, y \in X\,, s,\, t > 0)\,,$$

for some $0 < p \leq 1$ is called a quasi fuzzy p-norm.

Definition 1.5. *Let X be a real vector space. A quasi fuzzy p-norm $N : X \times \mathbb{R} \to [0,\, 1]$ is called a quasi fuzzy $(\beta,\, p)$-norm on X if (N_3) in Definition 1.3 takes the form*

$$(N_3') \quad N(cx,\, t) = N(x,\, \frac{t}{|c|^{\beta}}) \quad (c \neq 0,\, 0 < \beta \leq 1)\,.$$

Example 1.6. *Let $(X,\, ||\cdot||)$ be a real normed space. Define*

$$N(x,\, t) = \begin{cases} \frac{t}{t+||x||} & \text{when } t > 0,\, t \in \mathbb{R} \\ 0 & \text{when } t \leq 0\,, \end{cases}$$

where $x \in X$. Then (X, N) is a quasi fuzzy $(\beta\,, p)$-normed space.

Note that when $p = 1$, we call the quasi fuzzy $(\beta\,, p)$-norm a quasi fuzzy β-norm.

Definition 1.7. *Let $(X,\, N)$ be a quasi fuzzy β-normed vector space. A sequence $\{x_n\}$ in X is said to be convergent or converge if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x,\, t) = 1$ for all $t > 0$. In this case, x is called the limit of the sequence $\{x_n\}$ and we denote it by N-$\lim_{n\to\infty} x_n = x$.*

Definition 1.8. *Let $(X,\, N)$ be a quasi fuzzy β-normed vector space. A sequence $\{x_n\}$ in X is called Cauchy if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all integer $d > 0$, we have $N(x_{n+d} - x_n,\, t) > 1 - \varepsilon$.*

It is well-known that every convergent sequence in a quasi fuzzy β-normed vector space is Cauchy. If each Cauchy sequaence is convergent, then the quasi fuzzy β-normed space is said to be *quasi fuzzy complete* and the quasi fuzzy β-normed vector space is called a *quasi fuzzy Banach space*.

Now, we will state the theorem, the alternative of fixed point in a generalized metric space.

Definition 1.9. *Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if d satisfies*

(1) $d(x,\, y) = 0$ if and only if $x = y$;
(2) $d(x,\, y) = d(y,\, x)$ for all $x,\, y \in X$;
(3) $d(x,\, z) \leq d(x,\, y) + d(y,\, z)$ for all $x,\, y,\, z \in X$.

Theorem 1.10 (The alternative of fixed point [21], [32])**.** *Suppose that we are given a complete generalized metric space (X, d) and a strictly contractive mapping $J : X \to X$ with Lipschitz constant $0 < L < 1$. Then for each given $x \in X$, either*

$$d(J^n x,\, J^{n+1} x) = \infty \text{ for all } n \geq 0\,,$$

or there exists a natural number n_0 such that

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(1) $d(J^n x,\, J^{n+1} x) < \infty$ *for all* $n \geq n_0$;
(2) *The sequence* $\{J^n x\}$ *is convergent to a fixed point* y^* *of* J ;
(3) y^* *is the unique fixed point of* J *in the set*

$$Y = \{y \in X | d(J^{n_0} x, y) < \infty\} ;$$

(4) $d(y, y^*) \leq \frac{1}{1-L}\, d(y, Jy)$ *for all* $y \in Y$.

In this paper, we investigate the Hyers-Ulam-Rassias stability in quasi β-normed space and then the fuzzy stability by using a fixed point in fuzzy Banach space for the generalized quintic function $f : X \to Y$ satisfying the equation (1.2). Let us fix some notations which will be used throughout this paper. Let $a \in \mathbb{Z}\, (a \neq 0\,, \pm 1)$.

Before we proceed with stability problems, we will study the generalized quintic function.

Lemma 1.11. *Let X and Y be real linear spaces. Suppose $f : X \to Y$ is a generalized quintic mapping satisfying (1.2). Then*
(1) $f(a^n x) = a^{5n} f(x)$, *for all* $x \in X$ *and* $n \in \mathbb{N}$.
(2) $f(0) = 0$.
(3) f *is an odd mapping.*

Proof. (1) Letting $y = 0$ in the equation (1.2), we have $2a^6 f(x) - 2af(ax) = 0$, that is, $f(ax) = a^5 f(x)$, for all $x \in X$. Now inductively replacing x by ax , we have the desired result. (2) Putting $x = y = 0$ in the equation (1.2), $(a^5 - 1)f(0) = 0$. Hence $f(0) = 0$. (3) Letting $x = 0$ in the equation (1.2), we get $(a^4 - a^3 - a + 1)\Big(f(y) + f(-y)\Big) = 0$, for all $y \in X$. Hence we have $f(y) = -f(-y)$ for all $y \in X$. Thus it is an odd mapping. □

Note that $f(x) = \frac{1}{a^{5n}} f(a^n x)$, for all $x \in X$ and $n \in \mathbb{N}$.

2. Hyers-Ulam-Rassias stability over a quasi β-Banach space

Throughout this section, let X be a real linear space and let Y be a quasi β-Banach space with a quasi β-norm $||\cdot||_Y$. Let K be the modulus of concavity of $||\cdot||_Y$. We will investigate the Hyers-Ulam-Rassias stability for the functional equation (1.2); see also the paper [11].

For a given mapping $f : X \to Y$ and all fixed integer a ($a \neq 0, \pm 1$) , let

$$\begin{aligned} D_a f(x,y) := {} & af(ax+y) + af(ax-y) + f(x+ay) + f(x-ay) \\ & -a^2(a^2+1)\Big(f(x+y) + f(x-y)\Big) - 2(a^2-1)(a^4-1)f(x) \end{aligned} \tag{2.1}$$

for all $x,\, y \in X$.

Theorem 2.1. *Suppose that there exists a mapping $\phi : X^2 \to [0, \infty)$ for which a mapping $f : X \to Y$ satisfies $f(0) = 0$,*

$$||D_a f(x,y)||_Y \leq \phi(x,y) \tag{2.2}$$

and the series $\sum_{j=0}^{\infty} \left(\frac{K}{|a|^{5\beta}}\right)^j \phi(a^j x, a^j y)$ *converges for all* $x,\, y \in X$. *Then there exists a unique generalized quintic mapping $Q : X \to Y$ satisfying the equation (1.2) and the inequality*

$$||f(x) - Q(x)||_Y \leq \frac{K}{2^\beta |a|^{6\beta}} \sum_{j=0}^{\infty} \left(\frac{K}{|a|^{5\beta}}\right)^j \phi(a^j x, 0)\,, \tag{2.3}$$

for all $x \in X$.

Proof. By letting $y = 0$ in inequality (2.2), since $f(0) = 0$ we have

$$\begin{aligned} ||D_a f(x,0)||_Y &= ||2af(ax) + 2f(x) - 2a^2(a^2+1)f(x) - 2(a^2-1)(a^4-1)f(x)||_Y \\ &= 2^\beta |a|^{6\beta} ||f(x) - \frac{1}{a^5} f(ax)||_Y \le \phi(x,0)\,, \end{aligned}$$

that is,

$$||f(x) - \frac{1}{a^5} f(ax)||_Y \le \frac{1}{2^\beta |a|^{6\beta}}\, \phi(x,0)\,, \tag{2.4}$$

for all $x \in X$.

We note that putting $x = ax$ and multiplying $\frac{1}{|a|^{5\beta}}$ in the inequality (2.4), we get

$$\frac{1}{|a|^{5\beta}} ||f(ax) - \frac{1}{a^5} f(a^2x)||_Y \le \frac{1}{2^\beta |a|^{6\beta}}\, \frac{1}{|a|^{5\beta}} \phi(ax,0)\,, \tag{2.5}$$

for all $x \in X$.

Combining two inequalities (2.4) and (2.5), we have

$$||f(x) - \Big(\frac{1}{a^5}\Big)^2 f(a^2x)||_Y \le \frac{K}{2^\beta |a|^{6\beta}} \Big(\phi(x,0) + \frac{1}{|a|^{5\beta}} \phi(ax,0)\Big)\,, \tag{2.6}$$

for all $x \in X$.

Since $K \ge 1$, inductively using the previous note we have the following inequalities

$$||f(x) - \Big(\frac{1}{a^5}\Big)^k f(a^kx)||_Y \le \frac{K}{2^\beta |a|^{6\beta}} \sum_{j=0}^{k-1} \Big(\frac{K}{|a|^{5\beta}}\Big)^j \phi(a^jx,0)\,, \tag{2.7}$$

for all $x \in X$, $k \in \mathbb{N}$ and also

$$||\Big(\frac{1}{a^5}\Big)^k f(a^kx) - \Big(\frac{1}{a^5}\Big)^t f(a^tx)||_Y \le \frac{K}{2^\beta |a|^{6\beta}} \sum_{j=k}^{t} \Big(\frac{K}{|a|^{5\beta}}\Big)^j \phi(a^jx,0)\,, \tag{2.8}$$

for all $x \in X$ and $k\,,\ t \in \mathbb{N}\,(k < t)$.

Since the right-hand side of the previous inequality (2.8) tends to 0 as $t \to \infty$, hence $\{\big(\frac{1}{a^5}\big)^n f(a^nx)\}$ is a Cauchy sequence in the quasi β-Banach space Y . Thus we may define

$$Q(x) = \lim_{n\to\infty} \Big(\frac{1}{a^5}\Big)^n f(a^nx)\,,$$

for all $x \in X$. Since $K \ge 1$, replacing x and y by a^nx and a^ny respectively and dividing by $|a|^{5\beta n}$ in the inequality (2.2) , we have

$$\begin{aligned} &\Big(\frac{1}{|a|^{5\beta}}\Big)^n ||D_a f(a^nx, a^ny)||_Y \\ &= \Big(\frac{1}{|a|^{5\beta}}\Big)^n ||af(a^n(ax+y)) + af(a^n(ax-y)) + f(a^n(x+ay)) + f(a^n(x-ay)) \\ &\qquad -a^2(a^2+1)\Big(f(a^n(x+y)) + f(a^n(x-y))\Big) - 2(a^2-1)(a^4-1)f(a^nx)||_Y \\ &\le \Big(\frac{K}{|a|^{5\beta}}\Big)^n \phi(a^nx, a^ny) \end{aligned}$$

for all $x, y \in X$.

By taking $n \to \infty$, the definition of Q implies that Q satisfies (1.2) for all $x, y \in X$, that is, Q is the generalized quintic mapping. Also, the inequality (2.7) implies the inequality (2.3).

Now, it remains to show the uniqueness. Assume that there exists $T : X \to Y$ satisfying (1.2) and (2.3). Then

$$\begin{aligned}
||T(x) - Q(x)||_Y &= \Big(\frac{1}{|a|^{5\beta}}\Big)^n ||T(a^n x) - Q(a^n x)||_Y \\
&\leq \Big(\frac{1}{|a|^{5\beta}}\Big)^n K\Big(||T(a^n x) - f(a^n x)||_Y + ||f(a^n x) - Q(a^n x)||_Y\Big) \\
&\leq \frac{2K^2}{2^\beta |a|^{6\beta} K^n} \sum_{j=n}^{\infty} \Big(\frac{K}{|a|^{5\beta}}\Big)^j \phi(a^j x, 0)
\end{aligned}$$

for all $x \in X$. By letting $n \to \infty$, we immediately have the uniqueness of Q. □

Theorem 2.2. *Suppose that there exists a mapping $\phi : X^2 \to [0, \infty)$ for which a mapping $f : X \to Y$ satisfies $f(0) = 0$,*

$$||D_a f(x, y)||_Y \leq \phi(x, y) \tag{2.9}$$

and the series $\sum_{j=1}^{\infty} \Big(|a|^{5\beta} K\Big)^j \phi(a^{-j}x, a^{-j}y)$ converges for all $x, y \in X$. Then there exists a unique generalized quintic mapping $Q : X \to Y$ which satisfies the equation (1.2) and the inequality

$$||f(x) - Q(x)||_Y \leq \frac{1}{2^\beta |a|^\beta} \sum_{j=1}^{\infty} \Big(|a|^{5\beta} K\Big)^j \phi(a^{-j}x, 0), \tag{2.10}$$

for all $x \in X$.

Proof. If x is replaced by $\frac{1}{a}x$ in the inequality (2.4), then the proof follows from the proof of Theorem 2.1. □

3. Fuzzy fixed point stability over a Fuzzy Banach space

Let us fix some notations which will be used throughout this section. We assume X is a vector space and (Y, N) is a fuzzy Banach space. Using fixed point method, we will prove the Hyers-Ulam stability of the functional equation satisfying equation (1.2) in fuzzy Banach space.

Theorem 3.1. *Let $\phi : X^2 \to [0, \infty)$ be a function such that there exists an $0 < L < 1$ with*

$$\phi(x, y) \leq \frac{L}{|a|^{5\beta}} \phi(ax, ay) \tag{3.1}$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$N(D_a f(x, y), t) \geq \frac{t}{t + \phi(x, y)} \tag{3.2}$$

for all $x, y \in X$ and all $t > 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} a^{5n} f\Big(\frac{x}{a^n}\Big)$ exists for each $x \in X$ and defines a generalized quintic mapping $Q : X \to Y$ such that

$$N(f(x) - Q(x), t) \geq \frac{2^\beta |a|^{6\beta}(1 - L)\, t}{2^\beta |a|^{6\beta}(1 - L)\, t + L\, \phi(x, 0)} \tag{3.3}$$

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for all $x \in X$ and all $t > 0$.

Proof. By letting $y = 0$ in the inequality (3.2), we have

$$N\Big(2af(ax) - 2a^6 f(x),\, t\Big) \;\geq\; \frac{t}{t + \phi(x,\, 0)} \tag{3.4}$$

for all $x \in X$ and all $t > 0$.

We note that by letting $x = \frac{x}{a}$ in the inequality (3.4) we have

$$N\Big(2af(x) - 2a^6 f\Big(\frac{x}{a}\Big),\, t\Big) \;\geq\; \frac{t}{t + \phi(\frac{x}{a},\, 0)}\,.$$

The inequality (3.1) implies that

$$N\Big(f(x) - a^5 f\Big(\frac{x}{a}\Big),\, \frac{t}{2^\beta |a|^\beta}\Big) \;\geq\; \frac{t}{t + \frac{L}{|a|^{5\beta}}\,\phi(x,\, 0)}\,.$$

By putting $t = \frac{L}{|a|^{5\beta}}\, t$, we have

$$N\Big(f(x) - a^5 f\Big(\frac{x}{a}\Big),\, \frac{L}{2^\beta |a|^{6\beta}}\, t\Big) \;\geq\; \frac{\frac{L}{|a|^{5\beta}}\, t}{\frac{L}{|a|^{5\beta}}\, t + \frac{L}{|a|^{5\beta}}\,\phi(x,\, 0)}\,,$$

that is,

$$N\Big(f(x) - a^5 f\Big(\frac{x}{a}\Big),\, \frac{L}{2^\beta |a|^{6\beta}}\, t\Big) \;\geq\; \frac{t}{t + \phi(x,\, 0)}\,, \tag{3.5}$$

for all $x \in X$ and all $t > 0$.

We consider the set

$$S := \{g : X \to X\}$$

and the mapping d defined on $S \times S$ by

$$d(g,\, h) = \inf\{\mu \in \mathbb{R}^+ \mid N\Big(g(x) - h(x),\, \mu t\Big) \geq \frac{t}{t + \phi(x,\, 0)}\,, \forall x \in X \text{ and } t > 0\}$$

where $\inf \emptyset = +\infty$, as usual. Then $(S,\, d)$ is a complete generalized metric space; see [22, Lemma 2.1]. Now let's consider the linear mapping $J : S \to S$ such that

$$Jg(x) := a^5 g\Big(\frac{x}{a}\Big)$$

for all $x \in X$. Let $g\,, h \in S$ be given such that $d(g\,, h) = \varepsilon$. Then

$$N\Big(g(x) - h(x),\, \varepsilon t\Big) \geq \frac{t}{t + \phi(x,\, 0)}$$

for all $x \in X$ and all $t > 0$.

$$\begin{aligned}
N\Big(Jg(x) - Jh(x),\, L\varepsilon t\Big) &= N\Big(a^5 g\Big(\frac{x}{a}\Big) - a^5 h\Big(\frac{x}{a}\Big),\, L\varepsilon t\Big) \\
= \quad & N\Big(g\Big(\frac{x}{a}\Big) - h\Big(\frac{x}{a}\Big),\, \frac{L}{|a|^{5\beta}}\varepsilon t\Big) \geq \frac{\frac{L}{|a|^{5\beta}} t}{\frac{L}{|a|^{5\beta}} t + \phi(\frac{x}{a},\, 0)} \\
\geq \quad & \frac{\frac{L}{|a|^{5\beta}} t}{\frac{L}{|a|^{5\beta}} t + \frac{L}{|a|^{5\beta}}\phi(x,\, 0)} = \frac{t}{t + \phi(x,\, 0)}
\end{aligned}$$

for all $x \in X$ and all $t > 0$. $d(g,\, h) = \varepsilon$ implies that $d(Jg,\, Jh) \leq L\varepsilon$. Hence we get

$$d(Jg,\, Jh) \leq L\, d(g,\, h)$$

for all $g,\, h \in S$. The inequality (3.5) implies that $d(f,\, Jf) \le \frac{L}{2^\beta |a|^{6\beta}}$. By Theorem 1.10, there exists a mapping $Q : X \to Y$ such that

(1) Q is a fixed point of J, that is,

$$Q\Big(\frac{x}{a}\Big) = \frac{1}{a^5}\, Q(x) \tag{3.6}$$

for all $x \in X$. The mapping Q is a unique fixed point of J in the set $M = \{g \in S \,|\, d(f,\, g) < \infty\}$. This means that Q is a unique mapping satisfying the equation (3.6) such that there exists a $\mu \in (0,\, \infty)$ satisfying

$$N\Big(f(x) - Q(x),\, \mu t\Big) \ge \frac{t}{t + \phi(x,\, 0)}$$

for all $x \in X$ and all $t > 0$;

(2) $d(J^n f,\, Q) \to 0$ as $n \to \infty$. This implies the following equality

$$\text{N-}\lim_{n\to\infty} a^{5n} f\Big(\frac{x}{a^n}\Big) = Q(x)$$

for all $x \in X$;

(3) $d(f,\, Q) \le \frac{1}{1-L}\, d(f,\, Jf)$, which implies the inequality

$$d(f,\, Q) \le \frac{1}{1-L} \cdot \frac{L}{2^\beta |a|^{6\beta}} = \frac{L}{2^\beta |a|^{6\beta}(1-L)}\,.$$

It implies that

$$N\Big(f(x) - Q(x),\, \frac{L}{2^\beta |a|^{6\beta}(1-L)} t\Big) \ge \frac{t}{t + \phi(x,\, 0)}$$

for all $x \in X$ and all $t > 0$. By replacing t by $\frac{2^\beta |a|^{6\beta}(1-L)}{L}\, t$, we have

$$N\Big(f(x) - Q(x),\, t\Big) \ge \frac{2^\beta |a|^{6\beta}(1-L)\, t}{2^\beta |a|^{6\beta}(1-L)\, t + L\phi(x,\, 0)}$$

for all $x \in X$ and all $t > 0$. That is, the inequality (3.3) holds. By letting $x = \frac{x}{a^n}$ and $y = \frac{y}{a^n}$ in the inequality (3.2), we have

$$N\Big(a^{5n} D_a f\Big(\frac{x}{a^n},\, \frac{y}{a^n}\Big),\, |a|^{5\beta n}\, t\Big) \ge \frac{t}{t + \phi(\frac{x}{a^n},\, \frac{y}{a^n})}$$

for all $x,\, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Replacing t by $\frac{t}{|a|^{5\beta n}}$,

$$N\Big(a^{5n} D_a f\Big(\frac{x}{a^n},\, \frac{y}{a^n}\Big),\, t\Big) \ge \frac{\frac{t}{|a|^{5\beta n}}}{\frac{t}{|a|^{5\beta n}} + \phi(\frac{x}{a^n},\, \frac{y}{a^n})} \ge \frac{t}{t + L^n\, \phi(x,\, y)}$$

for all $x,\, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{t}{t + L^n\, \phi(x,\, y)} = 1$ for all $x,\, y \in X$ and all $t > 0$, we may conclude that

$$N\Big(D_a Q(x,\, y),\, t\Big) = 1$$

for all $x,\, y \in X$ and all $t > 0$. Thus the mapping $Q : X \to Y$ is the generalized quintic mapping. $\square$

Corollary 3.2. *Let $\theta \ge 0$, $p > 5$ be a real number and X be a normed linear space with norm $||\cdot||$. Suppose $f : X \to Y$ is a mapping satisfying $f(0) = 0$ and*

$$N(D_a f(x,y),\, t) \ge \frac{t}{t + \theta(||x||^p + ||y||^p)} \tag{3.7}$$

for all $x,\, y \in X$ *and all* $t > 0$. *Then* $Q(x) := N\text{-}\lim_{n\to\infty} a^{5n} f\Big(\frac{x}{a^n}\Big)$ *exists for each* $x \in X$ *and defines a generalized quintic mapping* $Q : X \to Y$ *such that*

$$N(f(x) - Q(x),\, t) \geq \frac{2^\beta |a|^\beta (|a|^{p\beta} - |a|^{5\beta})\, t}{2^\beta |a|^\beta (|a|^{p\beta} - |a|^{5\beta})\, t + \theta\, ||x||^p}$$

for all $x \in X$ *and all* $t > 0$.

Proof. The proof follows from Theorem 3.1 by taking $\phi(x,\, y) = \theta(||x||^p + ||y||^p)$ for all $x\,, y \in X$ and $L = |a|^{(5-p)\beta}$. □

Theorem 3.3. *Let* $\phi : X^2 \to [0, \infty)$ *be a function such that there exists an* $0 < L < 1$ *with*

$$\phi(x, y) \leq |a|^{5\beta}\, L\, \phi(\frac{x}{a}, \frac{y}{a})$$

for all $x,\, y \in X$. *Let* $f : X \to Y$ *be a mapping satisfying* $f(0) = 0$ *and*

$$N(D_a f(x, y),\, t) \geq \frac{t}{t + \phi(x, y)} \tag{3.8}$$

for all $x,\, y \in X$ *and all* $t > 0$. *Then* $Q(x) := N\text{-}\lim_{n\to\infty} a^{-5n} f(a^n\, x)$ *exists for each* $x \in X$ *and defines a generalized quintic mapping* $Q : X \to Y$ *such that*

$$N(f(x) - Q(x),\, t) \geq \frac{2^\beta |a|^{6\beta} (1 - L)\, t}{2^\beta |a|^{6\beta} (1 - L)\, t + \phi(x, 0)} \tag{3.9}$$

for all $x \in X$ *and all* $t > 0$.

Proof. Let $(S,\, d)$ be the generalized metric space defined in the proof of Theorem 3.1. Consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{a^5} g(ax)$$

for all $x \in X$. Let $g\,, h \in S$ be given such that $d(g\,, h) = \varepsilon$. Then

$$N\Big(g(x) - h(x),\, \varepsilon t\Big) \geq \frac{t}{t + \phi(x,\, 0)}$$

for all $x \in X$ and all $t > 0$.

$$\begin{aligned} N\Big(Jg(x) - Jh(x),\, L\varepsilon t\Big) &= N\Big(\frac{1}{a^5} g(ax) - \frac{1}{a^5} h(ax),\, L\varepsilon t\Big) \\ = \quad N\Big(g(ax) - h(ax),\, |a|^{5\beta} L\varepsilon t\Big) &\geq \frac{|a|^{5\beta} L t}{|a|^{5\beta} L t + \phi(ax,\, 0)} \\ \geq \quad \frac{|a|^{5\beta} L t}{|a|^{5\beta} L t + |a|^{5\beta} L \phi(x,\, 0)} &= \frac{t}{t + \phi(x,\, 0)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. $d(g,\, h) = \varepsilon$ implies that $d(Jg,\, Jh) \leq L\varepsilon$. Hence we get

$$d(Jg,\, Jh) \leq L\, d(g,\, h)$$

for all $g,\, h \in S$. Similar to the note in proof of Theorem 3.1, we have

$$N\Big(f(x) - \frac{1}{a^5} f(ax),\, \frac{1}{2^\beta |a|^{6\beta}} t\Big) \geq \frac{t}{t + \phi(x,\, 0)}$$

for all $x \in X$ and all $t > 0$. Hence we have $d(f,\, Jf) \leq \frac{1}{2^\beta |a|^{6\beta}}$. By Theorem 1.10, there exists a mapping $Q : X \to Y$ such that

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(1) Q is a fixed point of J, that is,

$$Q(ax) = a^5\, Q(x) \tag{3.10}$$

for all $x \in X$. The mapping Q is a unique fixed point of J in the set $M = \{g \in S \,|\, d(f, g) < \infty\}$. This means that Q is a unique mapping satisfying the equation (3.10) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N\Big(f(x) - Q(x),\, \mu t\Big) \geq \frac{t}{t + \phi(x, 0)}$$

for all $x \in X$;

(2) $d(J^n f, Q) \to 0$ as $n \to \infty$. This implies the following equality

$$\text{N-}\lim_{n\to\infty} \frac{1}{a^{5n}} f(a^n x) = Q(x)$$

for all $x \in X$ and all $t > 0$;

(3) $d(f, Q) \leq \frac{1}{1-L}\, d(f, Jf)$, which implies the inequality

$$d(f, Q) \leq \frac{1}{1-L} \cdot \frac{1}{2^\beta |a|^{6\beta}} = \frac{1}{2^\beta |a|^{6\beta}(1-L)}\,.$$

This implies the inequality (3.9) holds. The remains of the proof is similar to the proof of Theorem 3.1. □

Corollary 3.4. *Let $\theta \geq 0$, $0 < p < 5$ be a real number and X be a normed linear space with norm $||\cdot||$. Suppose $f : X \to Y$ is a mapping satisfying $f(0) = 0$ and the inequality (3.7). Then $Q(x) := N\text{-}\lim_{n\to\infty} a^{-5n} f(a^n x)$ exists for each $x \in X$ and defines a generalized quintic mapping $Q : X \to Y$ such that*

$$N(f(x) - Q(x), t) \geq \frac{2^\beta |a|^\beta (|a|^{5\beta} - |a|^{p\beta})\, t}{2^\beta |a|^\beta (|a|^{5\beta} - |a|^{p\beta})\, t + \theta\, ||x||^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 3.3 by taking $\phi(x, y) = \theta(||x||^p + ||y||^p)$ for all $x, y \in X$ and $L = |a|^{(p-5)\beta}$. □

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